On strong hyperbolicity

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OUTLINE

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First Order Constant Coefficient Systems

\[ \partial_t u^\alpha = A^{\alpha i}_{\beta} \nabla_i u^\beta + B^\alpha_{\beta} u^\beta \]

- **Question:** When is the above system well posed in the $L^2$ sense?

\[ \|u^\alpha(t)\|_{L^2} \leq C(t)\|u^\alpha(0)\|_{L^2} \]

- **Answer:** It is well posed if and only if for all co-vectors $\omega_i$, the matrix $A^{\alpha i}_{\beta} \omega_i$ has only real eigen-values and a complete set of eigen-vectors.
First Order Constant Coefficient Systems II

\[ \partial_t \hat{u}^\alpha = i A^{\alpha i}_\beta \omega_i \hat{u}^\beta \]

\[ \hat{u}^\alpha(t) = (e^{i A^i \omega_i t})^\alpha_\beta \hat{u}^\beta(0) \]

\[ \|u^\alpha(t)\|_{L^2} = \|\hat{u}^\alpha(t)\|_{L^2} \leq C(t) \|\hat{u}^\alpha(0)\|_{L^2} \leq C(t) \|u^\alpha(0)\|_{L^2} \]

\[ C(t) = \sup_{\tau \in [0, t]} \sup_{\omega} \|(e^{i A^a \omega_a \tau})^\alpha_\beta\| \]
The above system is well posed (w.r.t. a Sobolev Norm) in a neighborhood of $u_0^\alpha$ if and only if for all $u^\alpha$ close enough to $u_0^\alpha$, all co-vectors $\omega_i$ and all points, the matrix $A^{\alpha i}_\beta(u, x, t)\omega_i$ has only real eigen-values and a complete set of eigen-vectors. Plus some "technical" condition.

We call such systems strongly hyperbolic.
If a system is strongly hyperbolic then there exists a positive definite bilinear form (a metric) $H_{\alpha\beta} = H_{\alpha\beta}(u, x, t, \omega_a)$ uniformly bounded by above and away from zero in $\omega_a$ such that:

$$H_{\alpha\gamma}A^{\gamma\alpha}_\beta \omega_a$$

is also symmetric. [Kreiss Matrix Theorem]

Technical condition requires $H$ to be smooth also on $\omega_i$.

If there exists a $H_{\alpha\beta}$ independent of $\omega_a$ we say that the system is symmetric hyperbolic.

If strong hyperbolicity fails it is easy to construct a sequence solutions whose initial data has norm one but whose norm at any future time tends to infinity. Non-linear behavior can not cure this.
Examples:

Example 1: Maxwell equations:

\[ W_{ij} := \partial_i A_j \]

\[ \partial_t E_i = \partial^j W_{ji} - \partial^j W_{ji} - \alpha(\partial^j W_{ij} - \partial_i W_{j}^j) \]

\[ \partial_t W_{ij} = \partial_i E_j - \frac{1}{2} \beta e_{ij} \partial^k E_k \]

This system is symmetric hyperbolic for \( \alpha < 0 \) and \( \beta < -\frac{2}{3} \) (most general symmetrizer built out of the 3-metric). But strongly hyperbolic for all \( (\alpha, \beta) \) such that \( \alpha \beta > 0 \).
Examples:

Example 2:

Consider the matrices,

\[
A^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}, \quad A^1 = \begin{pmatrix} -2 & 10s_1 & 0 \\ 0 & 1 & -2s_2 \\ 0 & 0 & 0 \end{pmatrix}, \quad A^2 = \begin{pmatrix} 0 & s_1 & 0 \\ 0 & \frac{1}{2} & 7s_2 \\ 0 & 0 & 1 \end{pmatrix}
\]

There is no positive definite \( h_{\alpha\beta} \) which would symmetrize \( A^i \omega_i \) for arbitrary \( \omega_i \). Nevertheless \( (A^0)^{-1}(A^1 + \lambda A^2) \) is diagonalizable.

\[
(A^0)^{-1}(A^1 + \lambda A^2) = \begin{pmatrix} -2 & (10 + \lambda)s_1 & 0 \\ 0 & \frac{1}{4}(2 + \lambda) & \frac{1}{2}(-2 + 7)s_2 \\ 0 & 0 & 2\lambda \end{pmatrix}
\]
Covariant definitions I:

\[ A^{\alpha\beta}(u, p) \nabla_{\alpha} u^\beta = J_\alpha(u, p) \]

The sum of two symmetric matrices is symmetric

Definition: The above system is symmetric hyperbolic if there exists \( h_{\alpha\beta}(u, p) \) such that:

- \( h_{\alpha\beta}(u, p) A^{\beta\gamma}(u, p) \) is symmetric.
- for some \( n_\alpha \), \( h_{\alpha\beta}(u, p) A^{\beta\gamma}(u, p) n_\alpha \) is positive definite.
- One can define an energy vector:

\[ E^\alpha := h_{\alpha\beta}(u, p) A^{\beta\gamma}(u, p) \delta u^\alpha \delta u^\gamma \]

\[ E^\alpha n_\alpha \geq 0 \]

- If \( n_\alpha \) is as above, then \( n_\alpha + \varepsilon w_\alpha \) is also as above, for \( \varepsilon \) small enough.
Covariant definitions II:

\[ A^{\alpha\beta}(u,p) \nabla_\alpha u^\beta = J_\alpha(u,p) \]

The sum of two diagonalizable matrices is not necessarily diagonalizable

**Definition A:** The above system is **strongly hyperbolic** if there exists \( n_\alpha \) such that:

- \( A^{\alpha\beta} n_\alpha \) is invertible, and
- for each loop \( \kappa(\lambda)_\alpha = \lambda n_\alpha + \omega_\alpha \lambda \in [-\infty, \infty] \) where \( \omega_\alpha \) is not proportional to \( n_\alpha \), \( \dim(\text{span}\{ \bigcup_{\lambda \in \mathbb{R}} \text{Kern}\{ A^{\alpha\beta} \kappa_\alpha(\lambda) \} \} ) = \dim\{ \text{manifold of fields} \} \)

**Definition B:** The above system is **strongly hyperbolic** if there exists \( n_\alpha \) such that for each co-vector \( \omega_\alpha \) there exists \( h_{\alpha\beta}(u,p,\omega) \) satisfying:

- \( h_{\alpha\beta}(u,p,\omega) A^{\beta\gamma}(u,p) \omega_\alpha \) is symmetric.
- \( h_{\alpha\beta}(u,p,\omega) A^{\beta\gamma}(u,p) n_\alpha \) is symmetric and positive definite.
- If \( n_\alpha \) is as above then \( n_\alpha + \varepsilon \omega_\alpha \) is also as above for \( \varepsilon \) small enough.
First Order Pseudo-Differential Systems

\[ \partial_t u^\alpha = P^\alpha_\beta(u, x, t, D)u^\beta := \int p(u, x, t, \omega_i)\alpha_\beta e^{i\omega_i x^i} \hat{u}^\beta d\Omega \]

The above system is said to be pseudo-differential of first order if the following limit exists,

\[ \lim_{\lambda \to \infty} \frac{1}{\lambda} p(u, x, t, \lambda \omega_i)\alpha_\beta := p_1(u, x, t, \omega_i)\alpha_\beta \]

If furthermore \( ip_1(u, x, t, \omega_i)\alpha_\beta \) has only real eigenvalues and a complete set of eigen-vectors we say the systems is strongly hyperbolic.

Strongly hyperbolic pseudo-differential operators plus technical condition are well posed. [Taylor, Kreiss-Ortiz-R]
We consider the domain of dependence of the linearized equation at a given background $u_0^\alpha$.

The domain of dependence of a region $\Sigma_0$ of a Cauchy surface is given by the maximal foliation of such region produced by hypersurfaces whose normal is such that:

\[ E^a(\delta u)n_a \geq 0 \quad \forall \delta u \leftrightarrow \det(A^{\alpha\alpha}_\beta n_\alpha) \neq 0 \]

Surfaces with normal such that the determinant vanishes are called characteristic surfaces.

For each co-vector $k_\alpha$ which is a characteristic there is a perturbation which in the high frequency limit moves along the integral lines of $V^a = \frac{\partial \det(A^c_k c)}{\partial k_\alpha}$ at points where $\det(A^c_k c) = 0$ ($V^a k_\alpha = 0$).

**Question:** what happens in the case of strongly hyperbolic systems?
Holmgren’s Theorem

- Given an analytic coefficient equation system (not necessarily hyperbolic!)

\[ \partial_i u^\alpha = A^{\alpha i}_{\beta}(x, t) \nabla_i u^\beta + B^{\alpha}_{\beta}(x, t) u^\beta \]

and assuming the solution vanishes in a hypersurface \( \Sigma_0 \) then the solution, if sufficiently smooth, vanishes in a whole neighborhood of it, given by the maximal foliations such that their normals do not become characteristics.

- Generalizable to the case of non-analytic coefficients for strongly hyperbolic systems.
  - Extend the space-time to \( R^n \) or \( T^n \).
  - Approximate the system by an analytic sequence of strongly hyperbolic systems.
  - Use Holmgren’s theorem on each one of them to conclude that the one parameter family of solutions so generated vanishes in some region \( \Omega_n \).
  - Use continuous dependence of solutions of strongly hyperbolic systems to show that the limiting solution would also vanish in a limiting set \( \Omega \).
Summary

- Strongly hyperbolic differential (and pseudo-differential) systems are well posed.
- There are global energy norms (pseudo-differential operators).
- There are covariant definitions. And open set of "space-like" hyper-surfaces.
- Strongly hyperbolic differential (and pseudo-differential) systems have finite propagation speeds. With domain of dependence given by their characteristic fields.
- Symmetric hyperbolic energy ↔ Summation by parts in finite differences
- Strongly hyperbolic pseudo-energy ↔ Pseudo-spectral methods.
Applications:

- ADM-BSSN first-second order systems [Frittelli-R, Sarbach-Calabrese-Pullin-Tiglio, Kreiss-Ortiz, Nagy-Ortiz-R]
- Constraint propagation. [Hyperbolicity properties of subsidiary systems of constraints.]
\[
G_{ab} = 0 \quad \Rightarrow \quad \begin{cases} 
\mathcal{L}_n h_{ab} = -2k_{ab}, \\
\mathcal{L}_n k_{ab} = (3) R_{ab} - 2k^c_a k_{bc} + k_{ab}k^c_c - \frac{D_a D_b N}{N}, \\
(3) R + (k^c_c)^2 - k_{ab}k^{ab} = 0, \\
D^b k_{ba} - D_a k^c = 0,
\end{cases}
\]
ADM equations II

\[ \mathcal{L}_{(t-\beta)} h_{ij} = -2Nk_{ij} \]

\[ \mathcal{L}_{(t-\beta)} k_{ij} = \frac{N}{2} h^{kl} \left[ -\partial_k \partial_i h_{lj} - \partial_l \partial_j h_{k} + 2\partial_k \partial_l (i h_{j}l) \right] + B_{ij} \]

where

\[ B_{ij} := N \left[ \gamma_{ikl} \gamma_{j}^{kl} - \gamma_{ij}^{k} \gamma_{kl}^{l} - 2k_{i}^{l} k_{j}^{l} + k_{ij}^{l} - A_{ij} \right], \]

\[ \gamma_{ij}^{k} := \frac{1}{2} h^{kl} (2\partial_l (i h_{j}k) - \partial_k h_{ij}), \]

\[ A_{ij} := a_i a_j - \gamma_{ij}^{k} a_k - 2\gamma_{ikl} \gamma_{j}^{(kl)} + \partial_i \left[ (\partial_j N) / N \right], \]

\[ a_i := (\partial_i N) / N. \]
ADM equations III

Hyperbolicity analysis: 1) consider only the principal part, 2) freeze coefficients, 3) substitute all derivatives by Fourier transforms ($\partial_k h_{ij} \rightarrow i\omega_k \hat{h}_{ik}$), and 4) define $\hat{e}_{ij} = i\omega \hat{h}_{ij}$. [Kreiss, Ortiz][Taylor]

The associated first order system is then

$$
\partial_t \hat{e}_{ij} = i\omega \left[ -2N \hat{k}_{ij} + \tilde{\omega}_k \beta^k \hat{e}_{ij} \right],
$$

$$
\partial_t \hat{k}_{ij} = i\omega \left[ -\frac{N}{2} \left( \hat{e}_{ij} + \tilde{\omega}_i \tilde{\omega}_j h^{kl} \hat{e}_{kl} - 2\tilde{\omega}^k \tilde{\omega}_{(i} \hat{e}_{j)k} \right) + \tilde{\omega}_k \beta^k \hat{k}_{ij} \right]
$$

with $\tilde{\omega}_i = \omega_i / \omega$.

Result:

- ADM equations are only weakly hyperbolic (3 eigenvectors missing).
ADM equations IV

\[ N = h^b Q \text{ (} h = \text{determinant of } h_{ij} \text{)} \]

The associated first order system is then

\[
\partial_t \hat{\ell}_{ij} \triangleq i\omega \left[ -2N\hat{k}_{ij} + \tilde{\omega}_k \beta^k \hat{\ell}_{ij} \right],
\]

\[
\partial_t \hat{k}_{ij} \triangleq i\omega \left[ -\frac{N}{2} \left( \hat{\ell}_{ij} + (1 + b)\tilde{\omega}_i \tilde{\omega}_j h^{kl} \hat{\ell}_{kl} - 2\tilde{\omega}^k \tilde{\omega}_{(i} \hat{\ell}_{j)k} \right) + \tilde{\omega}_k \beta^k \hat{k}_{ij} \right]
\]

Result:

- Modified ADM equations for \( b > 0 \) still weakly hyperbolic (2 eigenvectors missing).

- Adding Hamiltonian constraint does not change hyperbolicity, but does change characteristics.
BSSN equations I

\[ f^k = h^{ij} \gamma^k_{ij} + dh^{kl} \gamma^m_{lm} = h^{kl}(h^{ij} \partial_i h_{jl} + \partial_l \ln h) \]

\[ \mathcal{L}_{(t-\beta)} h_{ij} = -2Nk_{ij} \]

\[ \mathcal{L}_{(t-\beta)} k_{ij} = \frac{N}{2} h^{kl} \left[ -\partial_k \partial_l h_{ij} - b \partial_i \partial_j h_{kl} \right] + N \partial_i \xi_j + B_{ij} \]

\[ \mathcal{L}_{(t-\beta)} f_i = N \left[ -(2 - c) D^k k_{ki} + (1 - c) D_i k_k^k \right] + C_i \]
Hyperbolicity Analysis:

\[
\partial_t \hat{\ell}_{ij} \overset{\Delta}{=} i\omega \left[ -2\alpha \hat{k}_{ij} + \tilde{\omega}_k \beta^k \hat{\ell}_{ij} \right] \\
\partial_t \hat{k}_{ij} \overset{\Delta}{=} i\omega \left[ \frac{\alpha}{2} \left( -\hat{\ell}_{ij} - b \tilde{\omega}_i \tilde{\omega}_j h^{kl} \hat{\ell}_{kl} + 2\tilde{\omega}(\hat{f}_j) \right) + \tilde{\omega}_k \beta^k \hat{k}_{ij} \right] \\
\partial_t \hat{f}_i \overset{\Delta}{=} i\omega \left[ \alpha \left( (2 + 2c) \hat{k}_{ik} \tilde{\omega}^k + (1 - c) \tilde{\omega}_i h^{kl} \hat{k}_{kl} \right) + \tilde{\omega}_k \beta^k \hat{f}_i \right]
\]

Result: [Nagy-Ortiz-R]

- Modified BSSN equations for \( b > 0 \) \( c > 0 \) strongly hyperbolic.

- Eigenvalues: \((0, \pm 1, \pm \sqrt{b}, \pm \sqrt{c/2})\)
Constraint Propagation

Evolution System:

\[ \partial_t u^\alpha = A(u, t, x)^\alpha_{\beta} \partial_\alpha u^\beta + B(u, t, x)^\alpha, \]

Constraints:

\[ C^A = K(u, t, x)^A_{\alpha} \partial_\alpha u^\beta + L(u, t, x)^A, \]

Integrability condition (subsidiary system):

\[ \partial_t C^A = S(u, t, x)^A_{\alpha B} \partial_\alpha C^B + R(u, \partial u, t, x)^A_{\alpha B} C^B, \]

- Want to study what can we say about the properties of the subsidiary system from what we know from the evolution system.
Problem: In general $S(u, t, x)^{Aa} B$ is not unique if the constraint themselves satisfy certain identities.

For instance, if there is an $L_A(\omega)$ such that:

$$L_A(\omega) K^{An} \alpha_n = 0$$

we could add to $S(u, t, x)^{Aa} B$

$$M^{Aa} L_B$$

With this addition there are easy examples where one can get any sort of badly posed systems!
Assume: For any $\omega_i$, $K_{\alpha}^A \omega_n$ is surjective.

In general this is not satisfied, but in examples of interest one finds subset of constraints which do satisfy it. [Maxwell, EC].
Integrability condition implies:

\[ K^A(a_\alpha A^{[\alpha|b]}_\beta) - S^A(a_B K^{[B|b]}_\beta) = 0 \]

**Lemma 1:** Given any fixed non-vanishing co-vector \( \omega_a \). If \( (\sigma, u^\alpha) \) is an eigenvalue-eigenvector pair of \( A^{\alpha a}_\beta \omega_a \) then \( (\sigma, v^A = K^{Aa}_\alpha \omega_a u^\alpha) \), if \( v^A \) is non-vanishing, is an eigenvalue-eigenvector pair of \( S^{Aa}_B \omega_a \).
Integrability condition implies:

\[ K^{Aa} \alpha A^{\alpha b} \beta \omega_a \omega_b - S^{Aa} B K^{Bb} \beta \omega_a \omega_b = 0 \]

**Lemma 1:** Given any fixed non-vanishing co-vector \( \omega_a \). If \((\sigma, u^\alpha)\) is an eigenvalue-eigenvector pair of \( A^\alpha a \beta \omega_a \) then \((\sigma, u^A = K^{Aa} \alpha \omega_a u^\alpha)\), if \( u^A \) is non-vanishing, is an eigenvalue-eigenvector pair of \( S^{Aa} B \omega_a \).
Integrability condition implies:

\[ K^{Aa} \alpha A^{\alpha b} \beta \omega_a \omega_b - S^{Aa} B K^{Bb} \beta \omega_a \omega_b = 0 \]

- **Lemma 1**: Given any fixed non-vanishing co-vector \( \omega_a \). If \((\sigma, u^\alpha)\) is an eigenvalue-eigenvector pair of \( A^{\alpha a} \beta \omega_a \) then \((\sigma, u^A = K^{Aa} \alpha \omega_a u^\alpha)\), if \( u^A \) is non-vanishing, is an eigenvalue-eigenvector pair of \( S^{Aa} B \omega_a \).

- **Corollary 1**: If the evolution system is strongly hyperbolic then so is the subsidiary system. [It does not work symmetric \( \rightarrow \) symmetric].

- **Corollary 2**: The characteristics of the subsidiary system are a subset of the characteristics of the evolution system. The domain of dependence of the subsidiary system is at least as large as the domain of dependence of the evolution system.