Disordered systems, ground states and combinatorial optimization

Disordered Systems, Ground States and Combinatorial Optimization in Statistical Physics

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A simple combinatorial optimization problem: The (directed) polymer model:

Lattice graph  Non-directed  directed

Random bond energies: \( e_i \in [0,1] \)

Total energy:
\[
E = \sum_{\text{path}} e_i
\]

Find ground state – i.e. optimal path
From top node to a bottom nodes:
Collection of optimal directed polymers

Dijkstra's algorithm for shortest paths in general graphs

Start node: s
Minimal distance (energy) from s to j: d(j)
Predecessor of j: pred(j)

algorithm Dijkstra
begin
S := \{s\}, S' := N \setminus \{s\};
d(s) := 0, pred(s) := 0;
while |S| < |N| do
begin
choose (i,j);
d(j) := \min_{k,m} \{d(k) + c_{km} | k \in S, m \in S'\};
S' := S' \setminus \{j\}; S := S + \{j\};
pred(j) := i;
end
end

Performance \( O(N^2) \), with heap reshuffling \( O(N \log(N)) \)
Optimal paths with correlated disorder

Isotropically correlated disorder: \( \langle e_i e_{i+r} \rangle \sim r^{2\beta-1} \)

Universal geometrical properties:

Roughness:
\[ D(L) = \langle x^2 \rangle - \langle x \rangle^2 \sim L^\nu \]

Energy fluctuations:
\[ \delta E(L) = \langle E^2 \rangle - \langle E \rangle^2 \sim L^\sigma \]

Optimal paths with correlated disorder (2)

2d: Roughness exponent \( \nu \)  
Energy fluctuation exp. \( \sigma \)

\[ \langle e_i e_{i+r} \rangle \sim r^{2\beta-1} \]

Optimal paths with \( E < E_0 \)

- \( \beta < 0 \)
- 2d
- \( \beta = 0.4 \)

3d
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From one line to many lines

Continuum model for $N$ interacting elastic lines in a random potential

$$H = \sum_{i=1}^{N} \int_{\mathbf{y}} \int_{\mathbf{z}} \left( \frac{1}{2} \frac{dr}{dz} \right)^2 + V_{\text{rand}}(r(z), z) + \sum_{j=1}^{\infty} V_{\text{int}}(r_j(z), r_{j+1}(z))$$

Strong disorder: $V_{\text{rand}} >> V_{\text{int}}$ short ranged, hard core

$$H = \sum_{(\text{bond})} e_i n_i \quad n_i = 0, 1$$

Ground state of N-line problem: Minimum Cost Flow problem

Find successively shortest paths from $s$ to $t$ in the residual network $G_r(n)$:

- Use node potentials $p(i)$
- Reduced energies $e_p^{ij}(n) = e_{ij} + p(i) - p(j)$
- All positive

Use Dijkstra’s algorithm

Example conf. for 9 lines:

begin
\$n:=0; p(0)=0; G_r(0)=G;$
for line-counter = 1 to $N$ do
\$e_p^{ij}(n)=e_{ij}+p(i)-p(j);$  
\$d(i)\$ from $s$ to all other nodes $i$ in the residual network $G_r(n)$ w.r. to the reduced energies $e_p^{ij}(n)$;  
\$d(i)\$ from $s$ to $t$ by one unit;  
\$p(i)\$ from $d(i)$;  
end

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**Successive shortest path algorithm (2)**

**Entanglement of elastic lines in a disordered environment**

Magnetic flux lines in type II superconductors:

- **Pure case:** Abrikosov-FL lattice
- **Weak disorder:** Bragg glass phase
- **Strong disorder:** Topological defects
  - FL entanglement

**Definition of two-line-entanglement**

Check winding angle of line A and B:

- if $>2\pi$: A and B are entangled.

**Definition of entangled clusters:**

Entangled lines form clusters or bundles:

- $A \otimes B$ and $B \otimes C$ 
- $A \& B \& C$ in one bundle
Entanglement transition of elastic lines

Conventional 2d percolation transition

Entanglement transition of elastic lines (2)

Thick samples (H>H_c): Entanglement transition;

Thin samples (H<H_c): Disentangled lines – rods.

Magnetic field (B) driven Bragg glass (BG) to vortex glass (VG) transition in disordered high-T_c superconductors; n.b.: line density \[ \rho \] B
Loop percolation of magnetic flux lines in HTC (1)

**Vortex-Glass Model**

\[ H = \frac{1}{2} \sum_{i,j} (n_i \square b_j)G_D(i \square j)(n_j \square b_i) \]

- \( n_i = \ldots |2,1,0,1,2,\ldots | \), \( n_j = 0 \)
- \( b_j = \sum_{\text{plaqs}} A_j, A_y \in [0,2\pi] \)

\[ G_D(i \square j) \sim \exp(r_y / r_g) / r_g \]

Strong screening limit: \( \square 0 \)

\[ H = \sum_i (n_i \square b_i)^2 \]

Ground state: Minimum cost flow problem
Successive shortest path algorithm

Note: \( \square = \) strength of disorder

Loop percolation of magnetic flux lines in HTC (2)

2d

3d
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**Loop percolation of magnetic flux lines in HTC (3)**

- \( P_{\text{perc}} \)
- \( P_{\text{pc}} \)
- \( P(n) \)

2d:
- \( n = 3.3 \)
- \( b = 1.8 \)
- \( t = 2.45 \)

3d:
- \( n = 1.05 \)
- \( b = 1.4 \)
- \( t = 2.85 \)

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**Another combinatorial optimization problem: Interfaces in random bond Ising ferromagnets**

\[
H = \sum_i J_{ij} S_i S_j \quad J_{ij} \geq 0, \quad S_i = \pm 1
\]

Find for given random bonds \( J_{ij} \) the ground state configuration \( \{S_i\} \) with fixed +/- b.c.

Find interface (cut) with minimum energy

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Min-Cut-Max-Flow Problem

A network \( G(V,A) \), arcs (Bonds) \((i,j) \) have capacity \( u_{ij} \geq 0 \), flow \( 0 \leq n_{ij} \leq u_{ij} \) fulfills mass balance constraint:

\[
\begin{array}{c}
n_j - n_i = v_i & \text{for } i = s \\
-n_j + n_i = v_i & \text{for } i = t \\
0 & \text{else}
\end{array}
\]

Find the maximum flow \( n^* \) with value \( v \) from \( s \) to \( t \)

residual network \( G(n) \) with residual capacities:

\[
r_{ij} = u_{ij} - n_{ij} + n_{ji}
\]

\( n^* \) maximum flow \( v \) no directed path \( s \to t \) in \( G(n^*) \)

s-t cut \([S,S']\) is a partition of \( V \) in two disjoint sets with \( s \in S, t \in S', S \cup S' = V \), \( (S,S') = \{(i,j) \in A | i \in S, j \in S'\} \); capacity of the s-t-cut \[
|\{(i,j) \in A | i \in S, j \in S'\}|
\]

Min-Cut-Max-Flow-Theorem: \( \max_{n \in \mathbb{N}} v = \min_{S,S'} |\{(i,j) \in A | i \in S, j \in S'\}| \) and \( r_{ij}^* = 0 \) along \( (S,S') \)

Preflow-Push Algorithm

Strategy:
1) flood the network from source
2) propagate the flooding toward the target
3) push excess flow back towards source

excess flow \( e(i) = \sum_{j} n_{ij} - \sum_{j} n_{ji} \)

distance function \( d(i) \) (w.r.t. target)

begin
\[ d(i) = \text{exact distance from target, } d(s)=|V| \]
for all \((i,j) \) \( A: n_{ij} = u_{ij} \)
choose \( i \in V \setminus \{s,t\} \) with \( e(i) > 0 \)
if there is \((i,j) \) \( A \) with \( d(i)=d(j)+1 \):
\[ \begin{cases} 
\text{push } |\{e(i),r_{ij}^*\} \text{ flow units from } i \to j \\
\text{else} \\
\text{relabel } d(i) \max \{d(i)+1,(i,j) \in A, r_{ij}^*>0\}
\end{cases} \]
end
Preflow-Push-Algorithm (2)

Interprete distance function (or labels) as height of the nodes.

Problems that can be mapped on min-cut / max-flow

• Interfaces / wetting in random media
• Random field Ising model (in any dimension)
• Periodic media (flux lines, CDW, etc.) in disordered environments
• Elastic manifolds with periodic potential and disorder
**Example: Elastic manifolds / Elastic media**

Expansion of potential energy around equilibrium positions:

\[
H_{\text{elast}} = \sum_{r} [u(r)]^2 \sum_{(ij)} [u(r_i) - u(r_j)]^2
\]

Effects of impurities, i.e. disorder: lattice distortions:

\[
H_{\text{rand}} = \sum_{r} V[r, u(r)] \sum_{(ij)} V[r_i, u(r_i)]
\]

Manifolds: D=1 KPZ, in any D interface in D+1 RBIFM

\[
\mathbb{w}^2 = \sum_{i} (u_i - \langle u \rangle)^2 L^D
\]

\(D=2/3, 0.41, 0.22\) in D=2,3,4

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**Periodic elastic media**

Symmetry: \(H(u)=H(u+a \cdot n)\), e.g.: flux lines

Random pinning potential: first harmonics

\[
V[r, u(r)] = \cos(2\pi u(r)/a \cdot \langle r \rangle)
\]

\(\mathbb{a} \in [0,2\pi], \) random

\[
H = \sum_{r} [u(r)]^2 + \sum_{ij} \cos(u(r) \cdot \langle r \rangle) \sum_{(ij)} [u_i - u_j]^2 + \sum_{i} \cos(u_i \cdot \langle i \rangle)
\]

Flory argument: **Roughness** \(\langle u^2 \rangle \sim \ln \frac{L}{\mathbb{a}}\)

in 2d **RG**: \(\langle u^2 \rangle \sim (\ln \frac{L}{\mathbb{a}})^2\)
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**Periodic elastic media + periodic potential**

\[ H = \sum_{r} (u(r))^2 v \cos(p \cdot u(r)) + \sum_{r} \cos(u(r) \cdot \mathcal{D}(r)) \]

\[
\langle \mathcal{D}(r) \rangle = 0 \quad \langle \mathcal{D}(r)' \rangle = \mathcal{D}(r' \mathcal{D}(r)) \\
\langle \mathcal{D}(r) \rangle = 0 \quad \langle \mathcal{D}(r)' \rangle = \mathcal{D}(r' \mathcal{D}(r))
\]

2 periodicities with ratio \( p \) \quad T=0 roughening transition via \( \mathcal{D} \)

Charge density wave system \quad Flux line system

**Periodic elastic medium + periodic potential (2)**

Mapping to an RBIFM interface problem

Discrete interface hamiltonian:

\[ H = \sum_{i,j} (h_i \delta_{i,j})^2 \sum_{i} \cos(2 \delta_{i,j} / p \delta_{i,j}) \]

Ising model:

\[ H = \sum_{i,j} J_{ij} S_i S_j \]

\[ J_{h\text{direction}} = \langle \mathcal{D}(r) \rangle \cos(2 \sum_{i} h / p \mathcal{D}(r)) \]

\[ J_{r\text{direction}} = \text{const.} \langle \mathcal{D}(r) \rangle \]

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**Periodic elastic medium + periodic potential (3)**

The roughening transition in 3d:

Order parameter: $m_{p,q}(L,T) = |e^{2q/L^q}|$

<table>
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<tr>
<th>$p$</th>
<th>$D_k$</th>
<th>$D_{q,2}$</th>
<th>$D_{q,3}$</th>
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<td>0.033</td>
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</tr>
</tbody>
</table>

**Further applications of combinatorial optimization methods in Stat-Phys.**

- Flux lines with hard core interactions
- Vortex glass with strong screening
- Interfaces, elastic manifolds, periodic media
- Wetting phenomena in random systems
- Random field Ising systems
- Spin glasses (2d polynomial, d>2 NP complete)
- Statistical physics of complexity (K-Sat, vertex cover)
- Random bond Potts model at $T_c$ in the limit $q \to \infty$
- ...

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Further reading:

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A. Hartmann and H. Rieger,
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(Wiley VCH, Berlin, 2002)