EXACT COUNTING OF
4D BPS BLACK HOLES

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PLAN OF TALK

• 4D/5D Connection
• 1/4 BPS BHs
  in $\mathcal{N}=4$ string theory
• 1/8 BPS BHs
  in $\mathcal{N}=8$ string theory
• Comparison to topological strings
THE 4D/5D CONNECTION

IR^4 \rightarrow IR^4 \times IR^3 \times S^1

KK monopole

A 4D BH with 1 unit of KK monopole charge?

Spinning 5D Taub-NUT BH

IR^3 \rightarrow IR^4

Spinning 5D BH

BMPV

Type IIA String theory on a CY

BPS BH \leftrightarrow DO-D2-D4-D6 System

charge: (p_0, p_A; P^A, P^0)

D6 wrapped on CY

→ KK monopole charge

(M-theory on CY x S^1)

CONSIDER DO-D2-D6 with a single D6-brane,

PROPOSAL:

\[ Z_{4D}(p^0=1, q_0, q_A) = Z_{5D}(q_A, J_L = \frac{q_0}{2}) \]

Subtlety: 4D index \( \text{Tr}_{BPS} (-)^2 J^3 \)

5D index \( \text{Tr}_{BPS} (-)^2 J^3 + 2J^3_R \)

\[ J^3 \leftrightarrow J_A^3 \]

\[ \frac{q_0}{2} \leftrightarrow J_L^3 \]

\[ Z_{5D} = (-)^{q_0} Z_{4D} \]
CLASSICAL ENTROPY

\[ S_{4D} = 2\pi \sqrt{p^0 Q^3 - \frac{1}{4}(p^0 q^0)^2}, \]

\[ Q^3 \equiv (D_{abc} y^a y^b y^c)^2, \]

\[ q^0 = 3 D_{abc} y^b y^c. \]

\[ S_{4D} = 2\pi \sqrt{Q^3 - \frac{1}{4} q^0} \]

BMPV

\[ S_{5D} = 2\pi \sqrt{Q^3 - J^2} \]

\[ N = 4 \text{ String theory} \]

- IIA on \( K3 \times T^2 \)

A 1/4-BPS BH

\[ D0-D2-D6 \]

\[ (p^0 = 1) \]

\[ \text{loft to 5D} \]

\[ \text{M2 on } K3 \times T^2 \text{ w/ } J_L \]

\[ \text{compactify on } S^1 \subset T^2 \]

IA:

\[ D2 \text{ on } K3 \]

\[ + F1 \text{ on } S^1 \]

\[ + J_L \]

\[ \text{T-duality} \]

IB:

\[ D3 \text{ on } K3 \times S^1 \]

+ momentum along \( S^1 \)

+ \( J_L \)

\[ \text{U-duality} \]

IB:

\[ D1-D5 \text{ on } K3 \times S^1 \]

+ momentum along \( S^1 \)

+ \( J_L \)
\[ Z_{D1-D5} = \sum d_{5D}(L_0, N, J_L) e^{2\pi i (L_0 p + N \sigma + 2J_L \nu)} \]

\[ = \prod_{k \geq 0, \ell \geq 0} (1 - e^{2\pi i (k p + \ell \sigma + m \nu)})^{-c(4k \ell - n^2)} \]

Dijkgraaf, Moore, Verlinde, Verlinde

c(4k \ell - n^2) : coefficients of elliptic genus of a single K3

\[ Z_0 = (e^{\pi i \nu} - e^{-\pi i \nu})^{-2} e^{-2\pi i \sigma} \]

\[ \times \prod_{n \geq 1} (1 - e^{2\pi i (n \sigma + \nu)})^{-2} (1 - e^{2\pi i (n \sigma + \nu - 1)})^{-2} \]

\[ \times (1 - e^{2\pi i (n \sigma + \nu)})^{-20} \]

Antoniadis, Gava, Narain, Taylor

\[ Z_0 \cdot Z_{D1-D5} = \frac{e^{2\pi i \sigma}}{\Phi(p, \sigma, \nu)} \]

weight 10 modular form of \( \text{Sp}(2, \mathbb{Z}) \)

Dijkgraaf, Verlinde, Verlinde

\[ \text{D1-D5 on } K3 \times S^1 \rightarrow \text{Sym}^{a_1a_5}(K3) \text{ CFT} \]

elliptic genus:

\[ \chi_{N=a_1a_5}(y, \delta) = \text{Tr} (-)^F y^{F_L} \delta^{L_0} \]

4D/5D Connection

\[ Z_{4D} = Z_0 \cdot Z_{D1-D5} \]

partition function of D5 on K3xS^1

winding modes of a heterotic string
\( N = 4 \) string theory has U-duality group
\[ \text{SL}(2, \mathbb{Z}) \times O(6, 22; \mathbb{Z}) \]

Charge vectors: (ignoring \( F_1 \) and NS5 charges)
\[ g_e = (g_0, g_1, \ldots, g_{22}, p^{23}) \]
\[ g_m = (p^0, p^1, \ldots, p^{22}, q_{23}) \]

\( \text{BH degeneracy} \)
\[ d_{4D}(g_e, g_m) = d(g_e^2, g_m^2, g_e g_m) \]

\[ Z_{4D} = \sum_{k, l, m} d(k, l, m) \exp\left(2\pi i (k\phi + l\sigma + m\nu)\right) \]

\[ = \frac{1}{\Phi(\phi, \sigma, \nu)} \]
\textbf{ANSWER: (from 5D)}

\[ Z_{4D} = \eta(g)^{-6} \sum_{m \in \mathbb{Z}} g^{m^2} \]

\textbf{A 4D DERIVATION?}

- degeneracy depends only on \( J \)
- D4-D0 system
  \[ J = 4 g_0 D = 4 g_0 p^1 p^2 p^3 \]

\textbf{CONSIDER}

\[ p^1 = p^2 = p^3 = 1 \]

\[ J = 4 g_0 \]

- \( J \equiv 0 \) or \(-1 \) mod 4
  - The other case can be obtained by turning on D2-brane charge
- Do partition function is then
  \[ Z(g) = \sum_n d(J = 4n) g^n \]
  \[ = \eta(g)^{-6} \sum_{m \in \mathbb{Z}} g^{(m+\frac{1}{2})^2} \]

\textbf{D4 wrapped on (1,1,1) cycle \( P \subseteq T^6 \)}

\[ P \subseteq T^6 \quad \text{cplx submanifold for} \]
\[ \text{generic (tilted) } T^6 \]
\[ \text{(algebraic)} \]

\[
\begin{array}{cccc}
1 & 3 & 3 \\
3 & 10 & 3 \\
3 & 3 & 1 \\
\end{array}
\]

- \( \chi(p) = 6 \)
- \( \sigma(p) = -2 \)
- \( b_1(p) = 6 \)
- \( b_2(p) = 16 = b_2(T^6)+1 \)

- \( P \) has an extra 2-cycle \( \gamma \)
  - not induced from \( T^6 \)
  - can turn on gauge field flux
  - without induced D2-brane charge
  - but with induced D0-brane charge

- \( \alpha_1, \alpha_2, \alpha_3 \) 2-cycles induced from \( T^6 \)
  - \( \gamma \cdot \alpha_i = 1, \quad \gamma \cdot \gamma = 1 \)
  - \( \beta = 2 \gamma - \Sigma \alpha_i, \quad \beta \cdot \alpha_i = 0, \quad \beta \cdot \beta = -2 \)

- Freed-Witten anomaly:
  \[ F = \frac{c_1(P)}{2} + \text{integral} \]
  \[ c_1(P) = -\Sigma \alpha_i \]
Allowed fluxes that do not induce D2-charge:
\[ F = (m + \frac{1}{2}) \beta, \quad m \in \mathbb{Z} \]

Induce D0-brane charge
\[ \Delta q_0 = - \int \frac{F^2}{2} + \frac{c_2(P)}{24} \]
\[ = (m + \frac{1}{2})^2 - \frac{1}{4} \]

D4-D0 bound state:
D0 either dissolve into F
or be bound to D4 as instantons

\[ Z = 7(g)^2 \sum_{m \in \mathbb{Z}} \left( m + \frac{1}{2} \right)^2 \]
agree with 5D!

A little more on D4-D0 on T^6

- D4 wraps on a holomorphic (1,1,1) cycle in T^6 \(\Rightarrow T^6\) a principally polarized Abelian variety

\[ S_3: \text{genus 3 Riemann surface} \]
\[ S_3 \rightarrow J(S_3) \simeq T^6 \]

D4 is wrapped on divisor \( P_{(1,1,1)} = S_3 + S_3 \)
\[ \simeq \text{Sym}^2 S_3 \]

\[ S_3 \subset P_{(1,1,1)} \]
represents the class \( \gamma \)

- D4 wrapped on \( (P_1, P_2, P_3)? \)
moduli space \( \text{IP}^2 \times \text{IP}^2 \times (x T^6) \)

special locus in moduli space where D0's can dissolve into other kinds of F- flux...

work in progress
MACROSCOPIC ENTROPY?

\[ S = \sum_{S_H^2} E_{ab} E_{cd} \partial \frac{\mathcal{L}_{\text{eff}}}{\partial R_{abcd}} \]

Ward's formula

Correction from \( R^2 F^2 g^{-2} \) terms

- Cardoso, de Wit, Mohaupt

related to \( F_{\text{top}} (g_{\text{top}}, t^A) \) by

Legendre transform

- Ooguri, Strominger, Vafa
- Sen

OSV Conjecture:

\[ Z_{\text{BH}} (p, \phi) = \sum_{\mathcal{O}} \Omega (p, \mathcal{O}) e^{-g \cdot \phi} = |Z_{\text{top}}|^2 \]

What are these?

Strategy: compute \( Z_{\text{BH}} \) from the exact degeneracy as given by our indices for \( N=4 \) and \( N=8 \) BHs, and compare to \( Z_{\text{top}} \).
A simplification

We shall restrict to the case \( p^a = 0 \), i.e. no D6-brane charge
- simplifies calculations
- if \( p^a \neq 0 \), classical entropy
  \[ S_{cl} \sim 2\pi \sqrt{p^0 g^3} \]

Summation

\[ \sum e^{S_{cl} - g A} \]

"badly" divergent

if \( p^0 = 0 \), D4-\( D_2 \)-\( D_0 \)

\[ S_{cl} \sim 2\pi \sqrt{D\left( g_0 + \frac{1}{2} D_{AB} g_A g_B \right)} \]

OSV summation can be regularized.

\( N = 8 \)

\[ \Omega(p^0, g_A) = d(J(p^0, g_A)) \]

\[ \sum d(J) \frac{\delta J}{\delta g} = 7(8^4)^{-6} \sum g^{m^2} \]

electric \( g_0, g_A \)

charges:
- left \( \text{wrapped} \) D2,
- KK-monopole, NS5,
- magnetic \( p^0, p_A \)
- left \( \text{wrapped} \) D4,
- momenta, F1

\[ J = 4D_{ABC} p^A p^B p^C \left( g_0 + \frac{1}{2} D_{AB} g_A g_B \right) \]

Compute mixed partition sum \( Z_{BH} \)
(Poisson resummation)

\[ Z_{BH}(p, \phi) = \sum_{\phi^0 + 2\pi i k} 12 \exp \left( \frac{i V_{Tr}^6}{15 g_0^4} \right) \]

\[ 1 + O(e^{-V_{Tr}^6/15 g_0^4}) \]
Topological string on $T^6$

$$Z_{top} = e^{F_0}, \quad F_0 = \frac{i}{2\pi} D_{ABC} \tau^A \tau^B \tau^C$$

$$= i D_{ABC} \frac{X^A X^B X^C}{X^0}$$

$$V_{T^6} = D_{ABC} \text{Im} \tau^A \text{Im} \tau^B \text{Im} \tau^C$$

$$\mathcal{N} = 4$$

$$\Omega(p, q) = \oint dp \, d\sigma \, dv \frac{e^{i(2p^2 \sigma^2 + 2q^2 \sigma + (2n-1) \omega \bar{\omega})}}{\Phi(p, \sigma, v)}$$

DVV: calculate by contour integral around poles

- rational quadratic divisors

$Sp(2, \mathbb{Z})$ images of $\nu = 0$

$$\Omega = \begin{pmatrix} p & \nu \\ \nu & \sigma \end{pmatrix} \rightarrow (A \Omega + B)(C \Omega + D)^{-1}$$

$$(A \quad B) \in Sp(2, \mathbb{Z})$$

rqd's are of the form

$$a(p \sigma - \nu^2) + k\rho + l\sigma + m\nu + c = 0$$

$(k, l, m, a, c)$ integers

$$4ac - 4kl + m^2 - 1 = 0.$$
Exclude $a = 0$, $\Rightarrow \nu = 0$

$\sigma + \nu - \nu^2 = 0$ dominates the contribution.

Other rgs' suppressed by $O(e^{-Q^2})$
on-perturbative

Calculate......

$Z_{BH}(p, \phi) = \sum_{\phi^a \to \phi^a + 2\pi ik^a} |Z_{top}|^2 |g_{\phi'|\phi}|^2$

$Z_{top} = \epsilon F_0 + F_1 = \epsilon^{iD_{abc}} \frac{x^A x^B x^C}{x^0} \eta(\phi')^{-24}$

$e^{-K} = \text{Re}(\bar{X}^A \partial \alpha F_{\text{top}})$
a "quantum corrected" Kähler potential

A better (?) way to organize our answers (both $N=8$ and $N=4$)

$Z_{BH}(p, \phi) = \sum_{\phi \rightarrow \phi + 2\pi i k}$

$Z_{top}^2 = \left\{ \begin{array}{l}
|Z_{top}|^2 (1 + \text{im} t')^{-12} \quad \text{K3xT2} \\
|Z_{top}|^2 e^{4K} \quad \text{T6}
\end{array} \right.$

$\eta$ holomorphic anomaly

$g(\phi') = g(\phi') + \delta g$

$\delta g = \left\{ \begin{array}{l}
\text{1-loop, K3xT2}
\end{array} \right.$
What is the origin of mismatch?

- supersymmetric index

\[ \text{Ward's formula from } \text{Eff} \]

- The role of holomorphic anomaly?

- Fragmentation?

What's NEXT?

- $N=2$ orbifold of $N=4$ models?

- Precise counting of large $N=2$ CY BHs?

quivers? attempt in progress...