THE INITIAL-BOUNDARY VALUE PROBLEM FOR GENERAL RELATIVITY

\[ \begin{align*}
\epsilon &= 0 \\
C &= 0
\end{align*} \Rightarrow G_{\mu\nu} = 0
\]

EVOLUTION-CONSTRAINT SYSTEM:

ASSUME CAUCHY IS OK.
WHAT CAN GO WRONG IN \( D_2 \)?

MATHEMATICALLY:
- NOT WELL-POSED
- CONSTRAINTS NOT SATISFIED

NUMERICALLY:
- UNSTABLE

PHYSICALLY:
- BOUNDARY DATA WRONG
- CAN'T EXTRACT WAVEFORM
THE DETAILS DEPEND UPON THE SYSTEM

SIMPLE EXAMPLE: NULL EVOLUTION IN $\mathcal{D}_2$

INTRODUCE NULL TETRAD ASSOCIATED WITH FOLIATION OF $\mathcal{B}$

$g_{\mu\nu} = -\ell_{(\mu}n_{\nu)} + m_{(\mu}m_{\nu)}$

ASSUME $\mathcal{B}$ EITHER EXPANDS OR CONTRACTS IN $\ell^\mu$ DIRECTION

EVOLUTION SYSTEM $\mathcal{E}$: $G_{\mu\nu}\ell^\nu = 0$ $G_{\mu\nu}m^\mu m^\nu = 0$

INTRODUCE PROJECTOR ASSOCIATED WITH UNIT SPATIAL NORMAL $N^\mu$ TO $\mathcal{B}$

$h^\mu_{\nu} = \delta^\mu_{\nu} - N^\mu N_{\nu}$

CONSTRAINTS (3) $\mathcal{C}$: $h^\rho_{\nu}G_{\rho\sigma}N^\sigma = 0$

CONSTRAINT PROPAGATION

$\mathcal{C} = \frac{1}{r^2} [r^2 C]_{\mathcal{B}}$

WELL-POSED ???

ROBUST ✓

CAUCHY BOUNDARY FOR A SCALAR FIELD ON A CURVED BACKGROUND

(SYMMETRIC) HYPERBOLIC SYSTEM: $g^{\mu\nu}\nabla_\mu \nabla_\nu \Phi = 0$

BOUNDARY FLUX: $\mathcal{F} = N^\mu T_{\mu\nu} = -(\partial_\mu \Phi)N^\mu \partial_\nu \Phi$

WELL-POSED BOUNDARY CONDITIONS: $\mathcal{F} \geq 0$

HOMOGENEOUS DIRICHLET BOUNDARY CONDITION:

$\partial_\nu \Phi = 0$

HOMOGENEOUS SOMMERMED BOUNDARY CONDITION:

$\partial_\nu \Phi + \sqrt{-g_{\mu\nu}} N^\mu \partial_\nu \Phi = 0$

HOMOGENEOUS NEUMANN BOUNDARY CONDITION:

$N^\mu \partial_\mu \Phi = 0$

INHOMOGENEOUS BOUNDARY DATA $q(x^\alpha)$

$\partial_\nu \Phi = q(x^\alpha)$, $\partial_\nu \Phi + \sqrt{-g_{\mu\nu}} N^\mu \partial_\nu \Phi = q(x^\alpha)$, $N^\mu \partial_\mu \Phi = q(x^\alpha)$
WELL-POSED INITIAL-BOUNDARY VALUE PROBLEM FOR THE LINEARIZED EINSTEIN EQUATIONS IN THE HARMONIC GAUGE

\[ g_{\mu\nu} = \eta_{\mu\nu} + \delta g_{\mu\nu} \]
\[ \gamma^{\mu\nu} = \delta(\sqrt{-g}g^{\mu\nu}) \]

EVOLUTION SYSTEM $\mathcal{E}$

\[ \Box \gamma^{ij} = 0 \]
\[ \partial_{\mu} \gamma^{\mu\nu} = \partial_{t} \gamma^{t\nu} + \partial_{\nu} \gamma^{t\nu} = 0 \]

CONSTRAINT SYSTEM $\mathcal{C}$

\[ \Box \gamma^{ij} = 0 \]

THE INITIAL-BOUNDARY VALUE PROBLEM FOR THIS SYSTEM IS WELL POSED FOR FREE DIRICHLET, SOMMERFELD OR NEUMANN BOUNDARY DATA FOR THE COMPONENTS $\gamma^{ij}$.

THE CORRESPONDING EVOLUTION CODE IS ROBUST

THE BAD NEWS:

- IT IS NOT KNOWN HOW TO GENERALIZE THIS SYSTEM TO THE NONLINEAR CASE.
- NO KNOWN NONLINEAR METRIC-CONNECTION SYSTEM HAS THIS FLEXIBILITY OF BOUNDARY DATA.

BOUNDARIES IN LINEARIZED APPROXIMATION

- G. Calabrese, L. Lehner, D. Nishida, J. Pullin, O. Reula, O. Sarbach, M. Tiglio, gr-qc
- G. Calabrese, J. Pullin, O. Reula, O. Sarbach, M.1 Tiglio , gr-qc
- S. Frittelli and R. Gómez, gr-qc

BOUNDARIES FOR NONLINEAR EINSTEIN EQUATIONS


EVOLUTION VARIABLES: $\epsilon_{\mu}, \Gamma_{\mu\nu}^{\rho}, C_{\mu\nu\rho}$

THIS IS THE ONLY NONLINEAR SYSTEM WHICH IS KNOWN TO ADMIT PHYSICALLY GENERAL BOUNDARY CONDITIONS: ONLY THE WEYL CURVATURE REQUIRES BOUNDARY CONDITION AND ANALOGUES OF DIRICHLET, NEUMANN AND SOMMERTFELD ARE ALLOWED.

POOR MAN'S VERSION

- B. Szelágyi and J. Winicour gr-qc/0205044

BASED ON

CAUCHY PROBLEM IN HARMONIC COORDINATES:

WELL-POSED INITIAL-BOUNDARY VALUE PROBLEM FOR NONLINEAR SYSTEMS WITH CHARACTERISTIC BOUNDARIES:
HARMONIC INITIAL-BOUNDARY VALUE PROBLEM

REDUCED EVOLUTION SYSTEM: \( \gamma^{\mu\nu} = \sqrt{-g} g^{\mu\nu} \)
\( \gamma^{\alpha\beta} \partial_\alpha \partial_\beta \gamma^{\mu\nu} + S^{\mu\nu}(\gamma, \partial \gamma) = 0 \)

WELL-POSED FOR ANY DISSIPATIVE BOUNDARY CONDITIONS, e.g. DIRICHLET, SOMMERFELD, NEUMANN

CONSTRAINTS: \( H^\mu = \partial_\nu \gamma^{\mu\nu} = \tilde{H}^\mu(x^\rho, \gamma) \)
FOR BREVITY SET \( \tilde{H}^\mu(x^\rho, \gamma) = 0 \)
REDUCED EQUATIONS IMPLY
\( \gamma^{\alpha\beta} \partial_\alpha \partial_\beta H^\mu + C^\mu_{\beta\alpha} \partial_\alpha H^\beta + D^\mu_{\beta} H^\beta = 0 \)

UNIQUENESS IMPLIES \( H^\mu = 0 \) IF IT SATISFIES A DISSIPATIVE HOMOGENEOUS BOUNDARY CONDITION.

THIS IS NOT EASY TO ARRANGE.
EXAMPLE: DIRICHLET CONDITION \( H^\mu|_B = 0 \).
LET BOUNDARY BE AT \( z = 0 \) WITH \( x^\mu = (x^a, z) \). THEN
\( H_z = \partial_a \gamma^{za} + \partial_z \gamma^{zz} = 0 \)
\( H^a = \partial_a \gamma^{ab} + \partial_{\gamma^{az}} = 0 \)

NAIVE BOUNDARY DATA FOR \( \gamma^{\mu\nu} \) IMPROPERLY POSES BOTH DIRICHLET AND NEUMANN CONDITIONS ON \( \gamma^{az} \)

WELL-POSED HOMOGENEOUS BOUNDARY DATA

ONE CHOICE THAT WORKS:
\[
\begin{align*}
\gamma^{za}|_B &= 0 \\
\partial_z \gamma^{zz}|_B &= 0 \\
\partial_z \gamma^{ab}|_B &= 0
\end{align*}
\]
\[\implies \] \[\begin{align*}
H^z|_B &= 0 \\
\partial_z H^a|_B &= 0
\end{align*}\]

\[\implies \text{CONSTRAINTS SATISFIED} \]

INHOMOGENEOUS BOUNDARY DATA \( q(x^a) \)

BOUNDARY HARMONIC GAUGE FREEDOM (SHIFT)
\( \gamma^{za}|_B = q^a(x^b) \gamma^{za}|_B \)

NOTE: BOUNDARY DATA FOR \( \gamma^{za}|_B \) DEPENDS ON DATA FOR \( \gamma^{zz}|_B \) WHICH CAN ONLY BE DETERMINED BY CARRYING OUT THE EVOLUTION

BOUNDARY NORMAL: \( \delta^n = \frac{1}{N^a} N^n \partial_\mu = \partial_z + q^a \partial_a \)
NEUMANN DATA: \( q^{zz} = \partial^n \gamma^{zz}|_B \)

\( H^z|_B = 0 \implies q^{zz} = -\partial_a q^a \gamma^{zz}|_B \)
REMAINING NEUMANN BOUNDARY DATA

\[ q^{ab} = \partial^n \gamma^{ab} |_B \]

RELATED TO EXTRINSIC CURVATURE \( K^{ab} \) OF BOUNDARY.

\[ \partial^n H_a |_B = 0 \implies \]

\[
\sqrt{-h} D_b (K^b_a - \delta^b_a K) + \sqrt{g^{zz}} K_{ab} H^b - \frac{g^{zz}}{2} H_b \partial_a q^b = 0
\]

HERE \( h_{ab} \) AND \( D_a \) ARE THE METRIC AND CONNECTION INTRINSIC TO \( B \)

This forms a symmetric hyperbolic system which determines the 6 pieces of Neumann data \( q^{ab} \) in terms of 3 free functions, as well as the free (boundary gauge) data \( q^a \) and boundary values of the variables \( \gamma^{zz}, \gamma^{ab} \) and \( \partial_z \gamma^{za} \) which must be determined by the evolution.

ANY SOLUTION OF THE REDUCED EQUATIONS WITH THIS BOUNDARY DATA SATISFIES THE CONSTRAINTS.

IS THE INITIAL-BOUNDARY PROBLEM WELL-POSED???

NUMERICAL IMPLEMENTATION

SOME DIFFICULT CHOICES

- FIRST DIFFERENTIAL ORDER OR SECOND
  SECOND ORDER IN TIME, FIRST ORDER IN SPACE

- CUBIC BOUNDARY OR SPHERICAL
  CUBIC

- GENERAL BOUNDARY GAUGE OR \( \gamma^{za} |_B = 0 \)
  \( \gamma^{za} |_B = q(x^b) \gamma^{zz} |_B \)

- HARMONIC FORCING TERMS OR \( \partial_\mu \gamma^{\mu\nu} = 0 \)
  \( \partial_\mu \gamma^{\mu\nu} = \dot{H}^{\mu}(x^\rho, \gamma) \)

- BOUNDARY ACCURACY
  1ST ORDER IN NORMAL DIRECTION, 2ND ORDER TANGENTIALLY

- NUMERICAL STENCILS, DISSIPATION, ...
  BAG OF "TRICKS"

TESTS OF NAIVE ALGORITHM

- ROBUST STABILITY
- LINEARIZED WAVE CONVERGENCE TESTS
- NONLINEAR GAUGE WAVE CONVERGENCE TESTS
FIG. 2. The $L_\infty$ norm of the finite-difference error, rescaled by a factor of $1/\Delta^2$, for a gauge-wave. The upper two (mostly overlapping) curves demonstrate convergence to the analytic solution for a wave with amplitude $A = 10^{-1}$ evolved for 30 crossing times with gridsizes $80^3$ and $120^3$. The lower curve represents evolution of the same gauge-wave with $A = 10^{-3}$ for 300 crossing times with gridsize $80^3$. 

NEUMANN BOUNDARY CONDITIONS FOR THE SCALAR WAVE EQUATION IN SECOND DIFFERENTIAL ORDER FORM

CURVED BOUNDARY AT REST IN A MINKOWSKI METRIC BACKGROUND


MOVING BOUNDARY ON A DYNAMIC BACKGROUND SPACETIME

WHEN THE BOUNDARY MOVES TOWARD THE CAUCHY INTERIOR THE INTERPOLATION STENCIL FOR $\phi_0$ INVOLVES FUTURE TIME LEVELS.
The bottom line is to compute waveforms from binary black holes...

But your code has to pass basic tests if the waveforms can be trusted.