Fermionic topological quantum states as tensor networks

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Joint work with Carolin Wille and Oliver Buerschaper

Symmetry, topology, and quantum phases of matter: From tensor networks to physical realizations, KITP program, December 2016
What this talk is about?

- Topological and symmetry protected quantum phases of matter

- Can methods like tensor networks be useful in modelling these phases?

Classification of phases in 1D

Chen, Gu, Wen, Phys Rev B 83, 035107 (2011)
Schuch, Perez-Garcia, Cirac, Phys Rev B 84, 165139 (2011)

RVB spin liquids as tensor networks


Shadows of anyons


- New twist: Put emphasis on quantum states, not so much Hamiltonians, which are reinserted in the picture by means of parent Hamiltonians
What this talk is about?

- Work on phases of matter using tensor networks exclusively on bosons/spins
- How about systems having a fermionic component?

An attempt:

- Area laws for entanglement entropies and tensor networks
- Topological order in PEPS
- MPO-injective PEPS
- Fermionic PEPS
Area laws for the entanglement entropy and tensor network states
• **Area law** for the entanglement entropy $S(\rho_A)$:

$$S(\rho_A) = O(|\partial A|)$$

• Scale like boundary area, not volume: Much less entangled than possible!

Eisert, Cramer, Plenio, Rev Mod Phys 82, 277 (2010)
Hastings, JSTAT, P08024 (2007)
**Area laws for entanglement entropies**

- **Theorem:** Area laws hold true for
  1. arbitrary gapped models in 1D
  2. free bosonic and fermionic gapped Hamiltonians in any D
  3. Stabiliser Hamiltonians

- **Topological entanglement entropy** $\gamma$

  $$S(\rho_A) = \alpha |\partial A| - \gamma + O(|\partial A|^{-\beta})$$


Eisert, Cramer, Plenio, Rev Mod Phys 82, 277 (2010)
Hastings, JSTAT, P08024 (2007)
Area laws for entanglement entropies

- **Area law** for the entanglement entropy $S(\mathcal{A})$ scales like the boundary area, not the volume:
  \[ S(\mathcal{A}) = O(|\partial\mathcal{A}|) \]
  
  Much less entangled than possible!

- **Entanglement** captures “essential degrees of freedom, hugely removes redundancy”

- Can this be used to largely “**parametrise**” states?
Projected entangled pair states

- Equip lattice system with local tensor structure
- Projected entangled pair states (PEPS) in 2D

Verstraete, Cirac, cond-mat/0407066
Martin-Delgado, cond-mat/9610196
Projected entangled pair states

- Equip lattice system with **local tensor structure**
- Projected entangled pair states (PEPS) in 2D

Verstraete, Cirac, cond-mat/0407066
Martin-Delgado, cond-mat/9610196
Virtual indices of "bond dimension" $D$

Drastic reduction of parameters, down to $O(n^2 d D^4)$

In TI, a single tensor captures Hamiltonian and all global state properties

Verstraete, Cirac, cond-mat/0407066
Martin-Delgado, cond-mat/9610196
Matrix product states

- Matrix product states (MPS)

- At basis of powerful DMRG (density-matrix renormalisation group)

  White, Phys Rev Lett 69, 2863 (1992)

**Theorem:** All states that satisfy area laws (for Renyi entropies $\alpha < 1$) have efficient approximation in $\| \cdot \|_1$-norm ($\text{poly}(n, 1/\epsilon)$)

Eisert, Cramer, Plenio, Rev Mod Phys 82, 277 (2010)
Two surprising observations

Similar intuition holds true, but...

- **Theorem**: There are states that satisfy all (Renyi entropy) area laws, yet cannot be efficiently approximated by any tensor network state

Good reasons to believe that they capture low energy physics
Two surprising observations

- Another cute twist

  - **Theorem:** PEPS contraction is \#P-complete
    

  - **Theorem:** PEPS that approximate gapped ground states well [in the sense that they are $\|\cdot\|_1$ norm close and have uniformly gapped parent], can be contracted in quasi-polynomial time
    
    Schwarz, Buerschaper, Eisert, arXiv:1606.06301

- Cannot efficiently compute expectation values in worst case complexity!
Numerical studies

- Finite PEPS and iPEPS: Excellent numerical performance

![Graphs showing numerical performance](image)

Shustry-Sutherland model
Corboz, arXiv:1605.03006

Spin-3/2 AKLT spin liquids
Lavoie at al, Nature Phys 6, 850 (2010)

Fermionic tensor networks
Corboz, Evenbly, Verstraete, Vidal, Phys Rev A 81, 010303 (2010)

Ground state energies in the t-J model
Corboz, arXiv:1605.03006

Orus, Ann Phys 349, 17 (2014)
Topological order in PEPS
Definition of topological order

- **Degeneracy** of the Hamiltonian constant and depends on topology
- Excitations behave like quasi-particles having anyonic statistics
- All GS are **locally indistinguishable** (no local order parameter)
- To map between them, one needs a **non-local operator**

How can it be captured in PEPS governed by single tensor?
Parent Hamiltonian

**Theorem:** All MPS and PEPS have frustration-free parent Hamiltonians

\[
H = \sum_j h_j , h_j |\psi\rangle = 0
\]

**Injective PEPS:** PEPS projection has left inverse

- Any action achievable on the virtual indices by acting on the physical spins

- But they are unique ground states of their parents
Toric code

- Capture systems like **toric code**
  - Star operators $\prod_{j \in +} X_j$
    (flux at plaquette)
  - Plaquette operators $\prod_{j \in \square} Z_j$
    (charge at vertex)
  - Frustration-free parent Hamiltonian (stars and plaquettes act trivially on GS)
    \[
    H = -J \sum_k \left[ \prod_{j \in \square_k} Z_j + \prod_{j \in +_k} X_j \right]
    \]

Kitaev, quant-ph/9707021
Toric code

- Capture systems like toric code

- Define string operators

Ground state formed by closed loop configurations

- Shows $\mathbb{Z}_2$-topological order

- Excitations behave like quasi-particles with anyonic statistics
  ($e$ - anyons on vertices, $m$ - anyons on plaquettes)

Kitaev, quant-ph/9707021
Let $G$ be any finite group, e.g., $G = \mathbb{Z}_2 = \{1, Z\}$.
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- How about loops that are non-contractible?
They can be arbitrarily deformed, but do not vanish.

- Gives new ground states of parent Hamiltonian
They can be arbitrarily deformed, but do not vanish

Gives new ground states of parent Hamiltonian
Topology in PEPS

- Open strings can be deformed, except from end points (quasi-particles)
Open strings can be deformed, except from end points (quasi-particles)
G-injective and G-isometric PEPS

- **G-injective PEPS**: Symmetry group $G$ is acting on virtual indices and PEPS tensors are left-invariant on the $G$-invariant subspace
- **G-isometric PEPS**: All PEPS tensors are isometries

- It is possible to unitarily transform between any two states in ground space by acting on two stripes wrapping around the torus
- ..., the states in the GS cannot be distinguished by local operations
- ... the *entanglement entropy* of any topologically trivial region is

$$S(\rho_A) = \log |G| |\partial A| - \log |G|$$

- Here $-\log |G|$ is the topological correction to the area law

We recover topological order

- **Degeneracy** of the Hamiltonian constant and depends on **topology**
- All GS are **locally indistinguishable** (no local order parameter)
- To map between them, you need a **non-local operator**
- Excitations behave like quasi-particles with **anyonic statistics**
A complete picture?

- Good enough to capture **toric code, quantum double models** etc
  

- Take $G = S_3$, suitable for universal **topological quantum computation**

- Not capturing **string net models**
  

- Can a complete understanding of topological order be achieved in terms of PEPS?
MPO-injective PEPS
Beyond G-injective PEPS

- Virtual symmetries

- $G$-symmetry
- Matrices
- Groups

- MPO-symmetry
- Matrix-product operator
- Twisted groups and more

$D = 1$

$D > 1$
Beyond G-injective PEPS
Beyond G-injective PEPS
Beyond G-injective PEPS

- Stable under concatenation

Stable under concatenation

\[ \text{Stable under concatenation} \]

Area laws and tensor networks
Topological order in PEPS
MPO-injective PEPS
Fermionic PEPS
Beyond G-injective PEPS

- Stable under concatenation
Axioms of MPO-injectivity

- MPO symmetry
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- MPO projector
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- Stability under concatenation
Axioms of MPO-injectivity

- MPO symmetry
- MPO projector
- MPO injectivity
- Stability under concatenation

Can compute:

- Topological correction to area law
  \[ S(\rho_A) = c|\partial A| - \gamma \]
- Ground state space
Axioms of MPO-injectivity

- MPO symmetry
- MPO projector
- MPO injectivity
- Stability under concatenation

Can compute:

- Topological correction to area law
  \[ S(\rho_A) = c|\partial A| - \gamma \]
- Ground state space
- Anyonic statistics: \( S \) and \( T \) matrices
Axioms of MPO-injectivity

- MPO symmetry
- MPO projector
- MPO injectivity
- Stability under concatenation

Can compute:

- Topological correction to area law
  \[ S(\rho_A) = c|\partial A| - \gamma \]
- Ground state space
- Anyonic statistics: \( S \) and \( T \) matrices
- Captures Levin-Wen string net models

Towards tensor networks for fermionic systems
Axioms of MPO-injectivity

- Tensors with physical fermions
  - Book-keeping of the order (manual)

Wille, Buerschaper, Eisert, arXiv:1609.02574
**Axioms of MPO-injectivity**

- Tensors with physical fermions
  - Book-keeping of the order (manual)
- Add virtual fermions
  - Book-keeping of the order (in-built)
    - Fermionic entangled pairs
    - Grassmann numbers

\[ \#(\bullet \bullet \bullet \bullet \bullet) \text{ even} \]

Wille, Buerschaper, Eisert, arXiv:1609.02574
Williamson, Bultinck, Haegeman, Verstraete, arXiv:1609.02897
Fermionic MPOs?

- Fermionic MPOs
- Axioms take analogous form
- Graded algebraic structure
- Axioms fulfillable?

Wille, Buerschaper, Eisert, arXiv:1609.02574
Fermionic toric code

- **Edges:** Spin 1/2
- **Vertices:** Fermions

\[ H = \sum_v Q_v + \sum_p Q_p \]

\[ Q_v = \frac{1}{2} (1 + \prod_{i \in v} \sigma_i^Z) F_v \]

\[ Q_p = \frac{1}{2} (1 + \prod_{i \in p} \sigma_i^X) F_p \]

Fermionic MPO-injectivity

- Dual lattice
- Grassmann numbers
Fermionic MPO-injectivity

- Dual lattice
- Grassmann numbers

\[ A = \sum A_{p\bar{f}_1 f_2 f_3}^{p_1 p_2 p_3 v_1 v_2 v_3} \theta^p \theta^f_{\bar{f}_1} \bar{\theta}^f_{f_2} \bar{\theta}^f_{f_3} |p_1, p_2, p_3\rangle \langle v_1, v_2, v_3| \]

Wille, Buerschaper, Eisert, arXiv:1609.02574
Fermionic MPO-injectivity

- Dual lattice
- Grassmann numbers
- Virtual symmetries with branching structure*

*Edges of PEPS tensor are oriented such that no cyclic orientation arises

Wille, Buerschaper, Eisert, arXiv:1609.02574
Fermionic MPO-injectivity

- Dual lattice
- Grassmann numbers
- Virtual symmetries with branching structure

**Theorem:** Construction satisfies axioms

- Can compute properties, e.g., ground state degeneracy
- Interesting physical models?

$T_+$ at edges parallel to MPO direction

$T_-$ at edges anti-parallel to MPO direction

Purely bosonic $Y$ to ensure concatenation

Wille, Buerschaper, Eisert, arXiv:1609.02574
Twisted fermionic double models

- Twisted fermionic quantum doubles (instances of fermionic string nets)

  - Graded group cohomology: $\text{Triple}(G, s, \omega)$
    - **Group** $G$, defining bosonic degrees of freedom
    - **2-cocycle** $\mathcal{H}^2(G, \mathbb{Z}_2)$, governing coupling
      $$s(a, b) + s(ab, c) + s(a, bc) + s(b, c) = 0$$
    - **Graded 3-cocycle** $\mathcal{H}^3_f(G, U(1), s)$
      $$\omega(a, b, c)\omega(a, bc, d)\omega(b, c, d) = (-1)^{s(a, b)s(c, d)}\omega(ab, c, d)\omega(a, b, cd)$$

- Can all be shown to satisfy framework (tedious)

- **Fermionic toric code**: Simplest triple
  - $G = \mathbb{Z}_2$
  - $s(1, 1) = 1, s = 0$ otherwise

Wille, Buerschaper, Eisert, arXiv:1609.02574
Bultinck, Williamson, Haegeman, Verstraete, arXiv:1610.07849
- Consistent framework of topological PEPS for fermionic systems

Capture them as fermionic PEPS?

Discrete spin structures and commuting projector models for 2d fermionic symmetry protected topological phases

Wille, Buerschaper, Eisert, arXiv:1609.02574
Williamson, Bultinck, Haegeman, Verstraete, arXiv:1609.02897

Ising anyons in frustration-free Majorana dimer models

Ware, Son, Cheng, Mishmash, Alicea, Bauer, arXiv:1605.06125
Area laws and tensor networks

Contraction of PEPS

G-injective PEPS

Fermionic topological PEPS and a crime story

MPO-injective PEPS

Thanks for your attention