The $M$-theory $C$-field and $E_8$ Gauge theory

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based on
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to appear...

1. Introduction: Statement of 3 problems
2. What is a “$C$-field”?
3. The $E_8$ model for the $C$-field.
5. The Gauss law and the electric charge: $\Theta_X(C)$.
6. Some Applications

Related works:

Witten 96, Witten 99
Diaconescu, Moore, and Witten 00
Hopkins and Singer, math.AT/0211216
Introduction

This talk is about the relation of $M$-theory - a hypothetical generalization of string theory - to $E_8$ gauge theory in 10, 11, and 12 dimensions.

In different “physical regimes” $M$ theory must be well approximated by

- 11-dimensional supergravity
- Type II string theory
- $E_8 \times E_8$ heterotic string theory

Basic Philosophy: Formulating these relationships precisely, in the presence of nontrivial topology, challenges our understanding of the fundamental formulation of these theories.

The example in this talk is the precise formulation of the $M$-theory $C$-field with nontrivial topology. This leads to $E_8$ gauge theory.

Let’s sharpen our approach with a statement of 3 motivating problems.
Problem 1: 11-dimensional supergravity

We will be considering physics on an 11-dimensional, oriented, spin manifold, $Y$.

The basic fields of 11-dimensional supergravity are

- A metric $g$, (Lorentzian or Euclidean)
- A “$C$-field.” Initially, it is a 3-form potential
  
  $$C \in \Omega^3(Y).$$

  This generalizes the Maxwell potential $A \in \Omega^1$ of electromagnetism. The fieldstrength is
  
  $$G := dC \in \Omega^4(Y).$$

- A gravitino field $\psi \in \Gamma(S \otimes T^*Y)$

The action for the theory is (schematically):

$$\exp\left[- \int_Y \text{vol}(g) \mathcal{R}(g) + G \wedge *G + \bar{\psi} \slashed{D} \psi \right] \Phi(C)$$

$$\Phi(C) \sim \exp \left(2\pi i \int_Y \frac{1}{6} CG^2 - CI_8(g) \right)$$

$I_8(g)$: quartic polynomial in $R$.

Find a mathematically precise definition of $\Phi(C)$ when $G$ is cohomologically nontrivial, and $\partial Y = X$ is nonempty.
Problem 2A: Gauss Law and $C$-field electric charge

When $\partial Y = X$ is nonempty, the path integral for the $C$-field on a manifold with boundary $Y$ defines a wavefunction of the boundary values $C_X$ of $C$: $\Psi(C_X)$.

Because of the Chern-Simons phase $\Phi(C)$, formulating the gauge invariance of $\Psi(C_X)$ is nontrivial.

**Problem 2A: What conditions on $C_X$ are necessary for a nonvanishing gauge invariant wavefunction?**

Close analogy: $D = 3$ $U(1)$ gauge theory

$$S = \int_Y \frac{1}{2g_{ym}^2} F \ast F + \frac{k}{4\pi} A \wedge F$$

Gauss law $\Rightarrow [F] = 0$.

$g_{ym}^2 \to \infty$ recovers Chern-Simons theory. The Gauss law is $F(A_X) = 0$, and leads directly to the mathematical interpretation:

$$\Psi \in \Gamma(\mathcal{L} \to \mathcal{M}(F = 0))$$
Problem 2B: Defining $C$-field electric charge

In electromagnetism in $n$-dimensions:

$$d \ast F = j_e$$

Then $[j_e] \in H_{DR}^{n-1}(X)$ is the electric charge.

In $M$-theory, the Chern-Simons term $\Phi(\hat{C})$ is cubic and hence the theory is nonlinear:

$$d \ast G = \frac{1}{2} G^2 - I_8(g)$$

Background metrics and $C$-fields induce electric charge, and

$$[\frac{1}{2} G^2 - I_8] \in H_{DR}^8(X)$$

is the induced charge, as a DeRham cohomology class.

However, a basic axiom of the theory is that $C$-field electric charge should live in $H^8(X, \mathbb{Z})$.

Problem 2B: Define the $C$-field electric charge

$$\Theta_X(C) \in H^8(X, \mathbb{Z})$$

This is an integral lift of $[\frac{1}{2} G^2 - I_8]$. 
Problem 3: The emergence of $E_8$ gauge symmetry

- $M/K3 = Het/T^3 \Rightarrow$ Enhanced $E_8$ gauge symmetry arises on singular K3 surfaces.

- Moreover, a construction of Horava-Witten shows that a quotient of $M$-theory on $X \times S^1$ by an orientation reversing isometry of $S^1$ leads to the $E_8 \times E_8$ heterotic string:

- In 1996 Witten gave a definition of $\Phi(C)$ that used $E_8$ gauge theory in 12 dimensions (reviewed below). This definition was then used to establish a connection to the $K$-theoretic classification of RR fluxes in the limit that $M$ theory reduces to type II string theory:

All this suggests a hidden $E_8$ structure in $M$ theory which might point the way to a useful reformulation of the theory.

What is the precise relation of the $C$-field of 11-dimensional supergravity to an $E_8$ gauge field?
What is a C-field?

First we will describe the gauge equivalence class (gec) of C-fields.

Warmup: How do you describe the gec of $U(1)$ gauge fields on a manifold $M$?

Answer: All gauge invariant information is encoded in the holonomy around 1-cycles $\gamma$:

$$\tilde{\chi}_A(\gamma) = \exp\left(i \oint_\gamma A\right)$$

Regard the holonomy as a map

$$\tilde{\chi}_A : Z_1(M) \to U(1)$$

How does $\tilde{\chi}_A$ differ from an arbitrary such map?

(*) There exists a closed 2-form with $2\pi\mathbb{Z}$-periods,

$$F \in \Omega^2_{2\pi\mathbb{Z}}(M),$$

such that if $\gamma = \partial B_2$ is a boundary then

$$\tilde{\chi}_A(\gamma) = \exp\left(i \int_{B_2} F\right)$$

Maps $\tilde{\chi}$ that satisfy (*) are in 1-1 correspondence with the gec of $U(1)$ bundles with connection.
Cheeger-Simons characters

The generalization to $U(1)$ $p$-form gauge fields:

Definition: A **Cheeger-Simons character** $\tilde{\chi} \in \tilde{H}^{p+1}(M)$ is a homomorphism

$$\tilde{\chi} : Z_p(M) \rightarrow U(1)$$

such that there is a **fieldstrength** $\omega(\tilde{\chi}) \in \Omega^{p+1}_\mathbb{Z}(M)$:

$$\Sigma_p = \partial B_{p+1} \Rightarrow \tilde{\chi}(\Sigma_p) = \exp \left( 2\pi i \int_{B_{p+1}} \omega(\tilde{\chi}) \right)$$

The gec of a $p$-form gaugefield is a character $\tilde{\chi} \in \tilde{H}^{p+1}(M)$. 
How to describe a Cheeger-Simons character

The gauge invariant information in a Cheeger-Simons character can be expressed in two distinct ways, each of which is summarized by an exact sequence.

Sequence for “flat characters”: $\omega(\tilde{\chi}) = 0$:

$$0 \to H^p(M, U(1)) \to \check{H}^{p+1}(M) \to \Omega^{p+1}_{\mathbb{Z}}(M) \to 0$$

Sequence for “topologically trivial characters”:

If $A \in \Omega^p(M)$ is globally defined, then we may define a differential character:

$$\tilde{\chi}_A(\Sigma_p) := \exp[2\pi i \int_{\Sigma_p} A].$$

a.) $\tilde{\chi}_A$ only depends on $A$ modulo $\Omega^p_{\mathbb{Z}}$.

b.) The fieldstrength $\omega(\tilde{\chi}_A) = dA$ is trivial in cohomology:

$$0 \to \Omega^p / \Omega^p_{\mathbb{Z}} \to \check{H}^{p+1}(M) \to H^{p+1}(M, \mathbb{Z}) \to 0$$

The projection defines the characteristic class of $\tilde{\chi}$:

$$a(\tilde{\chi}) \in H^{p+1}(M, \mathbb{Z}).$$

Relation between the sequences:

$$a(\tilde{\chi})_{\mathbb{R}} = [\omega(\tilde{\chi})]_{DR}$$
Why do we need it?

The notion of Cheeger-Simons character captures torsion information invisible to differential forms.

It can happen that

\[ [\Sigma_p] \neq 0 \quad \text{but} \quad N\Sigma_p = \partial B_{p+1} \]

\[
(\tilde{\chi}(\Sigma_p))^N = \exp 2\pi i \int_{B_{p+1}} \omega(\tilde{\chi})
\]

The extra data of a C-S character tells us how to take the \( N^{th} \) root.

Note:

\[
H^p(M, U(1)) = \underbrace{T \amalg T \amalg \cdots \amalg T}_{H^{p+1}(M, \mathbb{Z})_{\text{tors}}}
\]

\[
T = H^p(M, \mathbb{R})/\tilde{H}^p(M, \mathbb{Z})
\]

\[
\tilde{H}^p(M, \mathbb{Z}) = H^p(M, \mathbb{Z})/H^p(M, \mathbb{Z})_{\text{tors}}
\]
Examples in Physics

• $\tilde{H}^1(M)$: The space of $U(1)$-valued functions on $M$:

Suppose $\phi : M \to U(1)$. If $\Sigma_0 = \sum_{\alpha} n_{\alpha} P_{\alpha} \in Z_0(M)$,

$$\tilde{\phi}(\Sigma_0) := \prod_{\alpha} (\phi(P_{\alpha}))^{n_{\alpha}}$$

$$\omega(\tilde{\phi}) = \frac{1}{2\pi i} \phi^{-1} d\phi$$

$$a(\tilde{\phi}) = \phi^*([dt])$$

In physics: $\tilde{H}^1(M)$ is the gauge group of electromagnetism.

• $\tilde{H}^2(M)$ is the space of gec of line bundles with connection.

• $\tilde{H}^3(M)$ is the space of gec of “gerbe connections” or “$B$-fields.”

• $\tilde{H}^4(Y)$: The “membrane coupling of $M$-theory”
The membrane of $M$-theory

In Maxwell theory, a charged particle, of charge $e$, moving along a worldline $\gamma$, couples to the potential $A$ via the holonomy:

$$\exp[i \int_\gamma eA]$$

In $M$ theory, there are “electrically charged” membranes, whose 3-dimensional worldvolumes couple to the $C$-field via

$$\sim \exp[2\pi i \int_\Sigma C]$$

The coupling of $C$ to all possible membrane worldvolumes $\Sigma$ summarizes the gauge invariant information in $C$, and hence we conclude that

$$[C] \in \check{H}^4(Y)$$

up to a subtlety...
Shifted differential characters

More accurately, the coupling is

$$\sqrt{\text{Det}D_{S(N)}} \exp\left(2\pi i \int_{\Sigma} C\right)$$

$$\Rightarrow \quad [C_1 - C_2] \in \tilde{H}^4(Y)$$

Because of fermion determinants $[C]$ is actually a “shifted differential character” with quantization of its fieldstrength:

$$[G]_{DR} = a_{\mathbb{R}} - \frac{1}{2} \lambda_{\mathbb{R}}$$

$a \in H^4(Y, \mathbb{Z})$, $\lambda = \text{class of the spin bundle of } Y$.

Conclusion: “What is a $C$-field?” Partial answer:

$$[C] \in \tilde{H}^4_{\frac{1}{2} \lambda}(Y)$$
Models for the C-field

At this point we have said what the gauge equivalence class of a C-field is, but have NOT answered the question:

“What is a C-field?”

Analogy: Formulate nonabelian gauge theory on $A/G$. Locality forces us to introduce redundant variables, $A \in A$.

Physicists usually think of $C \in \Omega^3(Y)$.

- Small gauge transformations: $C \to C + d\Lambda$, $\Lambda \in \Omega^2(Y)$
- “All” gauge transformations: $C \to C + \omega$, $\omega \in \Omega^3_{\mathbb{Z}}(Y)$

This is OK if $[C] \in \Omega^3(Y)/\Omega^3_{\mathbb{Z}}(Y)$.

But many interesting nontrivial phenomena in string/M-theory involve topologically nontrivial characters $[C]$:

- 5-branes,
- AdS/CFT, ...

We will introduce the “$E_8$ model for the C-field,” which seems well-suited to describing the M-theory action.
**E₈ model for the C-field**

The E₈ model is motivated by Witten’s definition of the M-theory action as an integral in 12-dimensions.

Topological fact: There is a homotopy equivalence

\[ E₈ \sim K(\mathbb{Z}, 3) \]
\[ BE₈ \sim K(\mathbb{Z}, 4) \]

up to the 15-skeleton.

For \( \text{dim } M \leq 15 \), there is a one-one correspondence between

\[ a \in H^4(M, \mathbb{Z}) \]

and isomorphism classes of principal E₈ bundles over \( M \).

For each \( a \) we pick a specific bundle \( P(a) \to M \).

**Definition:** A “C-field” on \( Y \) with characteristic class \( a \) is an element of

\[ E_P(Y) := \mathcal{A}(P(a)) \times \Omega^3(Y) \]

That is, our “gauge potentials,” or “C-fields,” will be pairs \( (A, c) \in E_P(Y) \),

We will often denote C-fields by \( \tilde{C} = (A, c) \).

We call \( c \) the “little c-field.”
The gauge equivalence class of $\tilde{C}$

How do we go from $\tilde{C} = (A, c) \in \mathbf{E}_P(Y)$ to $[\tilde{C}] \in \tilde{H}_C^{4}(Y)$?

Answer: The gauge invariant information is summarized by the coupling to the membrane.

To a pair $(A, c) \in \mathbf{E}_P(Y)$ we associate a (shifted) differential character:

$$\tilde{\chi}_{A,c}(\Sigma) = \exp \left[ 2\pi i \left( \int_{\Sigma} CS(A) - \frac{1}{2} CS(\omega) + c \right) \right]$$

$$dCS(A) = \text{tr} F^2 := \frac{1}{60} \text{Tr}_{248} \left( \frac{F^2}{8\pi^2} \right)$$

$$dCS(\omega) = \text{tr} R^2 := -\frac{1}{16\pi^2} \text{Tr}_{11} R^2$$

so

$$[\text{tr} F^2]_{DR} = a_{\mathbb{R}} \quad [\text{tr} R^2]_{DR} = \frac{1}{2} (p_1(TY))_{\mathbb{R}}$$

- It follows immediately that

$$\omega(\tilde{\chi}_{A,c}) = G = \text{tr} F^2 - \frac{1}{2} \text{tr} R^2 + dc$$

- The $CS(\omega)$ term has a sign ambiguity, cancelled by

$$\sqrt{\text{Det}(\mathcal{D}_{S(N)})}$$
Slogan

Morally speaking

\[ C \sim TrAdA + \frac{2}{3}A^3 \]

But this cannot be taken literally for many reasons.
The gauge group

Next, we seek a gauge group $\mathcal{G}$ so that

$$E_P(Y)/\mathcal{G} = \tilde{H}_{\frac{1}{2}\lambda}(Y)$$

The gauge group orbit is

$$(A, c) \sim (A', c') \quad \Leftrightarrow \quad \tilde{\chi}_{A, c} = \tilde{\chi}_{A', c'}$$

The solution:

$$A' = A + \alpha \quad \alpha \in \Omega^1(\text{ad}P)$$

$$c' = c - CS(A, A + \alpha) + \omega \quad \omega \in \Omega^3_\mathbb{Z}(Y)$$

So, we might conclude, the “gauge group” is:

$$\mathcal{G} \cong \Omega^1(\text{ad}P) \ltimes \Omega^3_\mathbb{Z}(Y)$$

$$(\alpha_1, \omega_1)(\alpha_2, \omega_2) = (\alpha_1 + \alpha_2, \omega_1 + \omega_2 + d(\text{tr}\alpha_1 \wedge \alpha_2))$$

But this is unsatisfactory for 2 reasons: one physical, one mathematical.
Physical objection: Open membranes

In electromagnetism, if $\partial Y = X$, the worldline of a charged particle can end on $P \in X$. The coupling
\[ \exp[i \int_\gamma eA] \]
is not gauge invariant:
\[ \exp[i \int_\gamma eA] \rightarrow \tilde{\chi}(P) \exp[i \int_\gamma eA] \quad \tilde{\chi} \in \tilde{H}^1(X) \]

In $M$ theory, the worldvolume of a membrane $\Sigma$ can end on $\sigma \in Z_2(X)$. By analogy, $C$-field gauge transformations act as:
\[ \exp[i \int_\Sigma C] \rightarrow \tilde{\chi}(\sigma) \exp[i \int_\Sigma C] \quad \tilde{\chi} \in \tilde{H}^3(X) \]

Conclusion:
\[ G = \Omega^1(\text{ad}P) \times \tilde{H}^3(X) \]

Recall
\[ 0 \rightarrow H^2(X, U(1)) \rightarrow \tilde{H}^3(X) \rightarrow \Omega^3_{\mathbb{Z}}(X) \rightarrow 0 \]
\[ \Rightarrow \text{nontrivial extension of the naive gauge group by "micro gauge transformations"} \]
Relation to Differential cocycles

So the true gauge group is:

\[ \mathcal{G} = \Omega^1(\text{ad}P) \times \check{H}^3(X) \]

with action on \( E_P(Y) \) is:

\[ (\alpha, \check{\chi}) \cdot (A, c) = (A + \alpha, c - CS(A, A + \alpha) + \omega(\check{\chi})) \]

This is also **mathematically natural**: 

Since we have a \( \mathcal{G} \)-action on a space \( E_P(Y) \) we can form the associated groupoid, which we regard as a category

\[ \text{Objects}(E_P(Y) \rtrislash \mathcal{G}) = E_P(Y) \]

The objects \( \check{C} \) have automorphism group \( H^2(Y, U(1)) \).

Hopkins & Singer’s “differential cochains” are gauge potentials for abelian \( p \)-form gauge fields in all dimensions and degrees. They can be applied to the \( C \)-field of \( M \)-theory. The space of (shifted) differential cocycles \( \check{Z}^4 \) is also a category.

**Theorem** There is an equivalence of the two categories.

**Proof:** Both are groupoids with the same isomorphism class of objects, and each object has automorphism group \( H^2(Y, U(1)) \).
Remarks

• This formalism makes clear the physical role of the $E_8$ gauge field. It is a kind of topological field theory since we can shift $A$ to any other connection $A' \in \mathcal{A}(P(a))$, and hence $A$ is only constrained by topology.

(Compare $\delta A = \psi$, in Donaldson-Witten theory.)

• What about ordinary gauge transformations?

$$(A, c) \rightarrow (A^g, c) \quad g \in Aut(P)$$

Every transformation of this type is equivalent to $(\alpha, \tilde{\chi})$ with

$$\alpha = A^g - A = g^{-1}DAg$$

$$\tilde{\chi}(\sigma) = \exp \left[ 2\pi i \int_{\sigma \times S^1} CS(A) \right]$$

$$\mathcal{A} = (1 - t)A + tA^g$$

But $Aut(P)$ is not a subgroup! It is a sub-groupoid.
Definition of $C$-field measure for $Y$ without boundary

In terms of the $E_8$ formalism the $C$-field measure in the path integral is:

$$[dC] \cdot e^{-\int_Y G^\wedge G \cdot \Phi(\tilde{C}; Y)}$$

$$\Phi(\tilde{C}; Y) = \exp\left[i\pi \xi(\mathcal{D}_A) + \frac{i\pi}{2} \xi(\mathcal{D}_{RS}) + 2\pi i I_{\text{local}}\right]$$

$$\xi(\mathcal{D}) = \frac{1}{2} (\eta(\mathcal{D}) + h(\mathcal{D}))$$

$$I_{\text{local}} = \int_Y \left(\frac{1}{2} cG^2 - \frac{1}{2} cdcG + \frac{1}{6} c(d c)^2 - cI_8(g)\right)$$

**Theorem 1:** $\Phi$ is $G$-invariant:

$$\Phi(A, c; Y) = \Phi(A', c'; Y)$$

for $(A', c') = (\alpha, \omega) \cdot (A, c)$.

**Proof:** Variational formulae for the $\eta$ invariant. ♠
Relation to Witten’s definition

**Theorem 2:** Suppose $P(a) \to Y$ admits an extension

$$P_Z(a_Z) \to Z$$

for $Z$ a bounding spin manifold (hence $\partial Y = \emptyset$).

Suppose $(A_Z, c_Z)$ extends $(A_Y, c_Y) \in E_P(Y)$. Then

$$\Phi(\tilde{C}_Y; Y) = \exp \left\{ 2\pi i \int_Z \left[ \frac{1}{6} G_Z^3 - G_Z I_8(g_Z) \right] \right\} \left( -1 \right)^{I(D_{RS})/2}$$

Where

$$G_Z = \omega((A_Z, c_Z)) = \text{tr} F_Z^2 - \frac{1}{4} \text{tr} R_Z^2 + dc_Z$$

Proof: APS index theorem + Witten’s observation

$$\left[ \frac{1}{2} i(\mathcal{D}_A) + \frac{1}{4} i(\mathcal{D}_{RS}) \right] \overset{(12)}{=} \frac{1}{6} G_Z^3 - G_Z I_8(g_Z)$$

**Theorem 3:** (Witten 96) The factor

$$\sqrt{\text{Det} \mathcal{D}_{RS}} \Phi(\tilde{C}, Y)$$

is a well-defined and smooth complex-valued function on

$$A_{\text{spin}} \times E_P(Y)$$
C-field action on manifold with boundary

In the case \( \partial Y = X \) is nonempty the same formula applies

\[
\Phi(\tilde{C}, Y) = \exp \left[ i\pi \xi(\tilde{\mathcal{D}}_A) + \frac{i\pi}{2} \xi(\tilde{\mathcal{D}}_{RS}) + 2\pi i I_{\text{local}} \right]
\]

However, now there is a conceptually important distinction: the factor \( \exp[i\pi \xi(\tilde{\mathcal{D}}_A)] \) is a section of a \( U(1) \) bundle with connection.

As is well-known from Chern-Simons theory, \( \Phi \) is valued in a principal \( U(1) \) bundle

\[
\mathcal{Q} \rightarrow \mathbf{E}_P(X)
\]

Each extension \( \check{C}_Y \) of \( \check{C}_X \) defines an element:

\[
\Phi(\check{C}_Y, Y) \in \mathcal{Q}_{\check{C}_X}
\]

and two such extensions satisfy the “gluing law”

\[
\frac{\Phi(\check{C}_Y, Y)}{\Phi(\check{C}_{Y^0}, Y')} = \Phi(\check{C}_Y - \check{C}_{Y^0}, Y \cup \check{Y}')
\]

This property, in fact, defines \( \mathcal{Q} \).
Connection and Curvature

Because of gluing, the principal $U(1)$ bundle $Q$ carries a natural connection - that is, a lifting of paths:

$$p(t) \rightarrow \tilde{p}(t)$$

A path $p(t) = (A_X(t), c_X(t))$, $0 \leq t \leq T$ in $E_P(X)$ defines a $C$-field on the cylinder $\tilde{C}_p \in E_P(X \times [0, T])$. So

$$\Phi(\tilde{C}_p; X \times [0, T]) \in \mathcal{Q}_{(A,c)}(T) \otimes \mathcal{Q}_{(A,c)}^*(0) = \text{Hom}(\mathcal{Q}_{(A,c)}(0), \mathcal{Q}_{(A,c)}(T))$$

The curvature is easily computed: The holonomy around a small square $\diamond$ is the phase defined by the bounding 12-manifold $Z = \diamond \times X \Rightarrow$

$$\Omega = i\pi \int_X G \wedge (2\text{tr}(\delta AF) + \delta c) \wedge (2\text{tr}(\delta AF) + \delta c)$$

Note: We have now solved problem 1.
The physical wavefunction

As in 3D Chern-Simons theory we study the wavefunction associated with the manifold with boundary:

\[ X \quad Y \]

Fix an extension \( P_Y \) of \( P \) to \( Y \). For \( \tilde{C}_X \in \mathbf{E}_P(X) \) define:

\[ \mathbf{E}_{P_Y}(\tilde{C}_X) := \text{The set of all extensions } \tilde{C}_Y \text{ of } \tilde{C}_X. \]

\[
\Psi(\tilde{C}_X) = \int_{\mathbf{E}_{P_Y}(\tilde{C}_X)/\mathbb{G}(\tilde{C}_X)} [d\tilde{C}_Y] \exp\left[-|G(\tilde{C}_Y)|^2\right] \Phi(\tilde{C}_Y, Y)
\]

What can we say about \( \Psi \)?

- \( \Psi(\tilde{C}_X) \) is valued in \( \mathcal{L}_{\tilde{C}_X} \): The associated line to \( Q\tilde{C}_X \).

- Theory of determinant line bundles \( \Rightarrow \mathcal{L} = \operatorname{PFAFF}(\mathcal{D}_{\text{ad}P}) \), the Pfaffian line bundle for adjoint \( E_8 \) fermions on the boundary \( X \).
The Gauss Law

Physical wavefunctions should be gauge invariant:

\[ g \cdot \Psi(\tilde{C}_X) = \Psi(g \cdot \tilde{C}_X) \]

To formulate the Gauss law we need a lift:

\[ \mathcal{L} \xrightarrow{g} \mathcal{L} \]

\[ \downarrow \]

\[ \mathbf{E}_P(X) \xrightarrow{g} \mathbf{E}_P(X) \]

- We’ll define the lift.

- For micro gauge transformations \( g \in \text{Aut}(\tilde{C}_X) \) we must have \( g \cdot \Psi = \Psi \). This implies the vanishing of \( C \)-field electric charge.

- The full Gauss law is quantization of “Page charge”

In analogous 3D \( U(1) \) level \( k \) Chern-Simons theory, the Gauss law implies

a.) **Characteristic class:** \( c_1(P) = 0 \)

b.) **Fieldstrength:** \( F(A_X) = 0 \)

c.) **Holonomy:** Wilson lines are \( k \)-torsion.
Definition of the $G$ action on $\mathcal{L}$

**Theorem:** There exists a well-defined lift of $G$ to $\mathcal{L}$.

The hard part is constructing

$$(0, \tilde{\chi}) : \mathcal{L}_{A,c} \to \mathcal{L}_{A,c + \omega(\tilde{\chi})}$$

Consider the linear path

$$p(t) = (A, c + t\omega(\tilde{\chi}))$$

We define the group action on $\psi \in \mathcal{L}_{A,c}$:

$$(0, \tilde{\chi}) \cdot \psi := \varphi(\tilde{C}_X, \tilde{\chi}) \ U(p(t))\psi$$

The “correction factor” $\varphi$ is required in order to implement a well-defined group action:

$$(0, \tilde{\chi}_1)(0, \tilde{\chi}_2) = (0, \tilde{\chi}_1 + \tilde{\chi}_2)$$

The correction factor $\varphi$ is necessary because $\mathcal{L}$ has curvature

So

$$U(p_3) = U(p_1)U(p_2) \exp[i\pi \int_X G \wedge \omega_1 \wedge \omega_2]$$
Idea for the correction factor $\varphi$

We are given:

- $\tilde{C}_X$ on a 10-manifold $X$.
- A gauge transformation $\tilde{\chi} \in \tilde{H}^3(X)$.

Glue with a gauge transformation to construct a “twisted character” $\tilde{C}_Y$ on $Y = X \times S^1$:

\[\varphi(\tilde{C}_X, \tilde{\chi}) := \Phi(\tilde{C}_Y; X \times S^1)\]

Turns out:

\[\varphi(\tilde{C}_X, \tilde{\chi}_1 + \tilde{\chi}_2) = \varphi(\tilde{C}_X, \tilde{\chi}_1)\varphi(\tilde{C}_X, \tilde{\chi}_2)e^{-i\pi \int_X G \wedge \omega_1 \wedge \omega_2}\]
Formula for the correction factor $\varphi$

We use the data $\tilde{C}_X$ and $\tilde{\chi}$ to construct a nontrivial character on a closed 11-fold $Y = X \times S^1$ and take the $M$-theory phase:

$$\varphi(\tilde{C}_X, \tilde{\chi}) := \Phi([\tilde{C}_X] + \tilde{t} \cdot \tilde{\chi}, X \times S^1)$$

Using cobordism on a pair of pants we get:

$$\varphi(\tilde{C}_X, \tilde{\chi}_1 + \tilde{\chi}_2) = \varphi(\tilde{C}_X, \tilde{\chi}_1) \varphi(\tilde{C}_X, \tilde{\chi}_2)e^{-i\pi \int_X G \wedge \omega_1 \wedge \omega_2}$$
**Remark: $E_8$ interpretation of the correction factor**

In $E_8$ language, on $Y = X \times S^1$ we have a $C$-field:

$$\tilde{C}_X + \tilde{t} \cdot \tilde{\chi} = (\mathcal{A}, \gamma) \in \mathbf{E}_{P_g}(Y)$$

where

$$\mathcal{A} = (1 - t)A_X + tA_X^g \quad t \sim t + 1$$

is built from an $E_8$ gauge transformation $g \in \text{Aut}(P)$.

The $E_8$ gauge transformation $g$ is determined from

$$\tilde{\chi}(\sigma) \sim \exp[i \int_{\sigma \times S^1} C S(\mathcal{A})] = \exp[i WZ(g)]$$

Then

$$\varphi(\tilde{C}_X, \tilde{\chi}) = \exp[i \frac{\pi}{2} \eta(\Phi_A) + 2\pi i I_{\text{local}}]$$
The action of automorphisms on $\mathcal{L}$

Let us reconsider to the action

$$(0, \tilde{\chi}) : \mathcal{L}_{A,c} \rightarrow \mathcal{L}_{A,c+\omega(\tilde{\chi})}$$

When $\tilde{\chi}$ is flat, i.e., $\tilde{\chi} \in H^2(X, U(1))$ then $\omega(\tilde{\chi}) = 0$ and hence

$$(0, \tilde{\chi}) \cdot (A, c) = (A, c).$$

"Points $(A, c)$ have automorphisms."

However, if $\Psi \in \mathcal{L}_{A,c}$ then

$$(0, \tilde{\chi}) \cdot \Psi = \varphi(\tilde{C}_X, \tilde{\chi}) \Psi$$

and $\varphi(\tilde{C}_X, \tilde{\chi})$ can be a nontrivial phase.
The integral Gauss law

Gauge invariance requires $\varphi(\tilde{C}_X, \tilde{\chi}) = 1$ for all flat $\tilde{\chi}$.

Cocycle law $\Rightarrow$ on flat characters, for fixed $\tilde{C}_X$, $\varphi$ is a homomorphism

$$\varphi(\tilde{C}_X, \cdot): H^2(X, U(1)) \to U(1)$$

Poincare duality $\Rightarrow$ the existence of $\Theta_X(\tilde{C}_X) \in H^8(X, \mathbb{Z})$:

$$\varphi(\tilde{C}_X, \tilde{\chi}) = \exp[2\pi i \langle \Theta_X(\tilde{C}_X), \tilde{\chi} \rangle]$$

Conclusion: The integral Gauss law is:

$$\Theta_X(\tilde{C}_X) = 0$$

(This is the analog of $c_1(P) = 0$ in 3D Chern-Simons theory.)
\( \Theta_X(\tilde{C}) \) is C-field electric charge

To interpret \( \Theta \) let us “insert” an M2 brane wrapping

\[
\sigma \in Z_2(X).
\]

Consider the gauge invariance of \( \Psi_\sigma(\tilde{C}) \).

Flat gauge transformations \( \tilde{\chi} \) with \( \tilde{\chi} \in H^2(X, \mathbb{R}/\mathbb{Z}) \) act on the wavefunction as

\[
\tilde{\chi}(\sigma) = \exp(2\pi i \langle PD[\sigma], \tilde{\chi} \rangle)
\]

Therefore, we interpret \( \Theta_X \) as the C-field electric charge induced by the background metric and C-field.

Moreover, in the presence of membranes with spatial cycle \( \sigma \in Z_2(X) \), the Gauss law is:

\[
\Theta_X(\tilde{C}) + PD([\sigma]) = 0
\]
Properties of $\Theta_X(\tilde{C})$

Consistency check:

$$[\Theta_X(\tilde{C})]_R = \left[\frac{1}{2} G^2 - I_8\right]_{DR}$$

Easily proved.

What do we know about $\Theta_X(\tilde{C})$ as an **integral class**?

$\Theta_X(\tilde{C})$ only depends on the characteristic class $a$, so denote it as $\Theta_X(a)$.

Since $[G]_R = a_R - \frac{1}{2} \lambda_R$,

$$[\Theta_X(a)]_R = \frac{1}{2} a(a - \lambda) + 30 \hat{A}_8$$

But $\Theta_X(a)$ is an **integral** class.

(N.B. $\Rightarrow \exists$ integral lift of $30 \hat{A}_8$ on spin 10-folds...)

- **Cobordism invariance:** $\langle \Theta_X(a), \check{\chi} \rangle$ is a cobordism invariant.

- **Bilinear identity:**

  $$\Theta_X(a_1 + a_2) + \Theta_X(0) = \Theta_X(a_1) + \Theta_X(a_2) + a_1 \cup a_2$$

- **Parity invariance:**

  $$\Theta_X(\lambda - a) = \Theta_X(a)$$

  $\Rightarrow$ can compute $\Theta_X(a)$ in some simple examples.
Quantization of “Page charges”

Remainder of the Gauss law is the electromagnetic dual of 
\([G] = a_R - \frac{1}{2} \lambda_R\).

Equation of motion \(\Rightarrow\)

\[d \left( *G + \frac{1}{2} C \wedge G - I_7 \right) = 0\]

\(\Rightarrow\) conserved charge:

\[\left[ *G + \frac{1}{2} C \wedge G - I_7 \right] \in H^7_{DR}(X)\]

known in physics as the “Page charge.”

\(*G|_X \sim \text{canonical momentum } \Pi\)

\[e^{i \int \omega \Pi} \Psi(C) = \Psi(C + \omega)\]

Gauss law for large \(C\)-field gauge transformations \(\Rightarrow\) quantization of Page charge.

Surprise! Because \(\varphi(\tilde{C}_X, \tilde{\chi})\) is a quadratic function, topology can induce half-integer charges

\[\int_{\Sigma_7} \left( *G + \frac{1}{2} C \wedge G - I_7 \right) \in \frac{1}{2} \mathbb{Z}\]
Application 2: The 5-brane partition function

In addition to the electrically charged membranes, \( M \)-theory has magnetically charged 5-branes - with 6-dimensional world-volumes \( W_6 \).

Cut out the tubular neighborhood of a 5-brane worldvolume \( W_6 \leftrightarrow Y_{11} \). The boundary is a 10-fold \( X \) which is \( S^4 \) fibered over \( W_6 \):

\[
S^4 \rightarrow X \xrightarrow{\pi} W_6
\]

We have defined:

\[
\Psi_{\text{bulk}} \in \Gamma(\mathcal{L} \rightarrow \mathbf{E}_P(X))
\]

The 5-brane partition function is a wavefunction:

\[
\Psi_{M5} \in \Gamma(\mathcal{L}_{M5} \rightarrow "C - fields on W_6")
\]

"Anomaly inflow": \( \Psi_{M5} \Psi_{\text{bulk}} \) is gauge invariant, \( \Rightarrow \)

\[
\mathcal{L}_{M5} = \mathcal{L}^{-1}
\]

This is the first of two definitions Witten gave for the 5-brane partition function.
The 5-brane partition function - cont’d

He also gave a second, “intrinsic definition,” which has been subsequently studied by Hopkins & Singer.

However, to give an “intrinsic definition” the 5-brane must be decoupled from the bulk 11-dimensional M-theory....

If \( \Theta_X(\tilde{C}) \neq 0 \) then 2-branes must end on the 5-brane:

![Diagram](image)

Therefore, \( \Theta_X(\tilde{C}) = 0 \) is a necessary condition for decoupling of the \( M5 \) brane from the bulk, i.e. a necessary condition for the existence of a well-defined “5-brane partition function.”

Now consider \( \pi_\ast(\Theta_X(a)) \in H^4(W_6, \mathbb{Z}) \).

Gysin sequence: \( G = \pi^\ast(\bar{G}) + \text{global angular form} \).

\[ \pi_\ast(\Theta_X) = 0 \Rightarrow \]

a.) \( \bar{G} = dh \)

b.) If (a) is satisfied, then \( \pi_\ast(\Theta_X(\tilde{C})) \) is torsion. This is precisely the torsion class found by Witten and Hopkins-Singer whose vanishing is necessary for having a well-defined 5-brane partition function.
Application 3: Relation of $M$-theory to $K$-theory

Just as $\Theta_X = 0$ guarantees no 2-branes will end on 5-branes, there is a condition that no D-branes end on a D-brane wrapping a worldvolume $W$. This is the condition:

$$(Sq^3 + [H])PD(W) = 0$$

This condition leads to the K-theoretic classification of D-branes.

So when $Y_{11}$ is $S^1$-fibered over a 10-manifold $U_{10}$ we expect a relation of $\Theta_X = 0$ to the K-theory classification of RR fluxes on $U_{10}$.

For simplicity, suppose $Y_{11} = U_{10} \times S^1$. Then $\partial Y_{11} = X_{10}$, $\partial U_{10} = V_9$. Then

$$a = \pi^* (\bar{a}) + \pi^* ([H]) \cup [dx^{11}]$$

where $\bar{a}, H$ live on $\partial U_{10} = V_9$.

Morally:

$$\pi_*(\Theta_X(a)) = (Sq^3 + [H])(\bar{a} - \frac{1}{2}\bar{\lambda})$$

Precise result:

$$\pi_*(2\Theta_X(a)) = [H] \cup (2\bar{a} - \lambda)$$

$$\pi_*(\Theta_X(a_1) - \Theta_X(a_2)) = (Sq^3 + [H])(\bar{a}_1 - \bar{a}_2)$$

This generalizes the result of Diaconescu, Moore, Witten to manifolds with boundary.
Application 4: Spatial boundaries

Suppose now we have a spatial boundary $\iota : X \leftrightarrow Y$.

Impose boundary conditions on $(A, c)$ via: $\iota^*(c) = 0$.

This boundary condition breaks the topological gauge symmetry $G$ but is preserved by the usual gauge symmetry $\text{Aut}(P)$.

Therefore, $\int_X \text{tr} F \wedge *F$ is gauge invariant and therefore the $E_8$ gauge fields are indeed dynamical on the boundary!

- Note that this is spontaneous symmetry breaking of groupoids, not of groups.

In this way we can incorporate both Horava-Witten and Fabinger-Horava models.

The cancellation of $E_8$ gauge anomalies is manifest, and local.

Gravitational anomalies are more subtle.
Conclusions

There is plenty of work to do

- We expect many future applications, for example, to anomaly cancellation issues connected to 5-branes, $G_2$ singularities, frozen singularities, etc.

It would help if we had good ways to compute $\Theta_X(a)$.

- Proper inclusion of gravitino determinant

- Complete the derivation of K-theory from M-theory by comparing type II and M-theory wavefunctions.

- Include Supersymmetry

- M-theory is parity invariant. Therefore all the above should be repeated when $Y$ is un-oriented.

But ... what is the physical role of the $E_8$ gauge field, and does it point to a useful reformulation of $M$-theory?

We don’t know yet.