DISCRETE DIFFERENTIAL FORMS IN NUMERICAL GENERAL RELATIVITY

- Motivation
- Discrete differential forms
- Einstein equations as differential ideal
- Discrete formulation
- Outlook

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SFB 382: Methods and Algorithms to simulate physical processes on high performance computers
Motivation

Numerical relativity is based essentially on the following procedure:

- Set up geometric differential equations
- *Split* into evolution equations and constraints
- Verify that constraints propagate

- *Discretise* evolution equations and constraints
- Solve constraints to provide initial data
- Choose gauges (coordinates, etc.)
- Evolve
- Check constraints to control the quality of solution
- Extract physical (invariant) information
Problems

- discretisation after split
- independent discretisation of evolution equations and constraints
- discrete versions are in general not compatible
- discrete constraints are not propagated by the discrete evolution → severe violation of constraints during simulations

- Einstein equations are invariant under diffeomorphisms → simulations are coordinate dependent
- invariant information has to be determined after the simulation

- geometric character (vector, tensor) of the variables plays no role

- finite element methods are largely ignored
Discrete Differential Forms

**continuous**

$p$-dimensional submanifold $S_p$:
- (0) point, (1) curve, (2) surface

$p$-form:
$$\omega : S_p \mapsto \int_{S_p} \omega \in \mathbb{R}$$

exterior derivative $d$:
$$\int_{S_p} d\omega = \int_{\partial S_p} \omega$$

Stokes’ theorem

**discrete**

$p$-simplices $\mathcal{S}_p$:
- (0) node, (1) edge, (2) face

Discrete $p$-form:
$$\omega : \mathcal{S}_p \mapsto \omega[\mathcal{S}_p] \in \mathbb{R}$$

**Definition:**
$$d\omega[\mathcal{S}_p] = \omega[\partial \mathcal{S}_p]$$

**Example:**
$$d\omega_{123} = \omega_{12} + \omega_{23} + \omega_{31}$$
Grassmann (wedge) product:

\[(\alpha, \beta) \mapsto \alpha \wedge \beta,\]

graded algebra

\[\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha,\]

derivation:

\[d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta.\]

deRham cohomology

discrete Grassmann product:

\[(\alpha, \beta) \mapsto \alpha \wedge \beta\]

Example:

\[
\begin{align*}
(\alpha \wedge \beta)_{123} &= \frac{1}{2} \left[ \alpha_{12} \beta_{13} + \alpha_{23} \beta_{21} + \alpha_{31} \beta_{32} \\
& \quad - \beta_{12} \alpha_{13} - \beta_{23} \alpha_{21} - \beta_{31} \alpha_{32} \right]
\end{align*}
\]

discrete d is derivation

singular cohomology
Einstein equation as differential ideal

Variables:

- (covariant) tetrad: \( \theta^i \)
- \( so(1,3) \) connection form: \( \omega^i_k \)

In addition:

- space-time ‘metric’: \( \eta_{ik} = (+, -, -, -) \)

Cartan’s structure equation:

\[
\begin{align*}
\text{d} \theta^i + \omega^i_k \wedge \theta^k &= 0, \\
\text{d} \omega^i_k + \omega^i_l \wedge \omega^l_k &= \Omega^i_k
\end{align*}
\]

Bianchi identity:

\[
\begin{align*}
\text{d} \Omega^i_k + \omega^i_l \wedge \Omega^l_k - \omega^l_k \wedge \Omega^i_l &= 0.
\end{align*}
\]
Nester-Witten form: 
\[ L_i = \frac{1}{2} \varepsilon_{ijkl} \omega^{jk} \wedge \theta^l \]

identity: 
\[ dL_i = S_i \overset{\sim}{\sim} \omega^2 + E_i \overset{\sim}{\sim} G_{ab} \]

Sparling: 
\[ dS_i = 0 \Leftrightarrow G_{ab} = 0. \]

exterior system for the variables \( \theta^i, \omega^i_k \):

\[
\begin{align*}
d\theta^i + \omega^i_k \wedge \theta^k &= 0, \quad \text{(2-form)} \\
dL_i - S_i &= 0. \quad \text{(3-form)}
\end{align*}
\]

gauge freedom: Lorentz rotations of the tetrad
Applications of the exterior system

- Einstein’s energy balance
- Landau-Lifshitz and Einstein pseudo-tensor
- Bondi mass loss, light focussing
- Positive mass theorem, Penrose inequality
**Discrete formulation**

1. Choose the topology of the time slices
2. Triangulate with 4-simplices
3. Replace continuous by discrete forms

**Discrete variables:**
values of $\theta^i$ and $\omega^i_k$ on edges: $\theta^i[e], \omega^i_k[e]$

**Geometric meaning:**

\[ l[e]^2 = \eta_{ik} \theta^i[e] \theta^k[e] \]

squared length

\[ R^i_k(e) = \exp(\omega^i_k[e]) \]
holonomy
• Choose gauge: Lorentz-rotations of the tetrad
• Evaluate the discrete forms on 2- resp. 3-simplices → algebraic, non-linear system for \( \{\theta^i[e], \omega^i_k[e]\} \)
• Split into evolution equations and constraints determined by the causal character of the simplices
• Bianchi identity is satisfied also for discrete formulation → essential for consistency of the equations
• squared length of an edge \( e \), holonomy along an edge \( e \): coordinate independent description of space-time
**Stepping in time**

- Triangulation consists of tetrahedra
- Each tetrahedron determines a **unique** point in the future
  - ‘dual’ triangulation in the next time slice, staggering
- edges will connect **null separated** points
- **CFL condition** built in
- can be used to fix the Lorentz gauge
Outlook

- Implementation in simple cases (1+1-systems)
  - spherical symmetry
  - pp-waves
- Investigation of the properties of the equations
  - propagation
  - hyperbolic character
- tetrad gauge?
- boundary conditions?
- methods of solution?