The Nature of Turbulence

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Coexistence of a coherent vortex and turbulence in two dimensions

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Two-dimensional turbulence – thin liquid films, soap films, atmosphere.

$$\partial_t \omega + \boldsymbol{v}\nabla\omega = \phi + \nu\nabla^2\omega - \alpha\omega,$$

where ω is vorticity, $\omega = \nabla \times \boldsymbol{v}$, ϕ – pumping, $\phi = \nabla \times \boldsymbol{f}$, $\boldsymbol{f}$ – force, ν – viscosity and α – friction coefficients. We assume that the pumping force is correlated at a scale l and is random in time.

Two quadratic dissipationless integrals of motion – energy and enstrophy:

$$\int d^2r\ v^2, \qquad \int d^2r\ \omega^2.$$

Pumped turbulence – two cascades: enstrophy flows to small scales whereas energy flows to large scales from the pumping scale l, being dissipated by viscosity and friction, respectively (Kraichnan 1967, Leith 1968, Batchelor 1969).

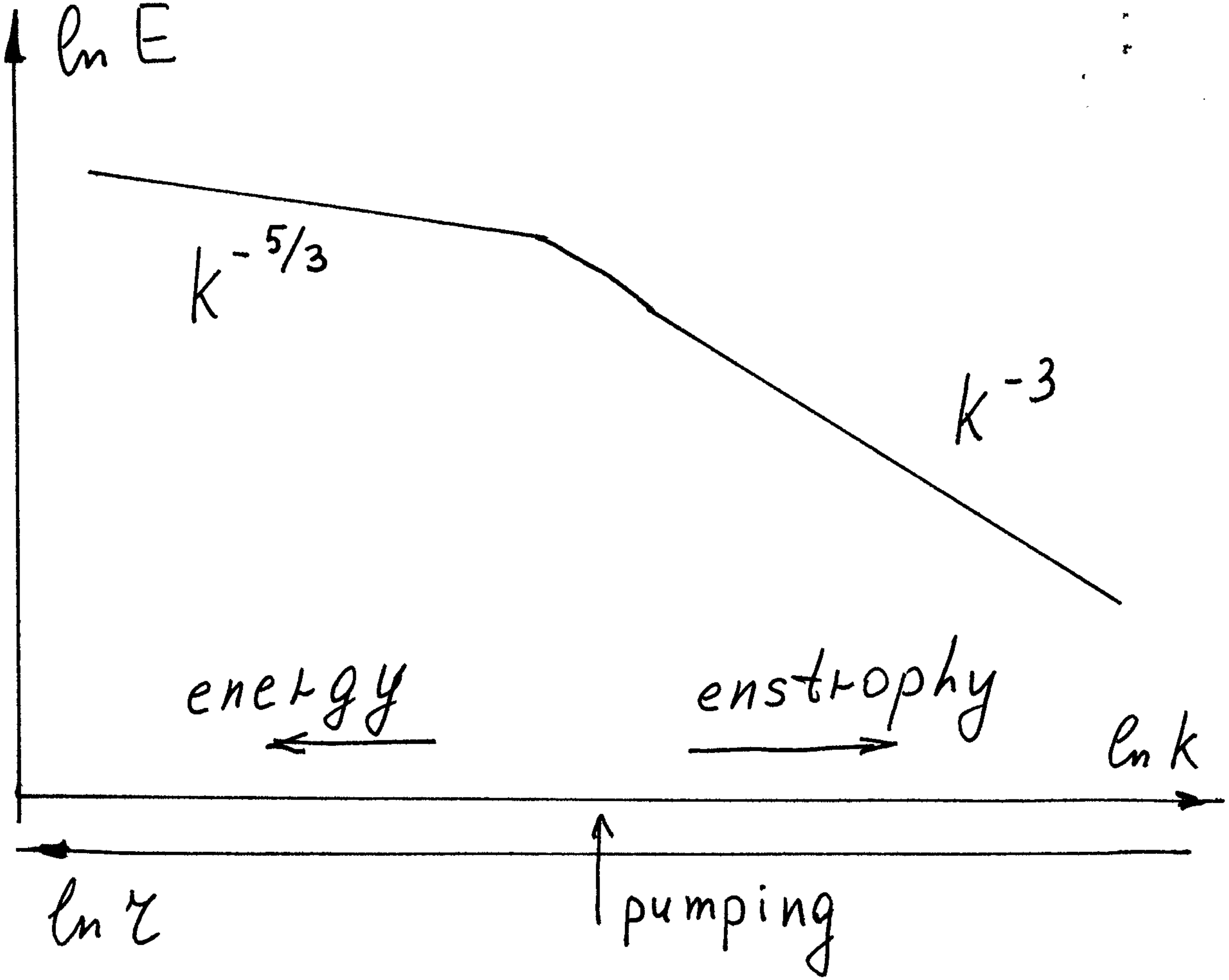
Constancy of the energy and enstrophy fluxes is expressed as follows

$$\langle (v_1 - v_2)\omega_1\omega_2 \rangle \propto r, \quad r \ll l;$$
$$\langle (v_1 - v_2)^3 \rangle = \epsilon r, \quad r \gg l.$$

Suggest the normal scaling $v_1 - v_2 \propto r$ in the direct cascade and $v_1 - v_2 \propto r^{1/3}$ in the inverse cascade. The spectrum

$$\langle v_1 v_2 \rangle = \int \frac{dk}{2\pi} e^{ikr} E(k),$$

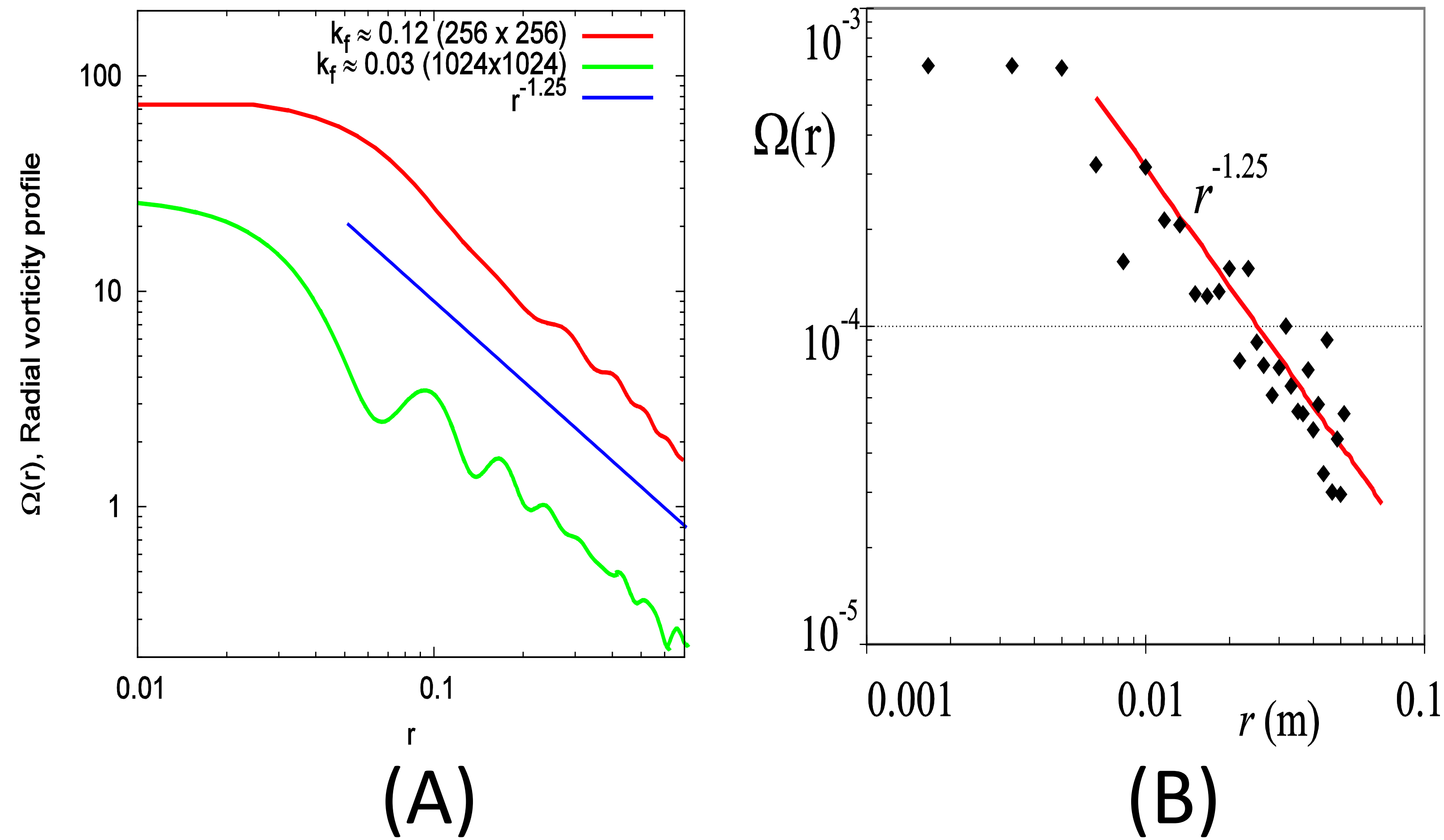
Then $E(k) \propto k^{-3}$ for the direct (enstrophy) cascade and $E(k) \propto k^{-5/3}$ for the inverse (energy) cascade. Direct cascade – logarithmic correlation functions of vorticity (Falkovich, Lebedev 1994). Inverse cascade – an absence of anomalous scaling (Paret and Tabeling 1998, Boffetta, Celani and Vergassola 2000).


ln E
k^{-5/3}
k^{-3}
energy
enstrophy
ln k
ln r
pumping

The inverse cascade is terminated by the friction at the scale $L_\alpha \sim \epsilon^{1/2}\alpha^{3/2}$ in an unbounded system. What will happen if the size box $L < L_\alpha$? Leads to the energy accumulation there and coherent structures. Experiment – single vortex (Shats, Xia, Punzmann and Falkovich 2007). Numerics – the vortex dipole (Chertkov, Connaughton, Kolokolov and Lebedev 2007).

Experiment – the vortex amplitude is determined by the bottom and wall friction. Numerics (frictionless) – the average velocity profile appears at a time $t \sim t_L = L^{2/3}\epsilon^{-1/3}$. We consider the case $\alpha = 0$, $t \gg t_L$. Then practically all the energy produced by pumping during the time t is accumulated in the coherent flow and its amplitude grows as $V \propto \sqrt{t}$.

Coherent structures – vortices with well-defined average velocity profile and relatively weak fluctuations on their background. Both, experiment and numerics, show that the vortices are isotropic and are characterized by power laws $V \propto r^{-1/4}$, $\Omega \propto r^{-5/4}$, where r is separation from the vortex center. Universality?



$k_f \approx 0.12$ (256 x 256)
$k_f \approx 0.03$ (1024x1024)
$r^{-1.25}$
Ω(r), Radial vorticity profile
100
10
1
0.01
0.1
r
(A)
10^{-3}
10^{-4}
10^{-5}
Ω(r)
$r^{-1.25}$
0.001
0.01
0.1
r (m)
(B)

The vorticity is a sum of the average component Ω and of the fluctuating components ω, $\langle\omega\rangle = 0$. Angular brackets mean averaging over times much larger than the characteristic turnover time and much smaller than the vortex evolution time t. The average and the fluctuating parts of the velocity are $\boldsymbol{V}$ and $\boldsymbol{v}$, respectively: $\Omega = \nabla \times \boldsymbol{V}$ and $\omega = \nabla \times \boldsymbol{v}$.

We are interested in scales larger than the pumping scale l, that is larger than the viscous scale. Then both, the viscosity and pumping terms are irrelevant. Separating the average and the fluctuating components, one finds the equations

$$\partial_t \Omega + \boldsymbol{V} \nabla \Omega + \nabla \langle \boldsymbol{v} \omega \rangle = 0,$$

$$\partial_t \omega + \boldsymbol{V} \nabla \omega + \boldsymbol{v} \nabla \Omega + \boldsymbol{v} \nabla \omega - \nabla \langle \boldsymbol{v} \omega \rangle = 0.$$

Experiment and numerics show that the vorticity profile Ω inside the vortex is highly isotropic. Then $\boldsymbol{V}$ has only polar component and $\boldsymbol{V}\nabla\Omega = 0$. Therefore

$$\partial_t \Omega + \nabla \langle v\omega \rangle = 0.$$

The last term in the equation reflects the fluctuation contribution to the average equation, supporting the average (coherent) velocity profile.

Next, we assume a scaling behavior $\Omega \propto r^{-1-\eta}$ where η is an exponent to be determined. Note that inside the vortex the separation into the average and fluctuating parts is equivalent to separation of angular harmonics: the zero harmonic corresponds to the average flow whereas higher harmonics correspond to its fluctuating part.

We expect that the fluctuation level is time-independent since it is determined by the energy flux ϵ, whereas the coherent part of the flow grows $\propto \sqrt{t}$. Thus, one can use the perturbation theory. Moreover ∂_t produces small factor. Adiabaticity. Thus, in the main approximation

$$(V/r)\partial_\varphi \omega + v_r \partial_r \Omega = 0.$$

The equation for Ω contains the object $\langle \omega v_r \rangle$ that can be expressed via the pair correlation function

$$\Phi(t, \varrho_1, \varrho_2, \varphi) = \langle v_r(t, r_1, \varphi_1) v_r(t, r_2, \varphi_2) \rangle \,,$$

where $\varphi = \varphi_1 - \varphi_2$, $\varrho = \ln(r/L)$. The pair correlation function, as well as higher correlation functions, is a subject of investigation.

In the main approximation

$$\left(\hat{\mathcal{N}}_1^{-1}\hat{\mathcal{K}}_1 - \hat{\mathcal{N}}_2^{-1}\hat{\mathcal{K}}_2\right)\Phi(r_1, r_2, \varphi) = 0\,,$$
$$\hat{\mathcal{N}} = (\partial_\varrho^2 + \partial_\varphi^2)r = r[(\partial_\varrho + 1)^2 + \partial_\varphi^2]\,,$$
$$\hat{\mathcal{K}} = V_0 \exp(-\eta\varrho)(\partial_\varrho^2 + 2\partial_\varrho + 2 + \partial_\varphi^2 - \eta^2)\,,$$

where we omitted the pumping and the time derivative that are small inside the big vortex. Here $V_0 \propto \sqrt{t}$ is an average velocity at the vortex periphery, at $r \sim L$.

Note that there are zero modes Z_m of the operator $\hat{\mathcal{K}}_m$ that are

$$Z_m = \exp(im\varphi + \beta_m \varrho) \propto r^{\beta_m},$$

with the exponents

$$\beta_m = \sqrt{m^2 + \eta^2 - 1} - 1\,.$$

Here $m = 1, 2, \ldots$ are numbers of angular harmonics. The sign is chosen to match to periphery.

We are looking for solutions of the equation for Φ that are analytic at close distances and angles. We begin with the following obvious solution

$$Z_m(r_1)Z_{-m}(r_2) + Z_m(r_2)Z_{-m}(r_1) \\ \propto r_1^{\beta_m} r_2^{\beta_m} \cos(m\varphi),$$

where Z_m are the above zero modes. Then one can construct a tower of more complicated constructions.

The next possible solution is

$$X_m(\boldsymbol{r}_1)Z_{-m}(\boldsymbol{r}_2) + X_m(\boldsymbol{r}_2)Z_{-m}(\boldsymbol{r}_1) + \\ Z_m(\boldsymbol{r}_1)X_{-m}(\boldsymbol{r}_2) + Z_m(\boldsymbol{r}_2)X_{-m}(\boldsymbol{r}_1).$$

Here the object X_m satisfies

$$X_m = \exp[im\varphi + (\beta_m + 1 + \eta)\varrho], \\ (\widehat{\mathcal{N}}_m)^{-1}\widehat{\mathcal{K}}_m X_m = A_m Z_m,$$

where A_m are real numbers.

All the terms in Φ do not contribute to $\langle \omega v_r \rangle$ since they are symmetric in φ. Thus we should find a correction $\delta\Phi$ to the pair correlation function related to the non-linear interaction of the fluctuations. The term is suppressed in comparison with the main contribution due to $V \gg v$.

Then we arrive at the equation

$$
\begin{aligned}
&\left(\widehat{\mathcal{N}}_1^{-1}\widehat{\mathcal{K}}_1 - \widehat{\mathcal{N}}_2^{-1}\widehat{\mathcal{K}}_2\right)\Phi(r_1, r_2, \varphi) \\
&= r_1^2\widehat{\mathcal{N}}_1^{-1}\left\langle \boldsymbol{v}(r_1, \varphi_1)\nabla\omega(r_1, \varphi_1) v_r(r_2, \varphi_2)\right\rangle \\
&-r_2^2\widehat{\mathcal{N}}_2^{-1}\left\langle v_r(r_1, \varphi_1)\boldsymbol{v}(r_2, \varphi_2)\nabla\omega(r_2, \varphi_2)\right\rangle .
\end{aligned}
$$

We are interested in the correction $\delta\Phi$ that is a forced solution of the equation related to the third-order correlation function.

The third-order velocity correlation function is defined as

$$F = \langle v_r(t, r_1, \varphi_1) v_r(t, r_2, \varphi_2) v_r(t, r_3, \varphi_3) \rangle .$$

The correlation function satisfies

$$\frac{\partial}{\partial \varphi_1} (\widehat{\mathcal{N}}_1)^{-1} \widehat{\mathcal{K}}_1 F + \cdots + \frac{\partial}{\partial \varphi_n} (\widehat{\mathcal{N}}_3)^{-1} \widehat{\mathcal{K}}_3 F = 0,$$

subscripts mean variables $\boldsymbol{r}_1$, $\boldsymbol{r}_2$, $\boldsymbol{r}_3$.

A simplest solution for the triple correlation function is

$$F \propto Z_m(\boldsymbol{r}_1) Z_k(\boldsymbol{r}_2) Z_{-m-k}(\boldsymbol{r}_3) + \text{permutations},$$

where permutations are produced over 1,2,3. However, it leads to a contribution to $\delta\Phi$ symmetric in φ that does not contribute to $\langle \omega v_r \rangle$.

The next solution in the tower can be constructed as follows

$$F = \alpha_m X_m(\boldsymbol{r}_1) Z_k(\boldsymbol{r}_2) Z_{-m-k}(\boldsymbol{r}_3) + \text{permutations}$$
$$+ \alpha_k Z_m(\boldsymbol{r}_1) X_k(\boldsymbol{r}_2) Z_{-m-k}(\boldsymbol{r}_3) + \text{permutations}$$
$$+ \alpha_{-k-m} Z_m(\boldsymbol{r}_1) Z_k(\boldsymbol{r}_2) X_{-m-k}(\boldsymbol{r}_3) + \text{permutations.}$$

The expression is a solution provided

$$\alpha_m m A_m + \alpha_k k A_k - \alpha_{-m-k}(m+k) A_{-m-k} = 0.$$

We should look for a solution with slowest decrease to the center. The expression with the lowest power of r corresponds to $m = k = 1$, in the case

$$\delta\Phi \propto r^{4\eta+\sqrt{3+\eta^2}-2}.$$

Then we find

$$\langle v_r \omega \rangle \propto r^{-1}\delta\Phi \propto r^{4\eta+\sqrt{3+\eta^2}-3}.$$

Substituting the result into the equation for Ω and accounting for $\Omega \propto r^{-1-\eta}$ one obtains the equation

$$5\eta + \sqrt{3+\eta^2} - 3 = 0.$$

The solution of the equation is $\eta = 1/4$. It corresponds both to experiment and numerics.

Note that the relation is time-independent since $\delta\Phi \propto t^{-1/2}$ and also $\partial_t\Omega \propto t^{-1/2}$. Probably, it is related to the non-linear mechanism of the energy transfer to large scales that is described by the third-order correlation function. The main contribution $\Phi \propto r^{-3/2}$. Equating the typical fluctuation and the average velocity, one gets $r_{\text{core}} \propto t^{-1}$.

Extensive numerics is needed to check our predictions.

Future developments: $3d$ effects; Coriolis forces; Passive scalar.

An interesting question concerns possibility/probability of appearing the coherent vortices in an unbounded system.