NONMODAL GROWTH OF THE MAGNETOROTATIONAL INSTABILITY

J. SQUIRE jsquire@pppl.gov
A. BHATTACHARJEE
Plasma Physics Laboratory, Princeton University

INTRODUCTION

We study the magnetorotational instability from using nonmodal stability methods [1].

Even for the unstable case, the non-normality of the system can have profound consequences.

The fastest growing structures can be very different from eigenmodes and grow at a much faster rate over reasonably long time-scales.

Over short time-scales the maximum MRI growth rate is *independent* of the vertical and azimuthal wave-numbers: always -dΩ/dlnr.

NONMODAL STABILITY THEORY

At some chosen time, what are the initial conditions that maximize the growth in the chosen norm?

If the system is not self-adjoint, this is *not* the least stable eigenmode [1].

Solve the local and global 1-D linearized MHD equations.

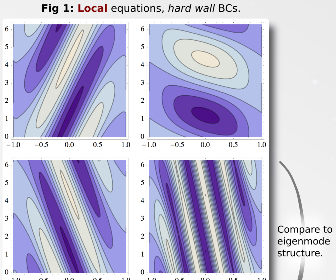
We use the perturbation energy as the norm $E = \|U\|_E^2 = \int d\boldsymbol{x} \left(|\boldsymbol{u}|^2 + |\boldsymbol{B}|^2\right)$ [2].

Changing to variables where $\|U\|_E^2 = \|U\|_2^2$ the energy is maximized using the singular value decomposition.

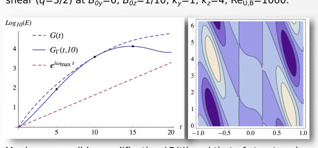
We discretize operators using Chebyshev polynomials in *Mathematica*.

The fastest growing solutions always resemble shearing waves

Fig 1: Local equations, *hard wall* BCs.

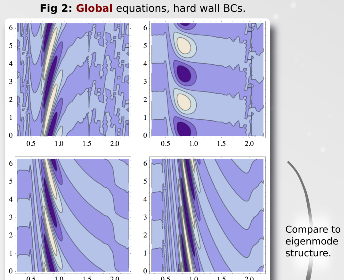


Fastest growing solution (B_x) for t=10 with Keplerian shear (q=3/2) at B_{0y}=0, B_{0z}=1/10, k_y=1, k_z=4, $Re_{U,B}$=1000.

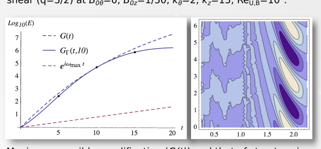


Maximum possible amplification ($G(t)$) and that of structure in Fig.1 (solid, blue) compared to least stable eigenmode (red).

Fig 2: Global equations, hard wall BCs.



Fastest growing solution (B_r) for t=10 with Keplerian shear (q=3/2) at $B_{0\theta}$=0, B_{0z}=1/30, k_θ=2, k_z=15, $Re_{U,B}$=10^4.



Maximum possible amplification ($G(t)$) and that of structure in Fig.2 (solid, blue) compared to least stable eigenmode (red).

COMPARISON TO SHEARING WAVES

The local MRI is commonly studied with a shearing wave decomposition [3].

Insert $f(\boldsymbol{x},t) = f(t)\exp\left(iqk_y\left(t-t_0\right)x + ik_y y + ik_z z\right)$ for each variable. All nonlinear terms cancel and one is left with a system of ODEs [4].

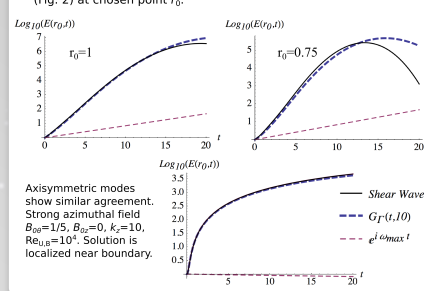
$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ \zeta \\ B \\ \eta \end{pmatrix} = \begin{pmatrix} -2qk_xk_y/k^2 - \frac{k^2}{\mathrm{Re}_U} & -2ik_z/k^2 & i\left(k_yB_{0y}+k_zB_{0z}\right) & 0 \\ i(q-2)k_z & -k^2/\mathrm{Re}_U & 0 & i\left(k_yB_{0y}+k_zB_{0z}\right) \\ i\left(k_yB_{0y}+k_zB_{0z}\right) & 0 & -k^2/\mathrm{Re}_B & 0 \\ 0 & i\left(k_yB_{0y}+k_zB_{0z}\right) & -iqk_z & -k^2/\mathrm{Re}_B \end{pmatrix}\cdot\begin{pmatrix} u \\ \zeta \\ B \\ \eta \end{pmatrix} \qquad \begin{aligned} &u = u_x,\ B = B_x, \\ &\zeta = k_z u_y - k_y u_z, \\ &\eta = k_z B_y - k_y B_z \end{aligned}$$

Compare global fastest growing solutions to shearing wave growth.

Get local parameters through expansion about chosen point r_0.

Shearing wave initial conditions obtained using same nonmodal techniques: *only matched parameter is the initial* k_x.

Comparison of shearing wave growth with growth of global solution (Fig. 2) at chosen point r_0.



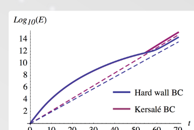
Axisymmetric modes show similar agreement. Strong azimuthal field $B_{0\theta}$=1/5, B_{0z}=0, k_z=10, $Re_{U,B}$=10^4. Solution is localized near boundary.

BOUNDARY CONDITIONS

General feature of nonmodal stability: results are less affected by changes to the system, e.g., boundary conditions [1].

A change to shearing BCs in Fig, 1 will not drastically change the nonmodal structure.

Example: compare hard wall BCs with those in Kersalé et al. [5].



Nonmodal growth is identical despite difference in spectrum. Global domain with $B_{0\theta}$=1/5, k_θ=2, k_z=15, $Re_{U,B}$=10^4.

Not so true for axisymmetric modes: less non-normality.

CONCLUSIONS

In many cases – particularly for non-axisymmetric MRI – the nonmodal growth can be far more physically relevant than eigenmodes.

Even in unstable situations, transient growth can last a sufficient time to give very large mode amplification.

Leads to a very natural connection between global modes and local shearing wave calculations [6].

Nonmodal techniques should always be used in considering the relative importance of different mode numbers. Eigenmodes will underestimate the importance of non-axisymmetric modes.

Over short time-scales shearing waves grow faster than static structures

Examine growth over infinitesimally short time-scales – opposite to *t*=∞ limit where eigenvalues apply.

Quantify these ideas using

$$G^+_{max} = \max_{U(0)} \|U(t)\|_E^{-2} \frac{d}{dt}\|U(t)\|_E^2\Big|_{t=0^+}$$

given by the largest eigenvalue of $\Lambda + \Lambda^\dagger$, with $\Lambda = F\mathcal{L}F^{-1} + \partial_t F F^{-1}\big|_{t=0}$, $\mathcal{L}(t)$ the linear operator, and $F(t)$ the inner product matrix $\|U\|_E^2 = U^\dagger \cdot F^\dagger F \cdot U$.

Compare growth of shearing wave at chosen $k_x(0)$, k_y, k_z, to that of static eigenmode-like structure at the same scale.

Use a WKB expansion for the static calculation in the local case. Have also carried out global calculation using a WKB-like time-dependent global shearing wave [7].

Remarkably, one obtains simple analytic results at arbitrary wavenumber and dissipation.

Shearing

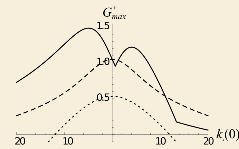
$$G^+_{max} = \max\begin{cases} q\Omega\left(\frac{1}{k}\sqrt{k_z^2 + \frac{k_x^2k_y^2}{k^2}} - \frac{k_xk_y}{k^2}\right) - 2\Omega\frac{k^2}{\mathrm{Re}_U} \\ q\Omega\left(\frac{1}{k}\sqrt{k_z^2 + \frac{k_x^2k_y^2}{k^2}} + \frac{k_xk_y}{k^2}\right) - 2\Omega\frac{k^2}{\mathrm{Re}_B} \end{cases}$$

Static

$$G^+_{max} = \max\begin{cases} q\Omega\frac{k_z}{k} - 2\Omega\frac{k^2}{\mathrm{Re}_U} \\ q\Omega\frac{k_z}{k} - 2\Omega\frac{k^2}{\mathrm{Re}_B} \end{cases}$$

In the shearing case, the growth rate has the *same* maximum, $q\Omega$, at *all* (k_y,k_z). The maximum is reached at $k_x(0)=\pm k_y$. This growth is obtained for eigenmodes only at k_z=0.968 B_{0z}, k_y=0 [3].

Growth as function of $k_x(0)$, at q=3/2, k_y=5, k_z=5, Re_U=10^4, Re_B=500. Solid, shear waves; dashed, static waves; dotted, least stable eigenvalue at B_{0z}=1/20. Note the high k_x region where static modes can grow faster, caused by low Re_B.



Strong linear growth over short times at all scales:

What is the role of linear physics in MRI turbulence and dynamo?

References

[1] L. Trefethen and M. Embree, *Spectra and Pseudospectra,* the behavior of nonnormal matrices and operators (Princeton University Press, 2005).

[2] P. J. Schmid, Ann. Rev. Fluid Mech., **39**, 129 (2007)

[3] S. A. Balbus and J. F. Hawley, Astrophys. J. **400**, 610 (1992)

[4] B. M. Johnson, Astrophys. J. **660**, 1375 (2007)

[5] E. Kersalé et al., Astrophys. J. **602**, 892 (2004).

[6] J. C. B. Papaloizou and C. Terquem, Mon. Not. R. Astron. Soc. **287**, 771

[8] Y. M. Shtemler, M. Mond, and E. Liverts, Mon. Not. R. Astron. Soc. **421**, 700 (2012)

[9] B. F. Farrell and P. J. Ioannou, J. Fluid Mech. **708**, 149 (2012).

We extend thanks to Dr. Jeremy Goodman for enlightening discussion. This work was supported by Max Planck/Princeton Center for Plasma Physics and U.S. DOE (DE-AC02-09CH11466)

PRINCETON PLASMA PHYSICS LABORATORY