Effective and exact dimensional reduction (and finite temperature topological orders)

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Effective dimensional reduction-Symmetries

Given a *D*-dim theory:

A *d*-dim **GLS** is a group of transformations that leave the theory invariant such that the minimum non empty set of fields that are changed under the symmetry operation occupies a *d*-dim region

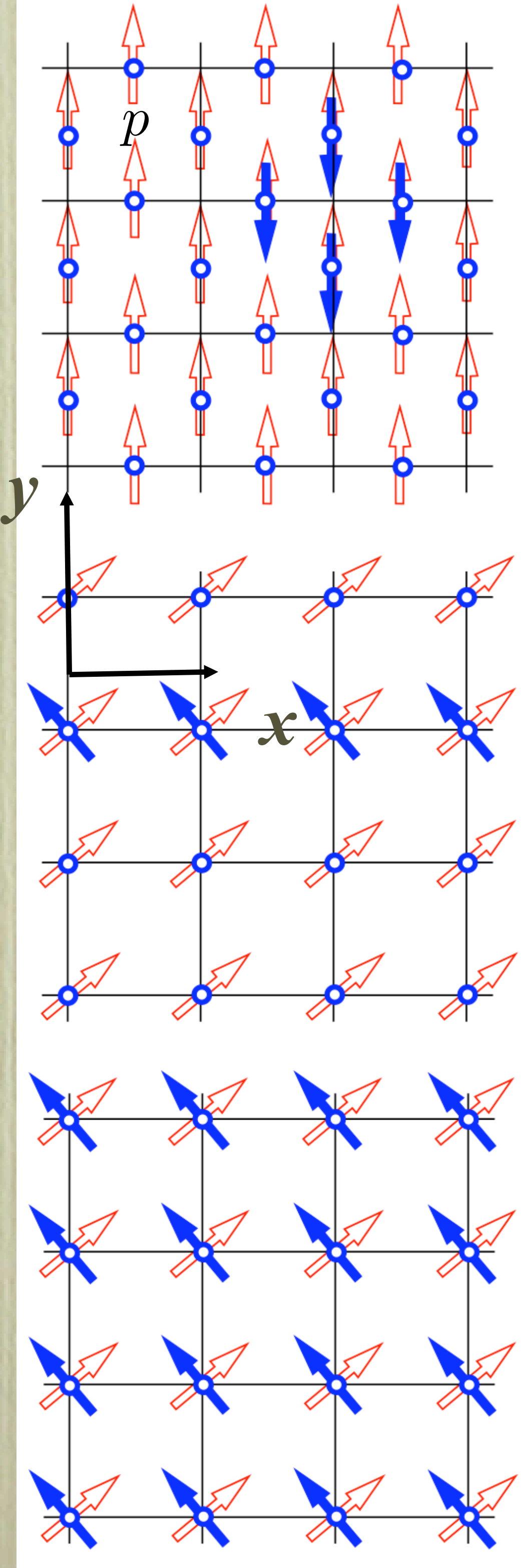
$$d \leq D$$

d=0 (Gauge)

d < *D* (Gauge-Like)

d=*D* (Global)

Group: $\mathcal{G}_d$



Gauge-Like-Symmetries $D=2$

$d=0$ (Ising Gauge Theory)

$$H=-K\sum_{p}\sigma^z_{ij}\sigma^z_{jk}\sigma^z_{kl}\sigma^z_{li} \qquad G_i=\prod_{s\in \mathrm{nn}}\sigma^x_{is}$$

$d=1$ (Orbital Compass Model)

$$H=-\sum_{i}[J_x\sigma^x_i\sigma^x_{i+\hat{e}_x}+J_z\sigma^z_i\sigma^z_{i+\hat{e}_y}]$$

$$O^x=\prod_{j\in C_x} i\sigma^x_j \qquad O^z=\prod_{j\in C_y} i\sigma^z_j$$

$d=D=2$ (XY model)

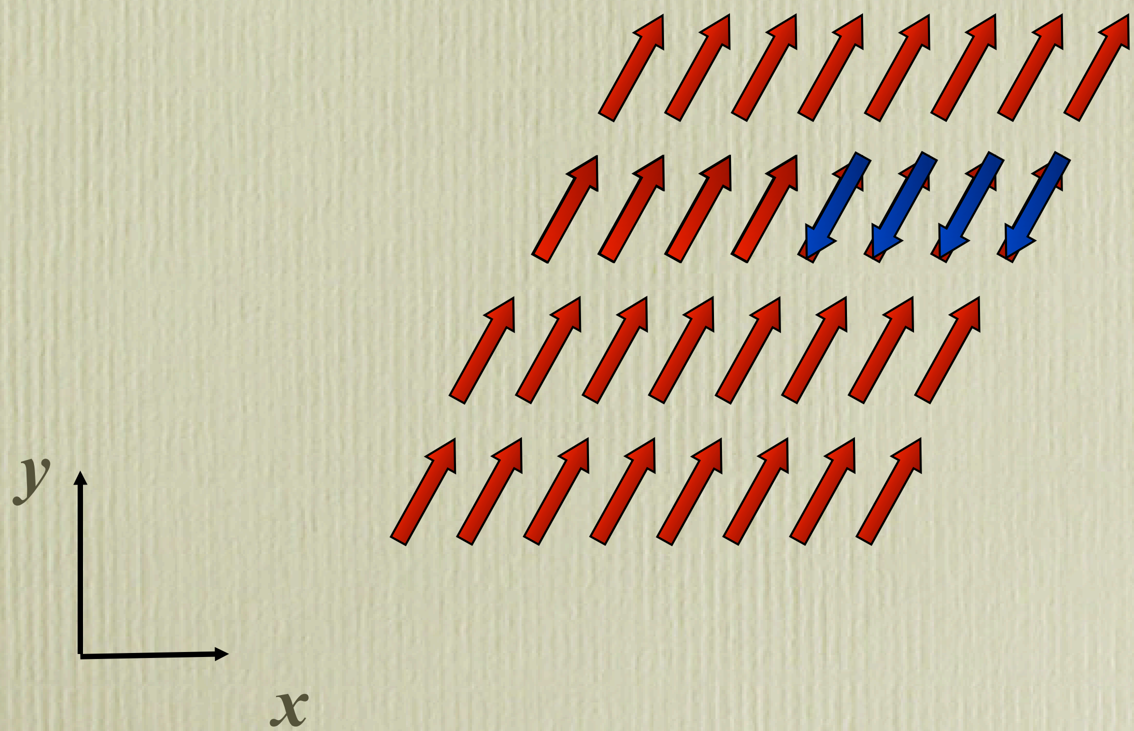
$$H=-J\sum_{\langle ij\rangle}[\sigma^x_i\sigma^x_j+\sigma^y_i\sigma^y_j]$$

$$U(\theta)=\prod_j \exp[-(i/2)\theta\sigma^z_j]$$

Intuitive Physical Picture

Orbital Compass Model

$$H = J \sum_{\vec{r}} (\sigma^x_{\vec{r}} \sigma^x_{\vec{r}+\hat{e}_x} + \sigma^y_{\vec{r}} \sigma^y_{\vec{r}+\hat{e}_y})$$



Associated
with the symmetry, a
soliton has
only a local energy
cost for constrained
motion
along one direction.

Symmetries

$$H = J\sum_{\vec{r}}(\sigma^x_{\vec{r}}\sigma^x_{\vec{r}+\hat{e}_x} + \sigma^y_{\vec{r}}\sigma^y_{\vec{r}+\hat{e}_y})$$

$$\vec{\phi}(\vec{x}) = (\phi_1(\vec{x}), \phi_2(\vec{x})) \qquad \vec{x} = (x_1, x_2)$$

$$W(\phi_\mu) = u(\sum_\mu |\phi_\mu|^2)^2 - \frac{1}{2}\sum_\mu m^2(|\phi_\mu|^2)$$

$$\mathcal{L} = \frac{1}{2}\sum_\mu |\partial_\mu\phi_\mu|^2 + \frac{1}{2}\sum_\mu |\partial_\tau\phi_\mu|^2 + W(\phi_\mu)$$

$$W(\phi_\mu) = u(\sum_\mu |\phi_\mu|^2)^2 - \frac{1}{2}\sum_\mu m^2(|\phi_\mu|^2)$$

$$\phi_\mu \rightarrow e^{i\psi_\mu(\{x_\nu\}_{\nu\neq\mu})}\phi_\mu$$

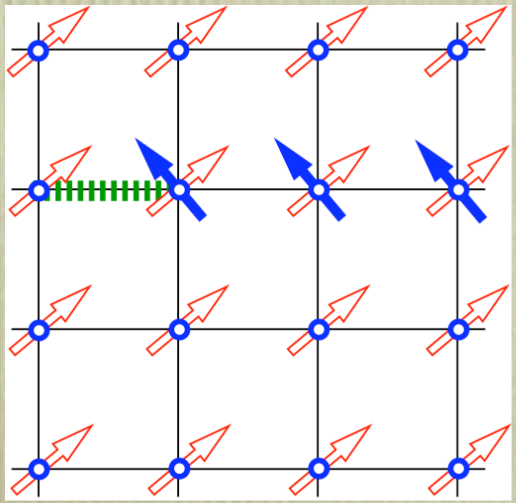
d-GLSs and Topological Phases

There is a connection between Topological Phases and the group generators of d-GLSs and its Topological defects

$d=1$ (D=2 Orbital Compass Model) C_x : closed path

$$O^x = e^{i\frac{\pi}{2}\sum_{j\in C_x}\sigma_j^x} = \mathcal{P}e^{i\oint_{C_x}\vec{A}\cdot\vec{ds}}$$

Symmetries are linking operators: $O^\mu|g_\alpha\rangle = |g_\beta\rangle$

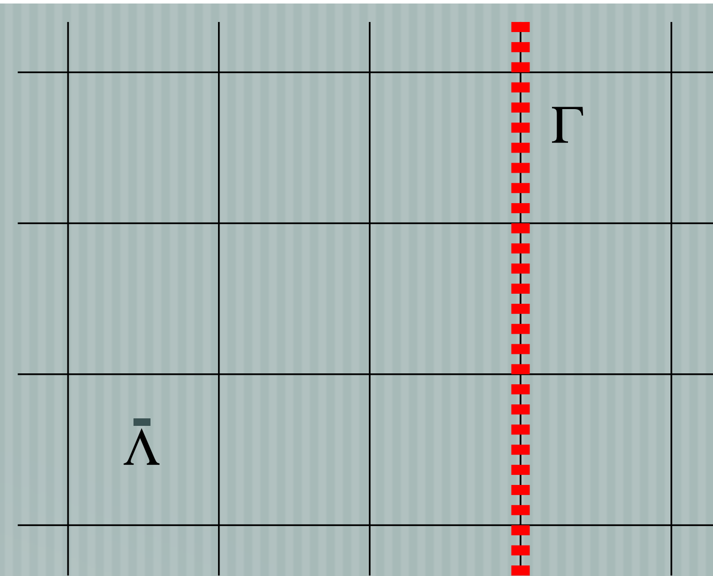


Topological defect: C_+ : open path

$$D^x = e^{i\frac{\pi}{2}\sum_{j\in C_+}\sigma_j^x} = \mathcal{P}e^{i\int_{C_+}\vec{A}\cdot\vec{ds}}$$

Defect-Antidefect pair creation

Dimensional reduction in classical systems:



$$\phi(\boldsymbol{x}) = \begin{cases} \phi_0(\boldsymbol{x}) & \text{if} \quad \boldsymbol{x} \in \Gamma \\ \psi(\boldsymbol{x}) & \text{if} \quad \boldsymbol{x} \in \bar{\Lambda} \end{cases}$$

$f[\phi] = f[\phi_0]$ localized observable

$$\langle f \rangle^D = \sum_{\{\psi\}} \sum_{\{\phi_0\}} f[\phi_0] \frac{e^{-\beta E[\phi_0,\psi]}}{\mathcal{Z}} = \sum_{\{\psi\}} \frac{z[\psi]}{\mathcal{Z}} \frac{\sum_{\{\phi_0\}} f(\phi_0) e^{-\beta E[\phi_0,\psi]}}{z[\psi]}$$

$$\langle f \rangle_l^d \equiv \mathsf{min}_\psi \, \langle f \rangle^d[\psi] = \langle f \rangle^d[\psi_{\mathsf{min}}], \qquad \langle f \rangle_u^d \equiv \mathsf{max}_\psi \, \langle f \rangle^d[\psi] = \langle f \rangle^d[\psi_{\mathsf{max}}]$$

$$\boxed{\langle f \rangle_l^d \le \langle f \rangle^D \le \langle f \rangle_u^d}$$

f may be any function.

$$\langle f \rangle_l^d \ : \ E_l[\phi_0, \psi_{\mathsf{min}}] \quad \text{and} \quad \langle f \rangle_u^d \ : \ E_u[\phi_0, \psi_{\mathsf{max}}]$$

Local lower dimensional theories

Lower dimensional bounds

D-dim system with Hamiltonian H_D and d-GLS group $\mathcal{G}_d$

The absolute value of the average of any quasi-local quantity f which is not invariant under d-GLS $\mathcal{G}_d$ is bounded from above by the absolute value of the mean of the same quantity when this quasi-local quantity is computed with a d-dim H_d that is globally invariant under $\mathcal{G}_d$ and preserves the range of the interactions in the original D-dim system

$\mathcal{C}_j$

η_i

$$\phi_i = \begin{cases} \eta_i \text{ if } i \in \mathcal{C}_j \\ \psi_i \text{ if } i \notin \mathcal{C}_j \end{cases}$$

ψ_i

$$|\langle f(\phi_i)\rangle_{H_D}| \leq |\langle f(\eta_i)\rangle_{H_d}|$$

Dimensional reduction

Phys. Rev. B 72, 045137 (2005); Annals of Phys. 327, 2491 (2012)

To Break or not to Break

Can we spontaneously break a d-GLS in a D-dim system ?

From the Generalized Elitzur's Theorem: (finite-range int.)

For non-$\mathcal{G}_d$-invariant quantities

- $d=0$ SSB is forbidden
- $d=1$ SSB is forbidden
- $d=2$ (continuous) SSB is forbidden

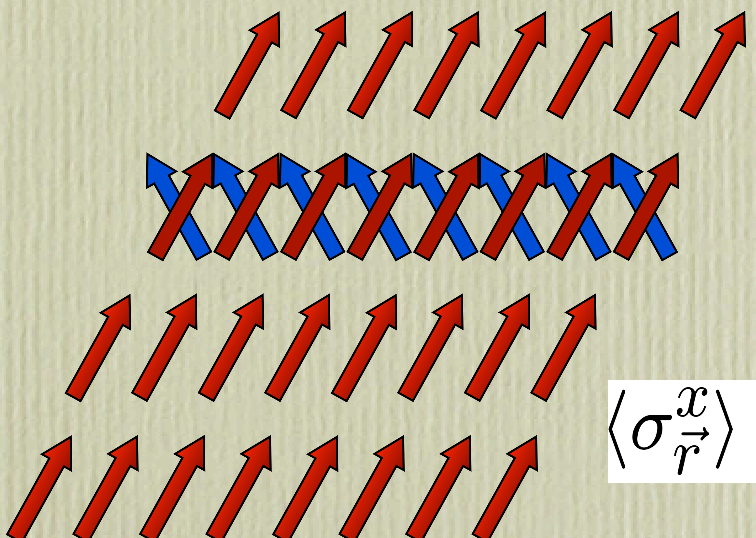
  $d=2$ (discrete) SSB may be broken
- $d=2$ (continuous with a gap) SSB is forbidden even at $T=0$

Transitions and crossovers are signaled by symmetry-invariant string/brane or Wilson-like loops

Example of application

Orbital Compass Model

$$H = J \sum_{\vec{r}} (\sigma^x_{\vec{r}} \sigma^x_{\vec{r}+\hat{e}_x} + \sigma^y_{\vec{r}} \sigma^y_{\vec{r}+\hat{e}_y})$$



Rotation by π
around the y-axis

$$\langle \sigma^x_{\vec{r}} \rangle = \langle \sigma^y_{\vec{r}} \rangle = \langle \sigma^z_{\vec{r}} \rangle = 0 \; for \; T > 0$$

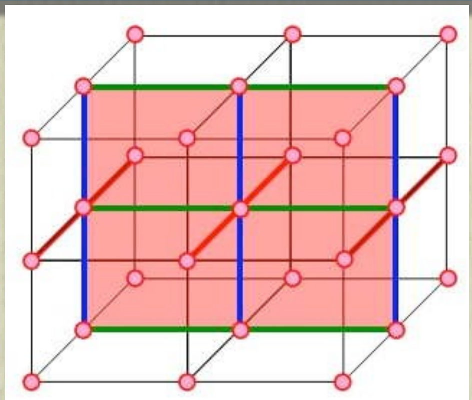
Lowest order allowed order paramete

Nematic

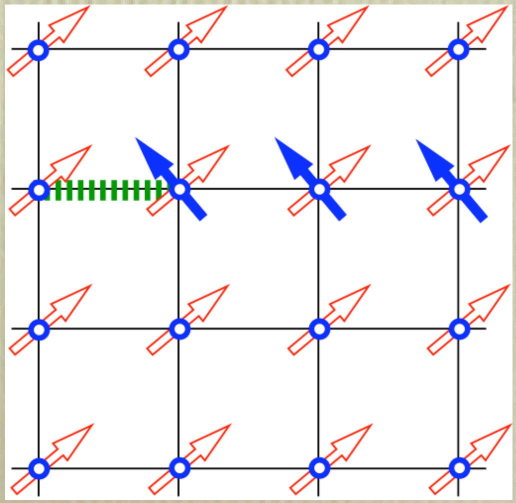
$$\langle \sigma^x_{\vec{r}} \sigma^x_{\vec{r}+\hat{e}_x} - \sigma^y_{\vec{r}} \sigma^y_{\vec{r}+\hat{e}_y} \rangle$$

Physical Consequences

For a D-dim system, $d < D$ GLSs lead to
dimensional reduction



- Conservation Laws within d-dim regions: Additional conserved currents
- Topological terms that appear in $d+1$ also appear in $D+1$
- Freely propagating d-dim topological defects



$d=1$ soliton in the $D=2$ orbital compass model
(Finite Energy cost)

PNAS 106, 16944 (2009), Annals of Physics 324, 977 (2009)
(on arXiv in 2006 and 2007: https://arxiv.org/pdf/cond-mat/0702377, https://arxiv.org/pdf/cond-mat/0605316.pdf)

Elasticity in space-time

Anisotropic derivatives (compass-like)

$$\mathcal{L} = \frac{1}{2} C_{\mu a \nu b} \partial^\mu u^a \partial^\nu u^b \qquad (u_\tau = 0)$$

$$J^a_{\mu_1 \ldots \mu_{D-1}} = \varepsilon_{\mu_1 \ldots \mu_{D-1} \nu \lambda} \partial^\nu \partial^\lambda u^a$$

$$\rho = \rho_0 [1 - \partial_i u^i]$$

$$\mathbf{j} = \rho_0 \partial_\tau \mathbf{u}$$

$$[\partial_\tau \partial_i u^i - \partial_i \partial_\tau u^i] = 0$$

$$\varepsilon_{\tau a i_1 \ldots i_{D-1}} J^a_{i_1 \ldots i_{D-1}} = 0$$

Glide constraint on dislocation motion

Elasticity in space-time

$$
\begin{aligned}
J_b^a &= \epsilon_{b\nu\lambda}\partial_\nu\partial_\lambda u_P^a \\
&= \delta_b^{(2)}(L) n_a \equiv n_a \int_L d\tau' \frac{d\bar{r}_b}{d\tau'} \delta^{(3)}(\vec{r}-\vec{\bar{r}}) \\
&= n_a \int_L d\tau' \dot{\bar{r}}_b(\tau') \delta^{(2)}(\vec{r}-\vec{\bar{r}}) \delta^{(1)}(\tau-\tau') \\
&= n_a \dot{\bar{x}}_b(\tau) \delta^{(2)}(\vec{r}-\vec{\bar{r}}(\tau))) \\
&\equiv n_a v_b \delta^{(2)}(\vec{x}_{1,2}-\vec{\bar{x}}_{1,2})
\end{aligned}
$$

$$J_y^x - J_x^y = (n_x v_y - n_y v_x)\, \delta^{(2)}(\vec{x}_{1,2}-\vec{\bar{x}}_{1,2}) = 0$$

$$\vec{v} \times \vec{n} = 0$$

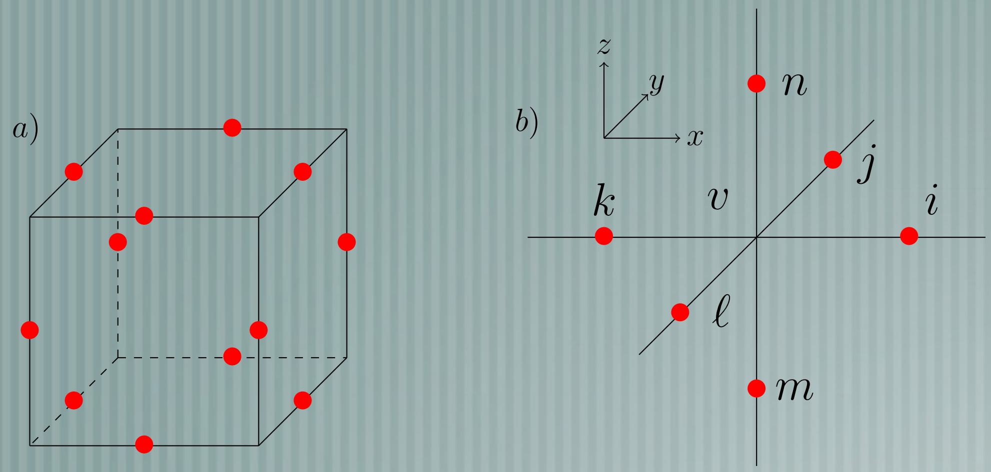
Glide constraint on dislocation motion

Annals of Physics 310, 181 (2004); Phil. Mag. 86, 2995 (2005); ...; a review of our approach: Phys. Rep. 683, 1 (2017)

More recent works by M. Pretko emphasizing relation to fractons

Exact dimensional reduction- dualities

The "X-cube model" and some of its generalizations are dual to classical Ising chains. (The same applies to the Toric Code.)



$$H = -a \sum_c A_c - b \sum_{\mu, v} B_v^{\mu},$$

$$A_c \equiv \prod_{n \in \partial c} \sigma_n^x,$$

$$B_v^x \equiv \sigma_j^z \sigma_n^z \sigma_\ell^z \sigma_m^z, \quad B_v^y \equiv \sigma_i^z \sigma_n^z \sigma_k^z \sigma_m^z, \quad B_v^z \equiv \sigma_i^z \sigma_j^z \sigma_k^z \sigma_\ell^z.$$

X-cube model = Ising chains

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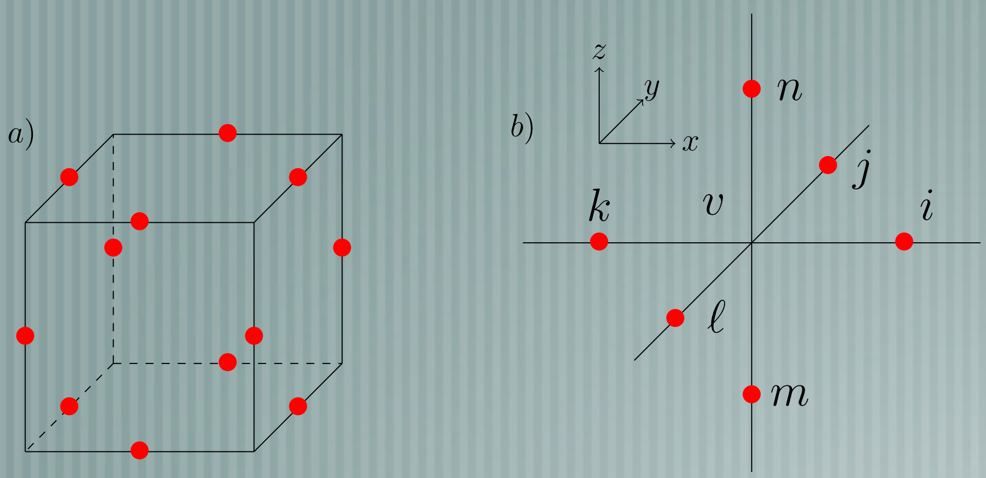

$$H = -a \sum_c A_c - b \sum_{\mu, v} B_v^\mu,$$

$$B_v^x B_v^y B_v^z = 1$$

A "bond algebraic duality": $A_c \to r_m, \quad 1 \leq m \leq L^3,$

$$B_v^x \to s_1^n, \quad B_v^y \to s_2^n, \quad B_v^z \to s_1^n s_2^n, \quad 1 \leq n \leq (L-1)^3$$

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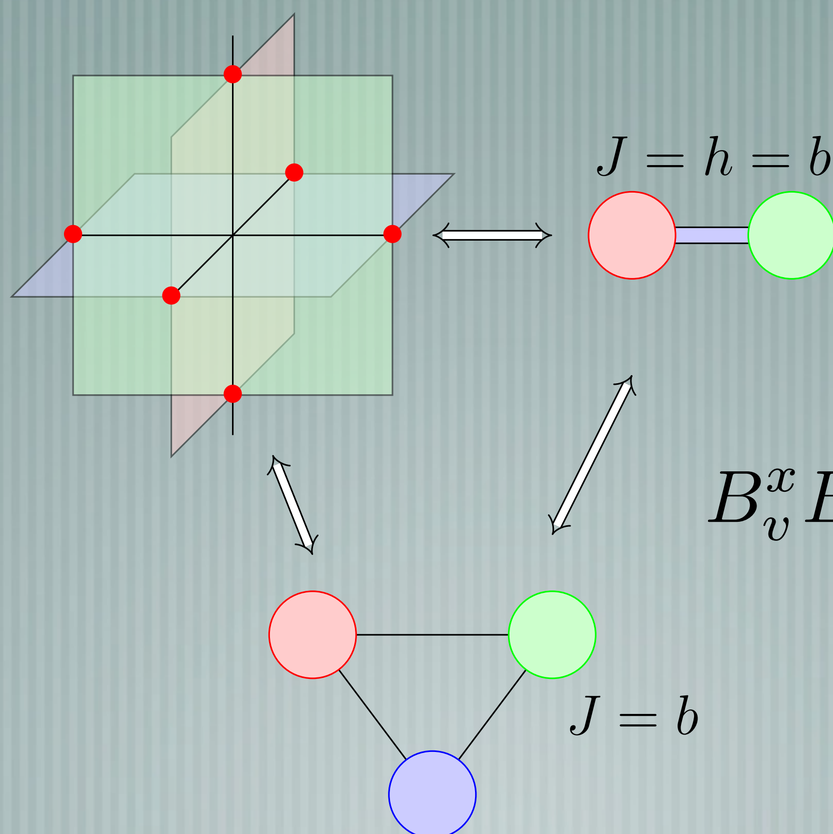

$$B_v^x B_v^y B_v^z = 1$$

$$B_v^x \equiv \sigma_j^z \sigma_n^z \sigma_\ell^z \sigma_m^z, \quad B_v^y \equiv \sigma_i^z \sigma_n^z \sigma_k^z \sigma_m^z, \quad B_v^z \equiv \sigma_i^z \sigma_j^z \sigma_k^z \sigma_\ell^z.$$

X-cube model = Ising chains

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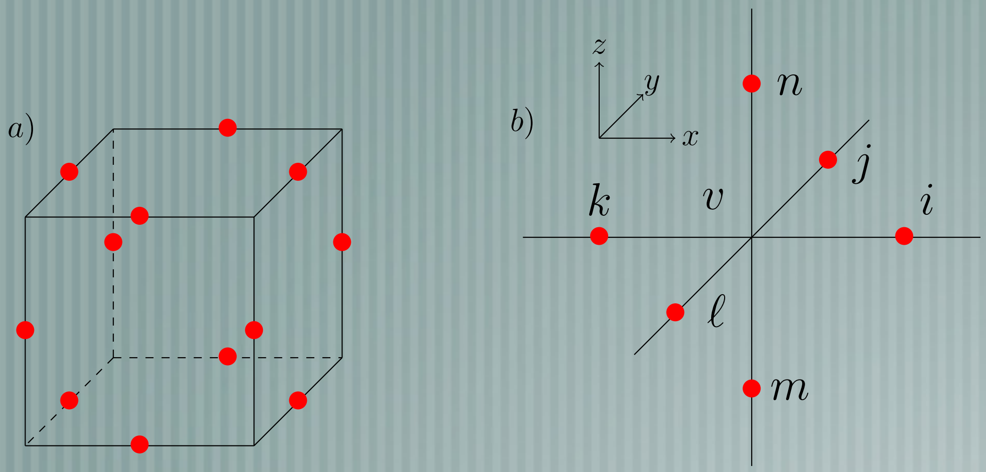


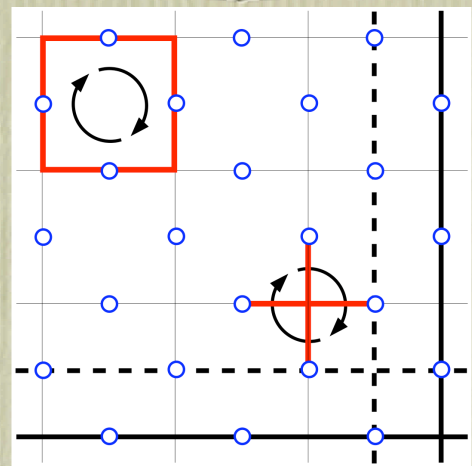
$$H = -a \sum_{c} A_c - b \sum_{\mu, v} B_v^{\mu},$$

$$\mathcal{Z}_{Open} = 2^{3L^3+6L^2+3L} (\cosh \beta a)^{L^3} [(\cosh \beta b)^3 + (\sinh \beta b)^3]^{(L-1)^3}$$

https://arxiv.org/pdf/1812.04561.pdf

Other dualities to Ising chains

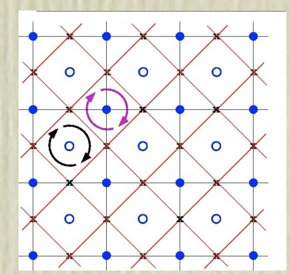
Kitaev's toric code model:



$$H_K = -\sum_s A_s - \sum_p B_p$$

$$A_s = \prod_{ij \in star(s)} \sigma^x_{ij}$$

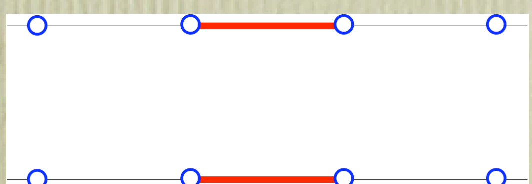
$$B_p = \prod_{ij \in boundary(p)} \sigma^z_{ij}$$



Duality mappings: Non-local

(Identical spectra)

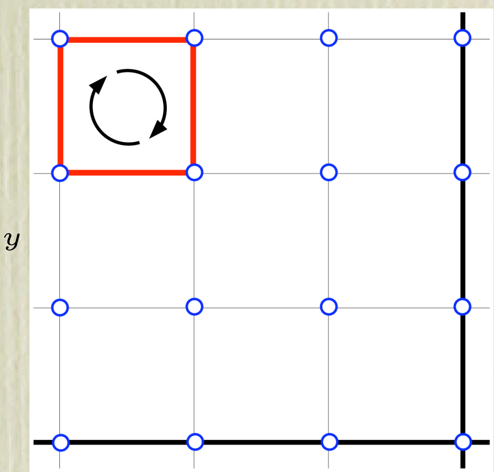
2 Ising chains:



$$H_I = -\sum_s \sigma^z_s \sigma^z_{s+1} - \sum_p \sigma^z_p \sigma^z_{p+1}$$

Wen's plaquette model:

$$H_W = -\sum_i \sigma^x_i \sigma^y_{i+\hat{e}_x} \sigma^x_{i+\hat{e}_x+\hat{e}_y} \sigma^y_{i+\hat{e}_y}$$



PNAS 106, 16944 (2009), Annals of Physics 324, 977 (2009)

(on arXiv in 2006 and 2007: https://arxiv.org/pdf/cond-mat/0702377, https://arxiv.org/pdf/cond-mat/0605316.pdf)

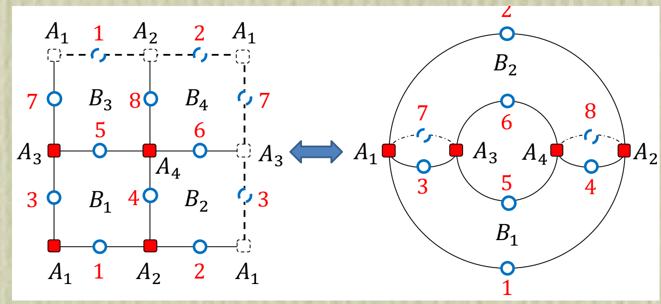
Other dimensional reductions

Model	D	d	Dual Model	Universality Class
2D Toric Code [6, 14]	2	1	Two decoupled 1D Ising chains	1D Ising
2D Honeycomb Toric Code [18, 31]	2	1	Two decoupled 1D Ising chains	1D Ising
Color Codes [18, 32]	2	1	Two decoupled 1D Ising chains	1D Ising
3D Toric Code [14, 33]	3	0, 1	Decoupled 1D Ising and 3D Ising models	3D Ising
X-Cube* [8, 34]	3	1, 2	Decoupled L 1D Ising and $L-1$ 1D Ising-gauge	1D Ising
Haah's Code** [12, 13, 30]	3	2	Two decoupled 1D Ising chains	1D Ising
4D Toric Code [7, 35]	4	2	Two decoupled 4D Ising models	4D Ising
Chamon's XXYYZZ [18, 27, 36]	3	1	Four decoupled 1D Ising chains	1D Ising

TABLE I: Universality classes of stabilizer code Hamiltonians. D is the spatial dimension of the lattice model. d is the dimension of the gauge-like symmetries. Dualities are defined as equivalence relations between partition functions: the 3DTC, for example, has a partition function proportional to the product of a 1D Ising and a 3D Ising partition function. While Chamon's XXYYZZ model is not an stabilizer code, it can also be shown by duality to exhibit dimensional reduction. Additionally, while all listed models above are constructed using Pauli operators, very similar results may be obtained for non-Pauli models, such as those with $\mathbb{Z}_p$ clock operators or $U(1)$ operators. *: While the X-Cube model's universality class does not depend on any choice of boundary conditions, the particular duality chosen holds for the case of cylindrical boundary conditions. **: The duality given below for Haah's code holds explicitly for those values of L for which the Ground State Degeneracy (GSD) is 4.

https://arxiv.org/pdf/1907.04180.pdf

Dependence of degeneracy of classical systems on topology



$$\mathcal{C}_+ : \quad \prod_s A_s^z = 1,$$

$$\mathcal{C}_- : \quad \prod_p B_p^z = 1.$$

$$n_{\text{g.s.}}^{\text{General Toric-Code}} = 4^g \times 2^{C_g^\Lambda - 2}.$$

$$\begin{cases} \text{Type I,} & L_x \neq L_y \text{ where at least} \\ & \text{one of } L_x \text{ or } L_y \text{ is odd} \\ \text{Type II,} & \text{otherwise.} \end{cases}$$

$$C_{g=1}^{\Lambda} = \begin{cases} 2, & \Lambda \text{ is a Type I lattice} \\ 2\min\{L_x, L_y\}, & \Lambda \text{ is a Type II lattice} \end{cases}$$

Phys. Rev. B 93, 205112 (2016)

Local order parameters

In a ferromagnet, a local expectation value is different for different orthogonal ground states (GSs)

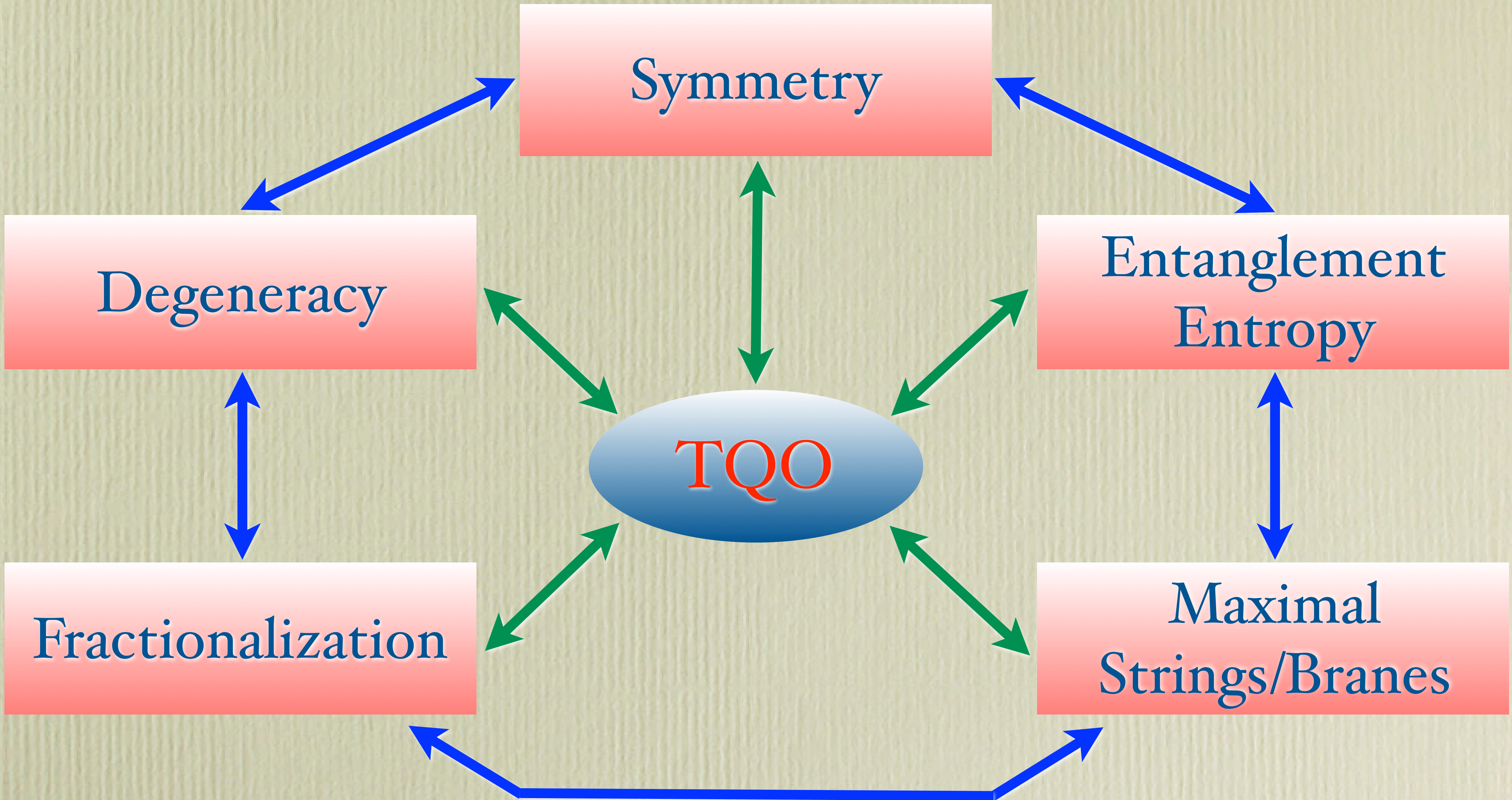
$$\langle g_\alpha|\hat{M}|g_\alpha\rangle \neq \langle g_\beta|\hat{M}|g_\beta\rangle \qquad T=0$$

Applying different boundary conditions can lead, at sufficiently low temperatures to spontaneous symmetry breaking

$$\langle \hat{M}\rangle_\alpha \neq \langle \hat{M}\rangle_\beta \qquad T\neq 0$$

Local Measurements can distinguish the GSs

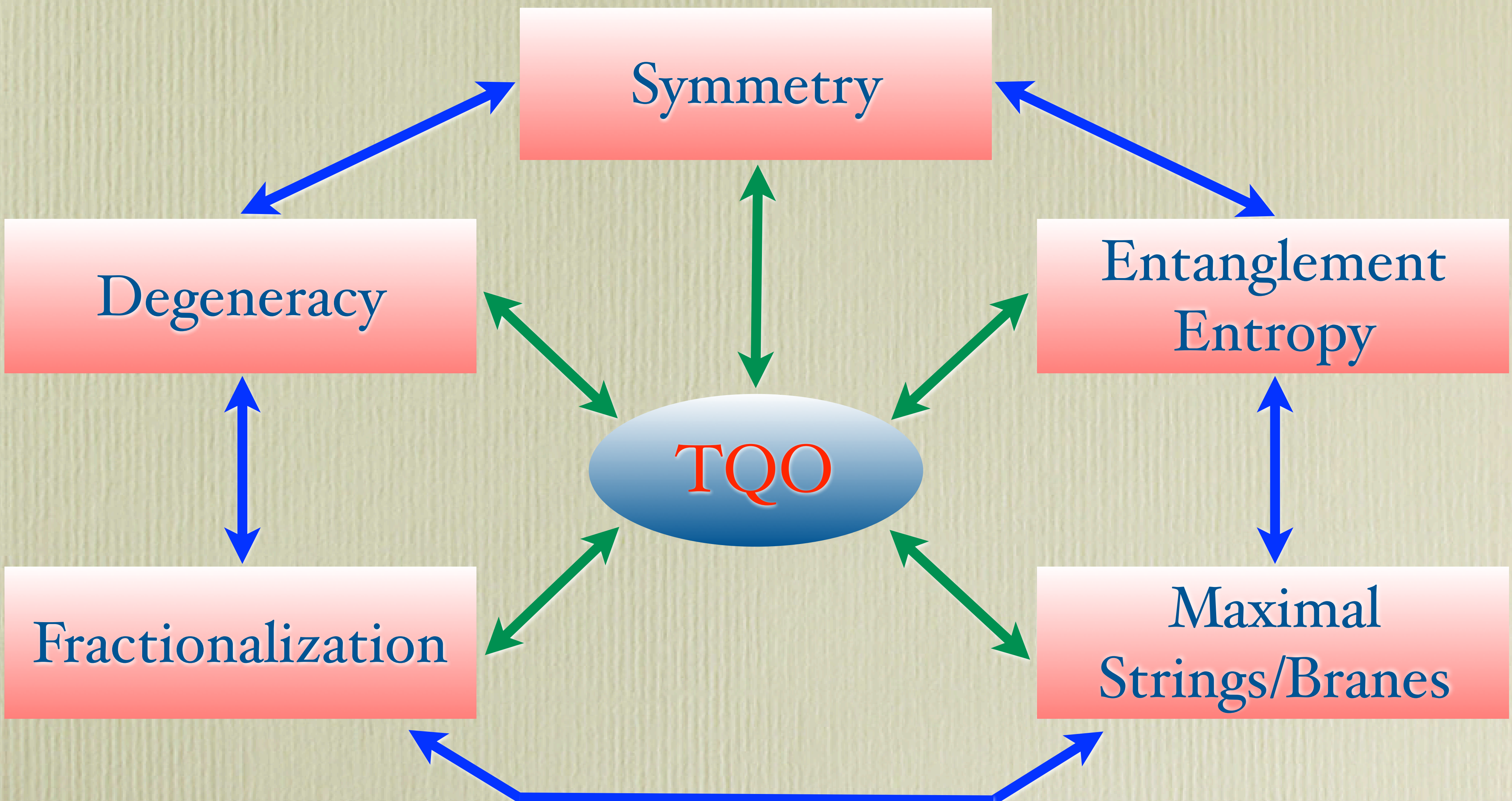
Concepts involved in TQO



In this "web", none of these features rigorously imply another. For instance, a system may have topological entanglement entropy yet not display topological order in the sense of stability. Similarly, string orders and fractionalization appear in systems displaying conventional symmetry breaking (e.g., the AKLT spin chain exhibits nematic order).

PNAS 106, 16944 (2009), Annals of Physics 324, 977 (2009)
(on arXiv in 2006 and 2007: https://arxiv.org/pdf/cond-mat/0702377, https://arxiv.org/pdf/cond-mat/0605316.pdf)

Concepts involved in TQO



It is important to establish
what is needed to display TQO

Topological Quantum Order

Colloquially, TQO is often very loosely referred to as order whose GS degeneracy depends on the surface topology of the manifold on which the physical system is embedded.

Order is evident only in non-local (topological)

Working definition: Robustness

Non-Distinguishability: Given a quasi-local operator $\hat{V}^m$

Kitaev:

$$\boxed{\langle g_\alpha|\hat{V}^m|g_\beta\rangle = c\,\delta_{\alpha\beta},\ \forall\,\alpha,\beta\in\mathcal{S}_0,}$$

Robustness:

To determine what is needed for TQO, we start by defining it. Given a set of N orthonormal ground states (GSs) $\{|g_\alpha\rangle\}_{\alpha=1,\dots,N}$ and a (uniform) gap to excited states, TQO exists iff for any bounded operator V with compact support (i.e. any *quasi-local operator*),

$$\langle g_\alpha|V|g_\beta\rangle = v\ \delta_{\alpha\beta} + c, \quad (1)$$

where v is a constant and c is a correction that it is either zero or vanishes exponentially in the thermodynamic limit. We will also examine a finite temperature ($T > 0$) extension for the diagonal elements of Eq. (1),

$$\langle V\rangle_\alpha \equiv \mathrm{tr}\ (\rho_\alpha V) = v + c \qquad \text{(independent of } \alpha\text{)}, \quad (2)$$

with $\rho_\alpha = \exp[-H_\alpha/(k_B T)]$ a density matrix corresponding to the Hamiltonian H endowed with an infinitesimal symmetry-breaking field favoring order in the state $|g_\alpha\rangle$. A system displays finite-T TQO if it satisfies both Eqs. (1), and (2).

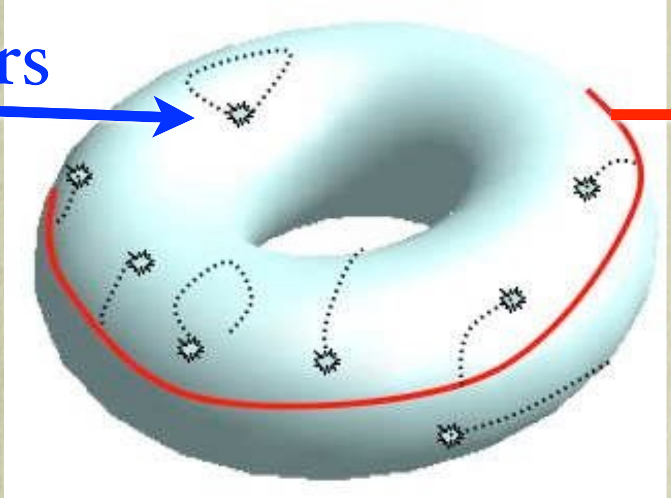
PNAS 106, 16944 (2009), Annals of Physics 324, 977 (2009)
(on arXiv in 2006 and 2007: https://arxiv.org/pdf/cond-mat/0702377, https://arxiv.org/pdf/cond-mat/0605316.pdf)

Error detection

Propagation of errors

V

(quasi-local)



$\hat{T}_\mu$

Logical operators (non-commuting braiding operations)

$[H, \hat{T}_\mu] = 0$

Protected subspace: $\hat{P}_0 = \sum_\alpha |g_\alpha\rangle\langle g_\alpha|$

As long as: $[\hat{P}_0 V \hat{P}_0, \hat{T}_\mu] = 0$ Causes no harm to $\hat{T}_\mu$

Non-distinguishability condition implies

$$[\hat{P}_0 V \hat{P}_0, \hat{T}_\mu] = 0$$

Physical Review B 77, 064302 (2008)

Theorem
Linking TQO and GLSs

Any physical system which displays T=0 TQO, and interactions of finite range and strength, in which all GSs (satisfying the non-distinguishability condition) can be linked by discrete d < 2 or continuous d <3 GLSs, has TQO at all temperatures.

(d-GLSs with $d < D$ can mandate the absence of SSB)

PNAS 106, 16944 (2009), Annals of Physics 324, 977 (2009)
(on arXiv in 2006 and 2007: https://arxiv.org/pdf/cond-mat/0702377, https://arxiv.org/pdf/cond-mat/0605316.pdf)

Stability and Protection of TQO

What happens when the *d*-GLSs $\mathcal{G}_d$
are not exact symmetries of the full *H* ?

(i.e., effect of perturbations)

As long as *d*-GLSs are **Emergent Symmetries**

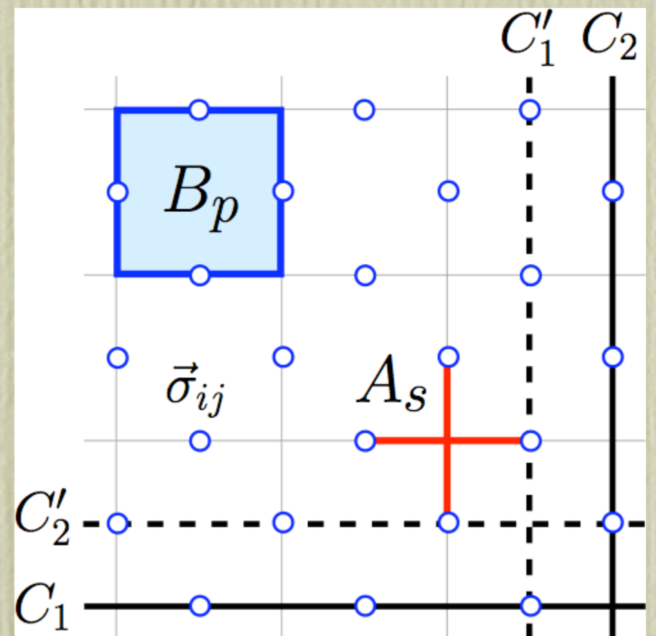
$\Downarrow$ **TQO is protected**

Case I: (Exact result) Continuous $d < 2$ emergent symmetry in a gapped system, TQO is protected

Case II: Numerous systems with exact discrete *d*-GLSs are adiabatically connected to TQO states where *d*-GLSs are emergent, i.e. TQO is protected

Thermal Fragility

In TQO systems, which have a gap,
does temperature preclude protection of information?



$$H = -\sum_s A_s - \sum_p B_p$$

$$A_s = \prod_{ij \in \mathsf{star}(s)} \sigma^x_{ij} \qquad B_p = \prod_{ij \in \mathsf{plaquette}(p)} \sigma^z_{ij}$$

$$X_{1,2} = \prod_{ij \in C'_{1,2}} \sigma^x_{ij} \qquad Z_{1,2} = \prod_{ij \in C_{1,2}} \sigma^z_{ij}$$

$$\{X_i, Z_i\} = 0 \ , \ [X_i, Z_j] = 0$$

Free-energy is analytic

No thermodynamic phase transition!

Thermal Fragility

For a finite size: By Symmetry

$$\langle Z_1 \rangle = \langle Z_2 \rangle = \langle X_1 \rangle = \langle X_2 \rangle = 0$$

Partition function (2 Ising chains):

$$Z = \mathrm{tr}\Big[\exp[-\beta(H - \sum_{i=1,2}(h_{x,i}X_i + h_{z,i}Z_i))]\Big]$$

$$= [(2\cosh\beta)^{N_s} + (2\sinh\beta)^{N_s}]^2 \cosh\beta h_1 \cosh\beta h_2$$

$$h_i = \sqrt{h_{x,i}^2 + h_{z,i}^2}$$

$$\langle Z_i \rangle = \lim_{h_{z,i}\to 0^+} \frac{\partial}{\partial(\beta h_{z,i})} \ln Z = \lim_{h_{z,i}\to 0^+} \frac{h_{z,i}}{h_i} \tanh(\beta h_i)$$

$$= 0$$

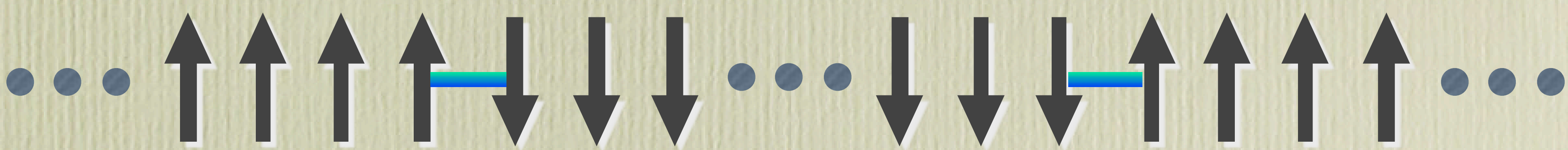
$$\langle X_i \rangle = \lim_{h_{x,i}\to 0^+} \frac{\partial}{\partial(\beta h_{x,i})} \ln Z = \lim_{h_{x,i}\to 0^+} \frac{h_{x,i}}{h_i} \tanh(\beta h_i)$$

Thermal Fragility: Energy-Entropy budget

From a Physics standpoint:

$$\langle Z_1 \rangle = \langle Z_2 \rangle = \langle X_1 \rangle = \langle X_2 \rangle = 0$$

Energy penalty for excitations: Independent of system size



Entropy: Log in the system size → Proliferation of defects

From loss of order: $$N_s \gtrsim \xi = \frac{1}{\ln \coth \beta J} \rightarrow_{\beta \rightarrow \infty} \frac{e^{2\beta J}}{2}$$

Crossover Temperature: $$\boxed{k_B T^* \sim \frac{2J}{\ln 2N_s}}$$

(similarly from energy-entropy considerations)

PNAS 106, 16944 (2009), https://arxiv.org/pdf/cond-mat/0605316.pdf

Thermal Fragility: Dynamical aspects

Time autocorrelation functions: Toric code with heat bath

$$G_{X_\mu}(t) \equiv \langle X_\mu(0) X_\mu(t) \rangle \sim e^{-(|t|/\tau)^\epsilon}$$

τ is independent of system size

1) Long-times: $|t| \gg \tau = \dfrac{\text{const.}}{1 - \tanh 2\beta J} \rightarrow_{\beta\rightarrow\infty} e^{\beta\Delta}$

$$\epsilon = 1$$

2) Intermediate-times: $\text{const.} \ll |t| \ll \tau = \dfrac{\text{const.}}{1 - \tanh 2\beta J}$

$$\epsilon = 1/2$$

Physical Review B 77, 064302 (2008)

Similar results for the autocorrelation functions apply to other stabilizers, fracton models, etc.

The X-cube model and the Haah code exhibit Ising chain type dynamics assuming a Glauber heat bath in the dual model

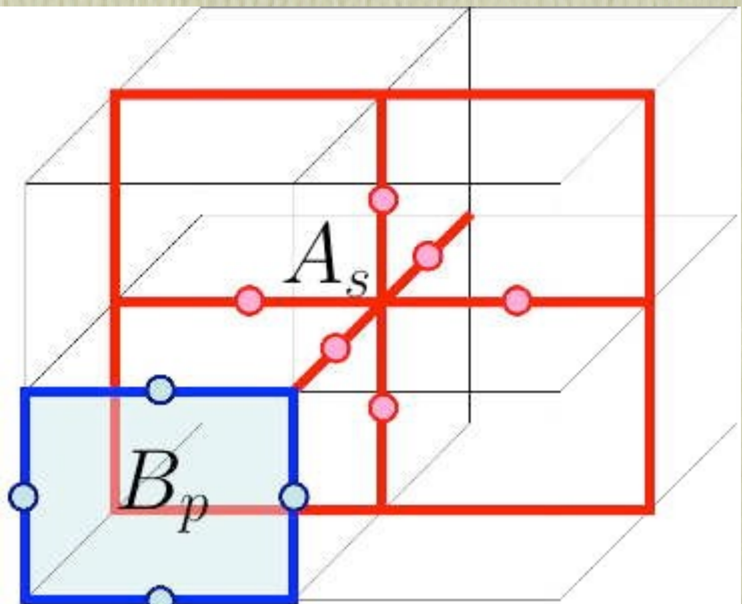
https://arxiv.org/pdf/1812.04561.pdf

https://arxiv.org/pdf/1907.04180.pdf

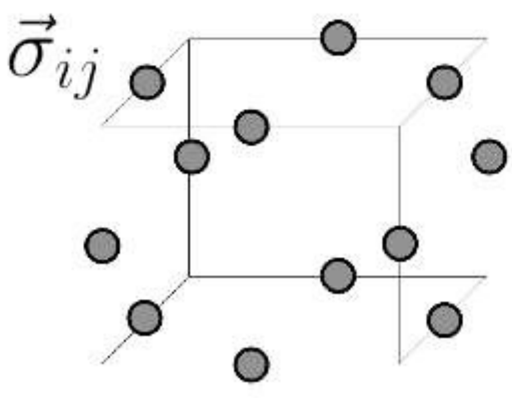
Thermal Fragility and Phase Transitions

What is the relation between the existence of a phase transition and TQO?

(Phase transitions are signaled by non-analyticities in the Free Energy)



Kitaev's toric in $3D$



Cube

It has a thermodynamic phase transition:

$$\mathcal{Z}_{3D} = \mathcal{Z}_{3D \text{ Ising gauge}} \times \mathcal{Z}_{1D \text{ Ising}}$$

$$(\beta_c = 0.761423)$$

It displays TQO

However, e.g.

$$\langle Z_{C_\mu} \rangle = \langle \prod_{(ij)\in C_\mu} \sigma^z_{ij} \rangle = 0$$

Loops around Toric cycles

Physical Review B 77, 064302 (2008)

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