Simulating Black Hole Spacetimes: Successes and Challenges (ITP Colloquium 3/13/02)

Simulating Black Hole Spacetimes
Successes and Challenges

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Outline

- 20th Century Gravitation: General Relativity
- Black Holes and Gravitational Radiation
- Challenges in Black Hole Simulations
- Critical Phenomena in Gravitational Collapse
- Towards Realistic Black Hole Simulations

General Relativity

- Einstein (1917)
- Gravitational effects consequence of curvature of spacetime; curvature consequence of matter-energy distribution in spacetime
- Spectacular predictions
  - Expanding universe
  - Black holes
  - Worm holes
  - Gravitational waves

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**Newtonian Gravitation**

- Gravitational force on object with gravitational mass $m_g$
  \[ \vec{F} = -m_g \nabla \phi \]
  \[ \nabla^2 \phi \propto \rho \]
- *Single Newtonian potential* (single field) $\phi$ describes gravitational interaction
- *Only objects with mass* contribute to mass density $\rho$
- *Action at a distance*: Changes in gravitational field propagate instantaneously to rest of universe

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**Newtonian Gravitation**

**Universality of Free Fall**

Assuming that inertial mass and gravitational mass are proportional, with the same proportionality constant for all substances:

\[ m_i \ddot{a} = \vec{F} = -m_g \nabla \phi \]
\[ \ddot{a} = -\nabla \phi \]
Relativistic Gravitation (GR)

Universality of free-fall elevated to Principle of Equivalence

- Locally, uniform gravitational field indistinguishable from uniform acceleration

- "Real" gravitational effects show up in non-uniformities of gravitational field (curvature of spacetime)

- Gravitational field much more complicated than in Newtonian case, essentially need four potentials plus two "wave fields"

- No action at a distance: disturbances in the gravitational field travel at most at the speed of light, c

- All forms of energy act as sources for gravitational field

The Metric

- The geometrical information about spacetime is completely encoded by the (symmetric) metric tensor

\[ g_{\mu\nu}(x^\alpha) = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix} \]

- Spacetime distance (squared) between nearby events is given by

\[ ds^2 = \sum_{\mu=0}^{3} \sum_{\nu=0}^{3} g_{\mu\nu} dx^\mu dx^\nu \]
Causal Structure

- Metric can be used to map out causal structure of the spacetime
- Spacetime metric is non-definite, displacements between events can be
  - Time-like (-)
  - Space-like (+)
  - Null (0)
- Lightcones separate different classes of events

3+1 split

- Slice spacetime into stack of 3-d space-like hypersurfaces
- 4-geometry of spacetime (4-metric) becomes time-evolution of 3-geometry of initial hypersurface (dynamics of 3-metric)
- General covariance: coordinates on spacetime arbitrary, don't effect physics
- 4 degrees of coordinate-freedom ("gauge freedom")
- Coordinates must be fixed: HOW?

Dynamical Variables
\[ g_{ij}(t,x^k), K_{ij}(t,x^k) \quad i,j,k = 1,2,3 \]

Dynamical Degrees of Freedom

\[ 10 - 4 - 4 = 2 \]
\[ 6 - 4 = 2 \]
**Gravitational Collapse and Black Holes**

- **BLACK HOLE**: Region of spacetime from which no physical signal can escape.
- During collapse of matter and/or radiation, BH forms when gravitational field becomes strong enough to "trap" light rays.
- Surface of black hole is known as the **event horizon**.
- **Singularities** (infinite, crushing gravitational forces) **inevitable inside black holes**.

(From Wald, General Relativity, 1984)

**Gravitational Radiation**

- **Gravitational waves**: "ripples" in the curvature of spacetime.
- At least in weak field limit, very much analogous to electromagnetic radiation; propagate at speed of light, transverse, two polarizations, frequency set by dynamics of source.

The Laser Interferometer Gravitational Wave Observatory (LIGO) installation near Hanford WA. Each interferometer arm is 4 km long. A similar instrument is located near Livingston LA (www.ligo.caltech.edu)

Cause periodic, quadrupolar variations in distance between freely falling objects (or induce strains in objects with interactions).
Sources of Gravitational Radiation

- For efficient radiation, need (large) masses confined to regions comparable in size to their Schwarzschild radii, $R_s$.
- Also need redistribution of significant fraction of mass-energy at close to light speed.

Source Strengths and LIGO Sensitivity

- Compact binary systems (BHs, neutron stars) good candidates.

\[ R_s = \frac{2G M}{c^2} \]
\[ R_s = \frac{2G M}{c^2} \]
\[ \frac{L}{L_{\text{design}}} = 10^{-21} \]
\[ \delta L/L = 10^{-15} \]

\[ M \geq 1.5 \times 10^{-7} M_{\odot} \]
\[ L_{\text{design}} = 3 \times 10^{-44} \text{erg sec}^{-1} \text{Hz}^{-1} \]

\[ \text{LIGO strain sensitivity: } (30-1000 \text{ Hz}) \]

\[ \text{Phase 1: } \delta L/L = 10^{-21} \]

\[ \text{Phase 2: } \delta L/L = 10^{-15} \]

\[ c = 3 \times 10^5 \text{ km sec}^{-1} \]

\[ G = 6.67 \times 10^{-11} \text{ m}^3 \text{kg}^{-1} \text{sec}^{-2} \]

\[ M = \text{mass} \]

\[ L = \text{light} \]

\[ c = \text{speed of light} \]

\[ \odot = \text{Sun} \]

\[ \text{kg} = \text{mass in kilograms} \]

\[ \text{m} = \text{mass in meters} \]

\[ G = \text{Gravitational constant} \]

\[ M = \text{mass of object} \]

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Black Hole Simulations: Key Challenges

- Formulation and discretization of equations of motion
- Singularity avoidance
- Computational demands
- NUMERICAL STABILITY
- Tie-in to observations (gravitational wave extraction)
- Shortage of personnel

Singularity Avoidance: Black Hole Excision

- To avoid singularity within black hole, exclude interior of hole from computational domain (Unruh)
- Catch 22: event horizon is globally defined, location unknown until complete spacetime geometry is in hand
- Apparent horizon functions as "instantaneous horizon" can be located at any instant of time
- Excise somewhat within apparent horizon
- Crucial for long-time BH evolution
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**Computational Demands**

- Finite difference methods dominant
- $D$ spatial dimensions:
  - $O(N^{D+1})$ CPU time
  - $O(N^D)$ Memory
- Minimum $N$ for interesting problems:
  \[ N \geq 1000 \]

**Computational Demands**

- $D \rightarrow D+1$ thus requires of the order of 10 CPU-speed doublings, or about 15 years of hardware evolution
- Estimate is pessimistic in some ways, but optimistic in others (e.g. increase in per-grid point complexity for higher $D$)

Crude estimates suggest calculations of general BH collisions require many CPU days on a Terahop's, terabyte machine.

(S.R. White, IBM TJW Research Ctr.)

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Computational Demands

- Simple minded analysis also ignores two key facts about simulation-based science

1. When good solution method is not yet known, need turn-around time on the order of hours (at most) to make effective progress

2. A few simulations generally won’t cut it. Most interesting problems will have considerable parameter space to explore, which introduces additional effective dimensionality into the problem

- Lots of room for gains due to improved algorithms

Model Problems

- Reduce computational demands by imposing symmetry conditions which reduces effective $D$ (spatial dimension)
  - $D = 1$ spherical symmetry (symmetry under all rotations about a point)
  - $D = 2$ axisymmetry (symmetry under rotations about an axis)

- Models can reveal interesting physics which may not depend crucially on symmetry assumptions

- Use for development of techniques for more realistic calculations
Model Problems

- Spherically symmetric calculations were initiated in earnest circa 1990.

- Axisymmetric calculations are on-going; turn-around times are still a few times longer for the 1990 spherically symmetric runs at comparable resolution.

Black Hole Critical Phenomena

- Consider parameterized families of initial data representing gravitational collapse

- Family parameter, \( \eta \), controls strength of gravitational field during time evolution

\[ p < p^* \quad \text{No black hole forms} \]
\[ p > p^* \quad \text{Black hole forms} \]
\[ p = p^* \quad \text{Threshold of BH formation} \]

- What is nature of no-BH to BH transition? (Christodoulou)

- Can one, in principle, make finite with arbitrarily small mass?
**Black Hole Critical Phenomena**

- BH threshold generically characterized by *exponential sensitivity* to initial conditions
- For any collapse model, find (isolated) *universal solutions* with additional symmetry
  - Type I: static or periodic
  - Type II: continuously or discretely self-similar
- Find *universal scaling laws* for solutions near-criticality, for example, in Type-II collapse, black hole masses satisfy
  \[ M_{BH} \propto \left| \bar{p} - p^* \right|^\gamma \]
  where \( \gamma \) is a *universal* (independent of initial data family) exponent

**Massless Scalar Collapse**  
*(Spherical Symmetry, G=c=1)*

- Matter field: massless scalar field: \( \phi(r,t) \)
- Initial data: \( \phi(r,0), \frac{\partial \phi}{\partial t}(r,0) \)
- Dynamics: Imploding / exploding shells of radiation
- Fundamental non-linearity gives rise to competition
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**Competition**

Where can energy in system end up?

\[ KE \rightarrow r = \infty \text{ COMPLETE DISPERSAL} \]

\[ PE \rightarrow r < R_{BH} \text{ BLACK HOLE FORMATION} \]

*Play two effects off each other to generate critical solutions*

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**Scalar Collapse: Weak Field Regime**

• Metric (flat space-time; fixed, no dynamics)

\[ ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \]

• Scalar field equation of motion

\[ \frac{\partial^2}{\partial t^2}(r\phi) = \frac{\partial^2}{\partial r^2}(r\phi) \]

• General solution: ingoing & outgoing waves

\[ r\phi(r,t) \equiv u(r+t) + v(r-t) \]
Scalar Collapse: Weak Field Regime

- Initial data: purely ingoing pulse
  \[ r\phi(r,0) = \phi_0 f(r) \]
  \[ \frac{\partial}{\partial t} r\phi(r,0) = \phi_0 \frac{df}{dr} \]

End state of evolution is complete dispersal of scalar field

Scalar Field: Strong Field Regime

- Metric
  \[ ds^2 = -\alpha^2(r,t)dt^2 + a^2(r,t)dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \]

- Auxiliary scalar field variables
  \[ \Phi(r,t) = \frac{\partial \phi}{\partial r} \]
  \[ \Pi(r,t) = \frac{a}{\alpha} \frac{\partial \phi}{\partial t} \]

- Mass function, conserved “mass-at-infinity”
  \[ a^2(r,t) = \left(1 - \frac{2m(r,t)}{r}\right)^{\frac{3}{2}} \]
  \[ M = \int_0^\infty \frac{dm}{dr} dr \]
**Scalar Collapse: Strong Field Regime**

- Equations of motion

\[
\frac{\partial \Phi}{\partial t} = \frac{\partial}{\partial r} \left( \frac{\alpha}{a} \Phi \right)
\]

\[
\frac{\partial \alpha}{\partial t} = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\alpha}{a} \Phi \right)
\]

\[
1 \frac{da}{dr} + \frac{a^2 - 1}{r} \frac{da}{dr} = 0
\]

\[
1 \frac{da}{dr} + \frac{a^2 - 1}{r} \frac{da}{dr} - 2\pi r \left( \Pi^+ + \Phi^+ \right) = 0
\]

(Strong-field evolution. Scalar field still completely disperses, but non-linear self-gravitational effects are apparent in trailing edge of waveform.)

**Generating a Critical Solution**

- Choose *arbitrary* one parameter family of initial data

- Locate initial bracket
  - \( p < p^* \) no black hole
  - \( p > p^* \) black hole

- Refine bracket via bisection search

- Can tune to machine precision, \( 10^{-10} \) provided code has enough dynamic range (adaptive mesh refinement)

\[
\phi(r,0) = \phi_0 f(r) = p f(r)
\]
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Bracketing Solutions

Scalar Collapse: Near-Critical Regime

- Tune control parameter, $p$, to machine precision
- As $p \to p^*$, unique critical solution emerges (details of initial data "washed away"
- Critical solution is discretely self-similar
- Curvature grows without bound in precisely-critical limit, and no black hole forms: $\to$ naked singularity

Marginal sub-critical evolution: each oscillation represents same strong-field dynamics playing out on a spatio-temporal scale some 30 times smaller than its predecessor.
Logarithmic time, radial coordinates used
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**Self-Similarity of Type II Solutions**

- $T \equiv$ central proper time (measured from “accumulation event”)
- $R \equiv$ physical (e.g., “areal” radius)
- Similarity variable: $\zeta = R/T$
- Naked singularity at $(0,0)$

**Self-Similarity of Type II Solutions**

- Continuously self-similar (CSS)
  
  \[ g^*(\zeta, \tau) = g^*(\zeta, \tau') \quad \tau, \tau' \text{ arbitrary} \]

- Discretely self-similar (DSS)
  
  \[ g^*(\zeta, \tau) = g^*(\zeta, \tau + n\Delta) \quad n = 0, \pm 1, \pm 2, \cdots \]

$\Delta = \text{Model-dependent "echoing" exponent}$
Features of Type II Collapse

- **Arbitrarily small black holes possible** (by definition)

- Critical solution is **self-similar**, possibly discretely

- Precisely critical solution contains a **naked singularity** (but formation clearly not generic)

- **Scaling laws** for dimension-ful physical quantities, such as the black hole mass

- **Universality** Same results independent of specifics of initial data

\[ M_{\text{BH}} \propto |p - p^*|^{\frac{1}{2}} \]
BH Critical Phenomena
Perturbation Theory
(Kaiké et al., Evans, Gundlach ...)

- Black hole threshold solutions (critical solutions) are **unstable** by construction
- Turn out to be **minimally unstable**, generally have only **one unstable mode** in perturbation theory
- Scaling-law exponent is just the reciprocal Lyapunov exponent of the unstable mode
- Explains most of phenomenology and universality

Critical Yang-Mills Collapse

- Spherically-symmetric collapse of non-Abelian SU(2) gauge field
- Rich phenomenology observed, Type I *and* Type II transitions, as well as a generalized Type I where the critical solution is a "hairy" or "colored" black hole
- Black hole excision techniques **crucial** in latter case

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Critical Yang-Mills Collapse

1: Type II behaviour
2: Type I behaviour, no excision
3: Generalized Type I, excision
(Top/bottom: sub/super-critical)

\[ \tau \sigma \ln | p - p^* | \]

Critical “Neutron Stars” (Scott Noble)

- Perfect fluid coupled to gravity, spherically symmetry
- Ideal-gas (Synge) equation of state
  \[ P = (\Gamma - 1)\rho_{\text{tot}} E \]
  \[ 1 < \Gamma \leq 2 \]
- Family of static solutions, “stars”, parametrized by central density
- Stars to right of maximum are unstable

\( \Gamma = 2: \text{extremely stiff fluid} \)
Critical “Neutron Stars”

- “Perturb” stable, static, star via imploding pulse of scalar field
- Fluid/scalar interact only through gravitational field.
- By fine-tuning amplitude of scalar pulse, can drive fluid to unstable stellar state (+ perturbations)
- End state is either black hole or perturbed stable star

Beyond Spherical Symmetry

- Currently working on axisymmetric collapse code with Hirschmann, Liebling and Pretoius
- Want to study critical phenomena in more general context, as well as develop infrastructure for more realistic calculations in numerical relativity
- Again, couple a scalar field to gravitational field, but can also study dynamics of vacuum spacetimes (nothing but gravitational waves and black holes).
Near-critical Scalar Field Collapse
Spherically Symmetric Initial Data

Solution is good match to result from spherically-symmetric code

Near-critical Scalar Collapse
Spherically Symmetric Initial Data
(continuous zoom-in)
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**Boosted Merger of Two Scalar Pulses With Black Hole Excision**

\[ \phi(t, \rho, z) \]

\[ t=0.00 \text{ M} \]

**Highly Prolate Scalar Collapse With Black Hole Excision**

\[ \phi(t, \rho, z) \]

\[ t=0.0 \text{ M} \]
Realistic Black Hole Collisions

- Event rate of BH-BH, NS-NS, BH-NS mergers highly uncertain, but is expected that for "advanced LIGO", observed rate should be few per year, conservatively
- Requires fully 3-D computations
- Will probably require adaptive mesh refinement techniques and black hole excision
- Computational demands are enormous: Teraflop/s, Terabyte class

State-of-the-Art
"3D" Black hole collision
(Lehner et al, PSU, UT Austin, UBC)

Mesh size of order 100 x 100 x 100; other calculations approaching 400 x 400 x 400 (100 Gbyte!)

Single component of 3-metric on 2-D slice through grid is visualized, black hole excision techniques employed

STABILITY IS KEY ISSUE

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"3D" Black hole collision

Personnel Issues
(or “In case you’re looking for a PhD topic”)

- Field has serious shortage of young researchers
- Funding derivative for numerical relativity is decidedly positive
- Area will become even more attractive to universities (i.e. faculty positions) as instruments such as LIGO start detecting signals

Caltech Gets More from Moore
Caltech’s coffers are $600 million fatter, thanks to semiconductor pioneer Gordon Moore, his wife Betsy, and the foundation the couple created in 2000. Their combined gift—$400 million from the Moore and the same again from the Garden and Betsy Moore Foundation—is the largest-ever donation to a university, eclipsing last year’s record-breaking gifts of $400 million to Stanford University from the William and Flora Hewlett Foundation and an anonymous $300 million to Rensselaer Polytechnic Institute.

Among the specific projects and bread areas that might get some of the Moore money are the design of the California Extremely Large Telescope; a 30-meter ground-based optical and infrared telescope that Caltech is planning jointly with the University of California; measurements to test general relativity; nanomechanics, a subatomic theorem; beam lines; and facilities for cryoelectron microscopy and functional brain imaging.

(Physics Today, January 2002)
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AND THE MORAL IS...

Near-critical Scalar Collapse z-Antisymmetric Initial Data

\[ \phi(t, \rho, z) \]

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