Power-law decay of the energy spectrum in linearized perturbed systems

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Examples of temporal evolution

- Amplified wave;
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- Weakly Amplified wave;
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- **Amplified wave**;
- **Weakly Amplified wave**;
- **Damped wave**;
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- Random Wave Collection;
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- Simultaneous Wave Collection;
Energy spectrum in fully developed turbulence

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  - $-5/3$ power-law for the energy spectrum over the inertial range;
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Energy spectrum and linear stability analysis

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- The perturbative evolution is ruled out by the initial-value problem associated to the Navier-Stokes linearized formulation.
Spectral analysis through initial-value problem

We determine the **exponent of the inertial range of arbitrary longitudinal and transversal perturbations** acting on a typical **shear flow**, i.e. the bluff-body wake:
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- Base flow approximated through 2D asymptotic Navier-Stokes expansions (Tordella & Belan, Phys. Fluids, 2003; Tordella & Scarsoglio, Phys. Lett. A, 2009) ⇒ (U(x, y; Re), V(x, y; Re));


Variety of the transient linear dynamics ⇒ Understand how the energy spectrum behaves and compare the decay exponent to that of the corresponding developed turbulent state:

The difference is large ⇒ quantitative measure of the nonlinear interaction in spectral terms;

The difference is small ⇒ higher degree of universality on the value of the exponent of the inertial range, not necessarily associated to the nonlinear interaction.
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Perturbation scheme

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- Linear three-dimensional perturbative equations in terms of velocity and vorticity (Criminale & Drazin, Stud. Appl. Math., 1990);
- Base flow parametric in $x$ and $Re \Rightarrow U(y; x_0, Re)$;
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Perturbative equations

- **Perturbative linearized system:**

\[
\frac{\partial^2 \hat{v}}{\partial y^2} - (k^2 - \alpha_i^2 + 2i\alpha_r\alpha_i) \hat{v} = \hat{\Gamma}
\]

\[
\frac{\partial \hat{\Gamma}}{\partial t} = (i\alpha_r - \alpha_i) \left( \frac{d^2 U}{dy^2} \hat{v} - U \hat{\Gamma} \right) + \frac{1}{Re} \left[ \frac{\partial^2 \hat{\Gamma}}{\partial y^2} - (k^2 - \alpha_i^2 + 2i\alpha_r\alpha_i) \hat{\Gamma} \right]
\]

\[
\frac{\partial \hat{\omega}_y}{\partial t} = -(i\alpha_r - \alpha_i) U \hat{\omega}_y - i\gamma \frac{dU}{dy} \hat{v} + \frac{1}{Re} \left[ \frac{\partial^2 \hat{\omega}_y}{\partial y^2} - (k^2 - \alpha_i^2 + 2i\alpha_r\alpha_i) \hat{\omega}_y \right]
\]

The transversal velocity and vorticity components are \( \hat{v} \) and \( \hat{\omega}_y \) respectively, \( \hat{\Gamma} \) is defined as \( \hat{\Gamma} = \partial_x \hat{\omega}_z - \partial_z \hat{\omega}_x \).

- **Initial conditions:**
  - \( \hat{\omega}_y(0, y) = 0 \);
  - \( \hat{v}(0, y) = e^{-y^2} \sin(y) \) or \( \hat{v}(0, y) = e^{-y^2} \cos(y) \);

- **Boundary conditions:** \((\hat{u}, \hat{v}, \hat{w}) \to 0\) as \( y \to \infty \).
Perturbation energy

**Kinetic energy density** $e$:

$$e(t; \alpha, \gamma) = \int_{-y_d}^{+y_d} (|\hat{u}|^2 + |\hat{v}|^2 + |\hat{w}|^2) dy$$

$$= \frac{1}{|\alpha^2 + \gamma^2|} \int_{-y_d}^{+y_d} (|\frac{\partial \hat{v}}{\partial y}|^2 + |\alpha^2 + \gamma^2||\hat{v}|^2 + |\hat{\omega}_y|^2) dy$$
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- **Amplification factor $G$:**

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- Reynolds number
Transient dynamics example

Fixed Reynolds number and wake configuration filed, the transient observed in long and short waves with different initial conditions is very diversified.
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The amplification factor $G$, obtained at $Re = 100, x_0 = 10$, with different initial condition and obliquity angle for a long (on the left) and a short (on the right) waves.
We have different temporal scales associated to the different perturbation wavelengths ⇒ A continuous instantaneous normalization can be used by defining as 

\[ t^* = \frac{t}{\tau_G}, \quad \tau_G = \frac{G(t)}{|dG(t)/dt|} \]

The amplification factor \( G \), obtained at \( Re = 100, x_0 = 10 \), with symmetric initial condition, \( \phi = 0 \) as a function of \( t \) (on the left) and of \( t^* \) (on the right).
The energy spectrum is computed at the asymptotic state, since it can widely vary during the transient;
Stop criterion

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- The time that perturbations take to get in their asymptotic condition is defined time such that:
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Results

High Reynolds number and intermediate wake configuration

\[ \text{Re}=100, \ x_0=10 \]

\[ \text{Energy spectrum in linearized systems} \]
Results

High Reynolds number and far wake configuration

\[ \text{Re}=100, \ x_0=50 \]

\[ G \]

\[ 10^0 \]

\[ k \]

\[ 10^0 \quad 10^1 \quad 10^2 \]

\[ \text{Sym, } \phi=0 \]
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Energy spectrum in linearized systems
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Middle Reynolds number and intermediate wake configuration

\[ \text{Re}=50, \ x_0=10 \]

Graph showing energy spectrum for different cases:
- Sym, \( \phi=0 \)
- Sym, \( \phi=45 \)
- Sym, \( \phi=90 \)
- Asym, \( \phi=0 \)
- Asym, \( \phi=45 \)
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The energy spectrum follows a \(-5/3\) power law.
Results

Low Reynolds number and intermediate wake configuration

Re=30, x_0=10

Energy spectrum in linearized systems
Asymmetric initial condition case

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\[ G(t^*) \]

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Asymptotics self similar state

Transient self similar state

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Concluding remarks

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Coming next ⇒ Temporal observation window of a large number of small 3D perturbations injected in a statistical way into the system.