Exceptional Topology of Non-Hermitian Systems

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Credit to **Flore Kunst, Elisabet Edvardsson, Marcus Stålhammar, Johan Carlström and Jan Budich**

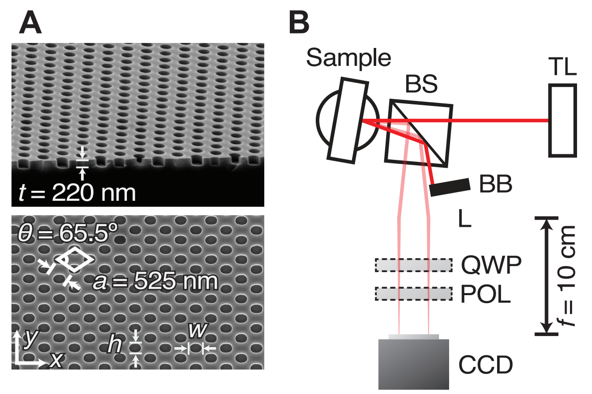
KITP, Santa Barbara, October 2019

Fantasy or reality?

- Complex energies, non-unitary time-evolution, …, Pandora's box!?

Reality:

- Dissipative systems — experiments!
- Photonic systems with gain and/or loss
- Various classical mechanical, electrical, robotic and optical metamaterials
- Open, non-equilibrium systems — toy alternative to the Lindblad master equation
- Effective description of systems with finite lifetime states $\mathrm{Im}[E] \sim 1/\tau$
- …



Need:

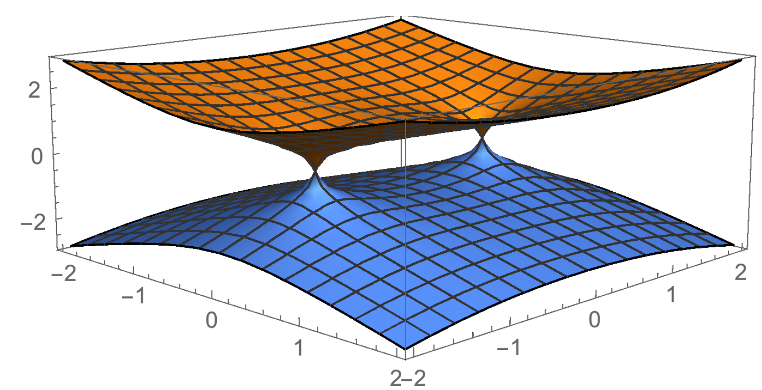
- Basic theory!

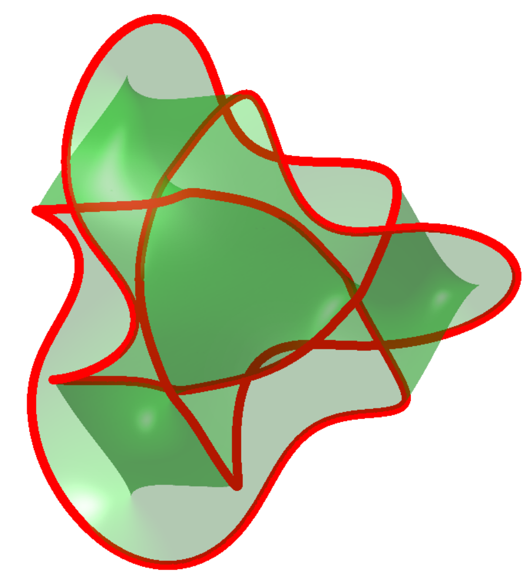
Today:

- Minimal example

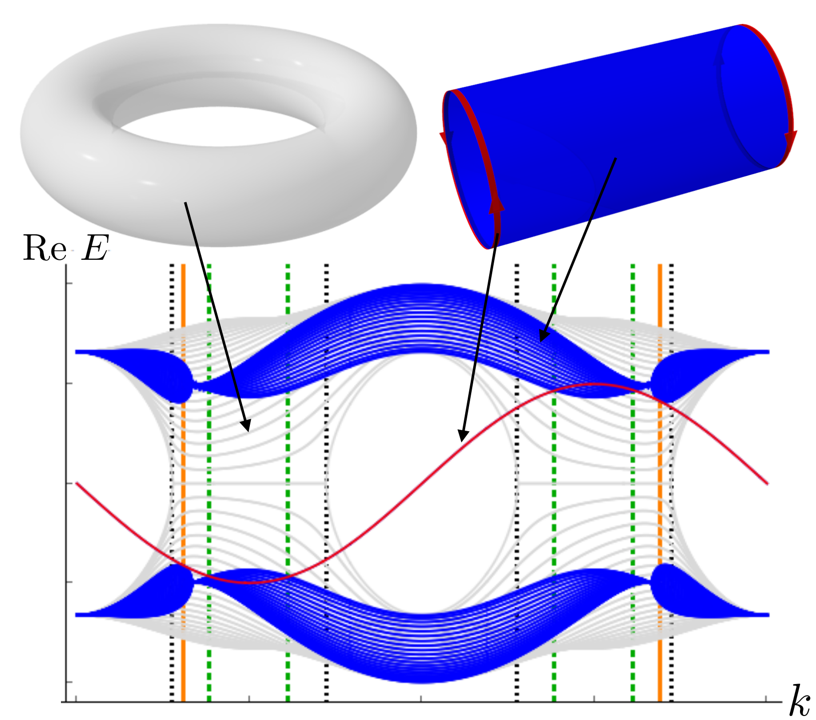
$$H = \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix} \qquad \alpha \neq 1$$

- Exceptional nodal phases





- Biorthogonal bulk-boundary correspondence



Minimal example:
a two-level system

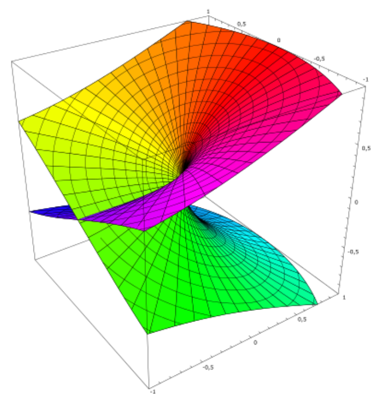
$$H = \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix} \qquad \alpha \neq 1$$

- Take home: Exceptional degeneracies & Square roots

A two-level system

$$H = \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix}$$

- Eigenvalues generally complex $E_{\pm} = \pm\sqrt{\alpha}$
  - Winding of α twice yield a winding of $E_{\pm}$ only once!



Note the branch point and branch cut

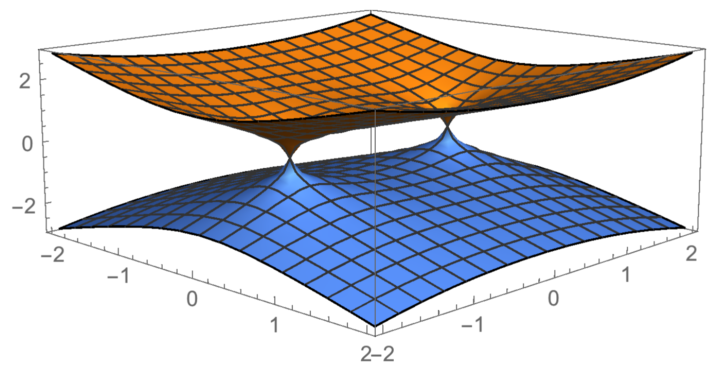
- Non-orthogonal eigenvectors $\Psi_{R,\pm} = \begin{pmatrix} \pm\sqrt{\alpha} \\ 1 \end{pmatrix}$
  - Left and right eigenvectors are different $\Psi_{L,\pm} = \begin{pmatrix} 1 & \pm\sqrt{\alpha} \end{pmatrix}$

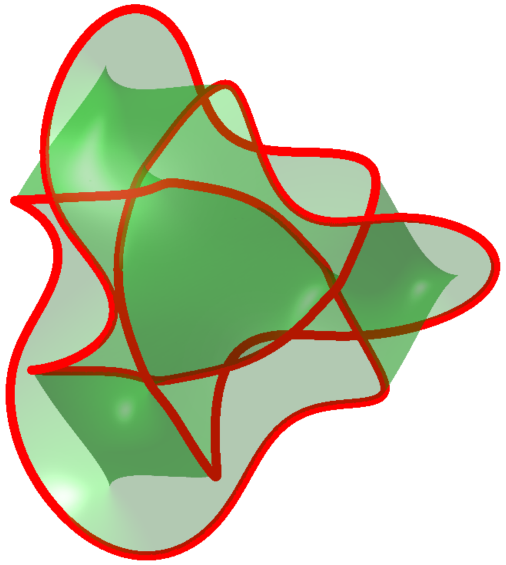
An exceptional point $(\alpha = 0)$

$$H = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

- Doubly degenerate eigenvalue $E_\pm = 0$
- But only one normalisable eigenvector! $\Psi_{R,\pm} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
  - The left eigenvector is the "opposite" $\Psi_{L,\pm} = \begin{pmatrix} 1 & 0 \end{pmatrix}$
- "*Exceptional points*" (EPs) with singular behaviour
  - Rare, measure zero in the space of matrices
  - Diverging response $|\partial_\alpha E(\alpha)| \to \infty$
- When can we expect EPs to occur and what are their consequences?

Exceptional nodal phases





- Take home: Abundant & conceptually rich

A step back:
Band crossings in Hermitian systems

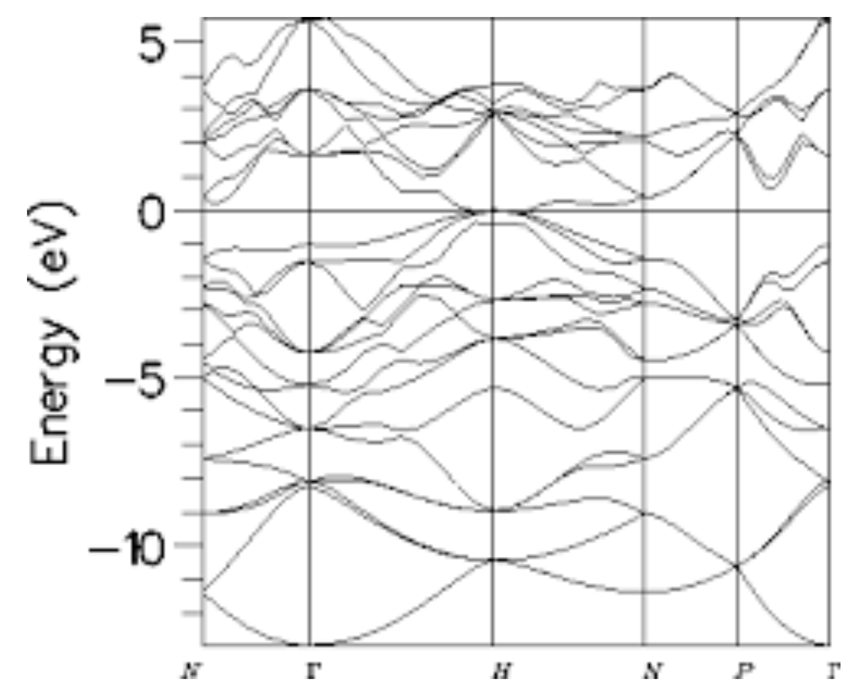
- When can we expect two energy bands to cross at a single point?

C. Herring, Phys. Rev. 52 365 (1937)

$$H(\mathbf{k}) = \begin{pmatrix} d_3(\mathbf{k}) + d_0(\mathbf{k}) & d_1(\mathbf{k}) - id_2(\mathbf{k}) \\ d_1(\mathbf{k}) + id_2(\mathbf{k}) & -d_3(\mathbf{k}) + d_0(\mathbf{k}) \end{pmatrix}$$

$$= \mathbf{d}(\mathbf{k}) \cdot \sigma + d_0(\mathbf{k})$$

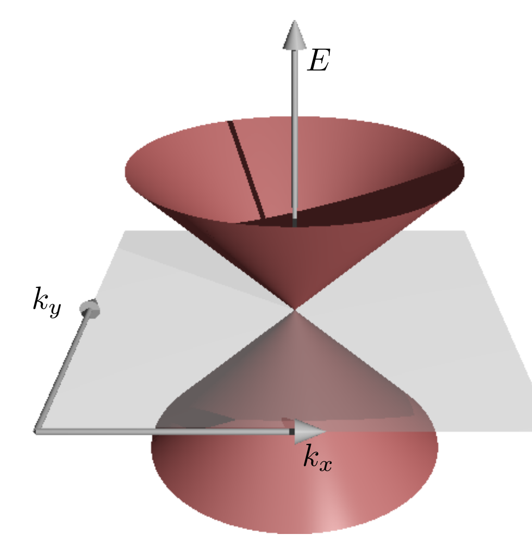
$$E(\mathbf{k}) = \pm\sqrt{d_1^2(\mathbf{k}) + d_2^2(\mathbf{k}) + d_3^2(\mathbf{k})} + d_0(\mathbf{k})$$



  - 3 equations
  - Fine-tuning in 2d
  - Stable and generic in 3d!

- Simplest case — the Weyl Hamiltonian

$$H = v\mathbf{k} \cdot \sigma$$

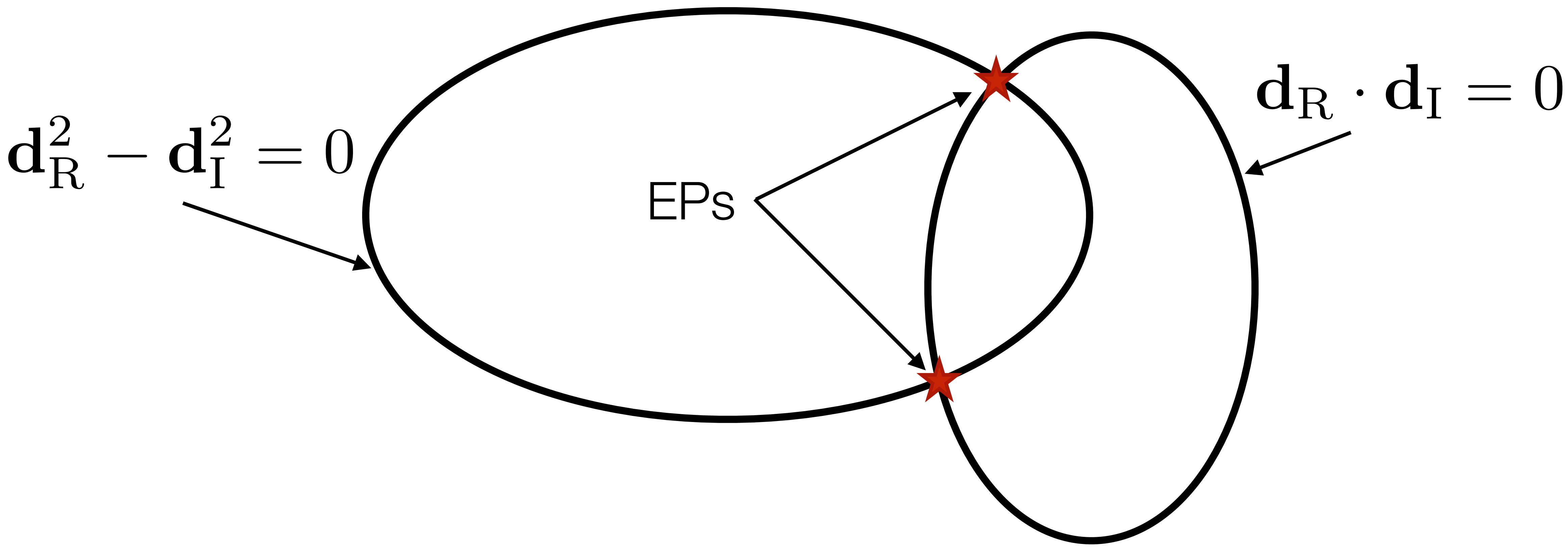


What about the non-Hermitian case?

$$H(\mathbf{k}) = \mathbf{d}(\mathbf{k}) \cdot \sigma$$ now with $$\mathbf{d}(\mathbf{k}) = \mathbf{d}_{\mathrm{R}}(\mathbf{k}) + i\mathbf{d}_{\mathrm{I}}(\mathbf{k})$$

$$E(\mathbf{k}) = \pm\sqrt{\mathbf{d}_{\mathrm{R}}(\mathbf{k})^2 - \mathbf{d}_{\mathrm{I}}(\mathbf{k})^2 + 2i\mathbf{d}_{\mathrm{R}}(\mathbf{k}) \cdot \mathbf{d}_{\mathrm{I}}(\mathbf{k})}$$

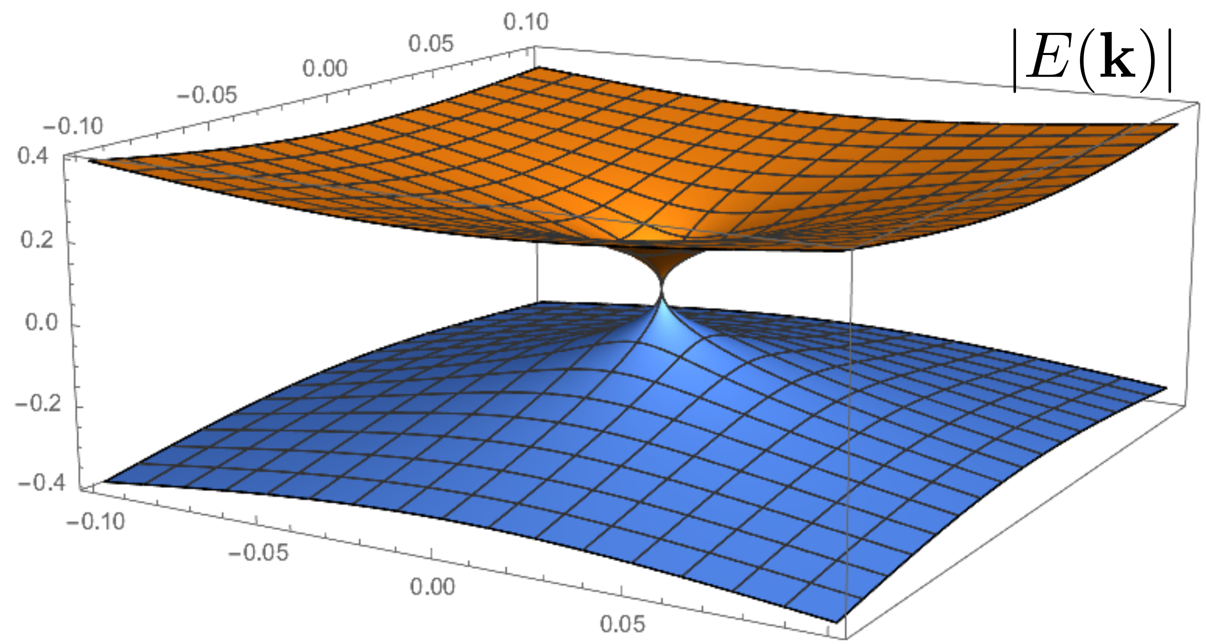
- Generic band crossings from tuning only *two parameters*! (Pancharatnam, Berry, …)
  - Look at $E^2(\mathbf{k})$ in 2d



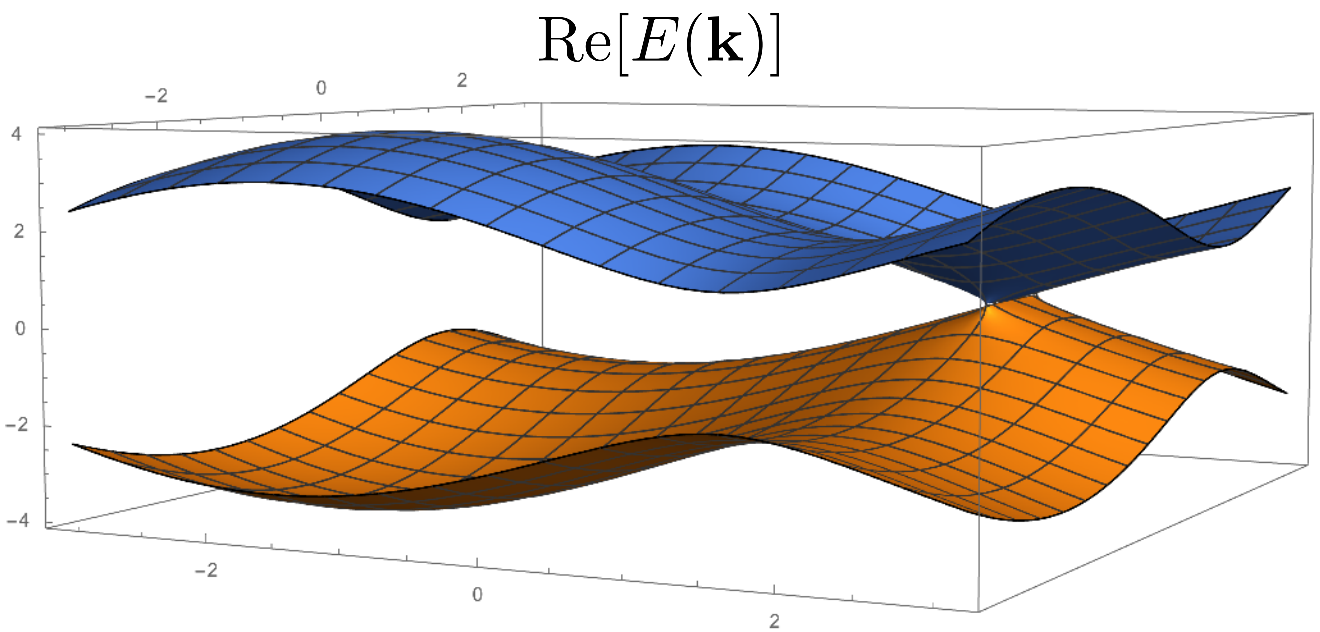
- EPs come in pairs and are generic in 2d, hence much more abundant than in the Hermitian case!

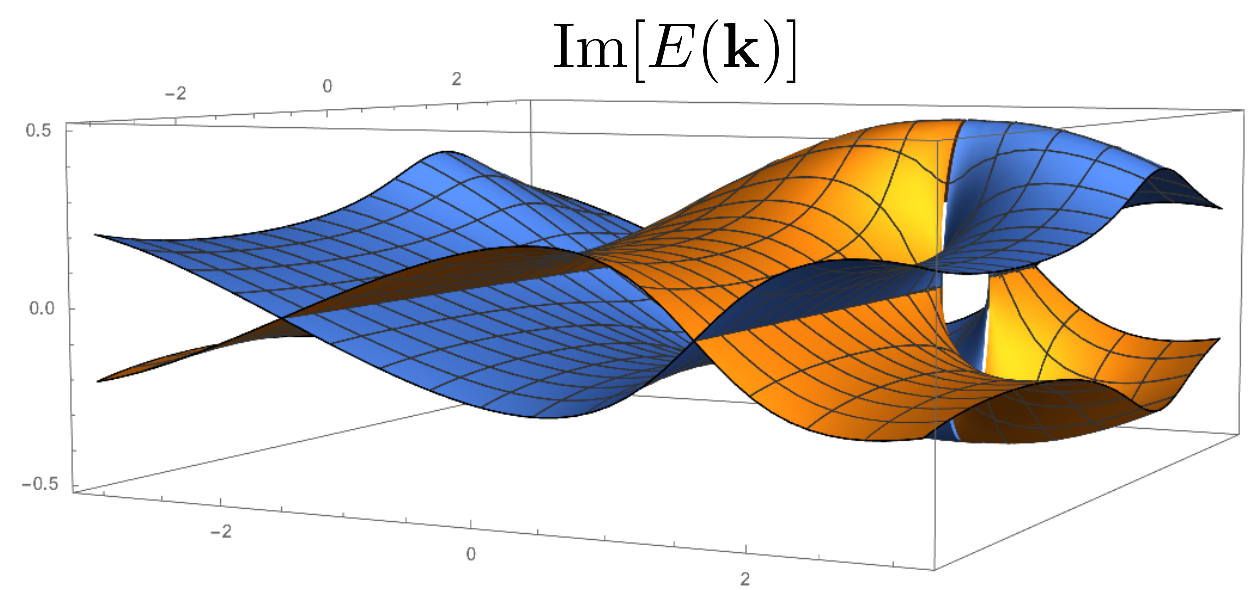
Spectral features

- EPs are non-analytical, "square roots of Weyl points"



- $E(\mathbf{k})$ is different than what one naively infers from $E^2(\mathbf{k})$!



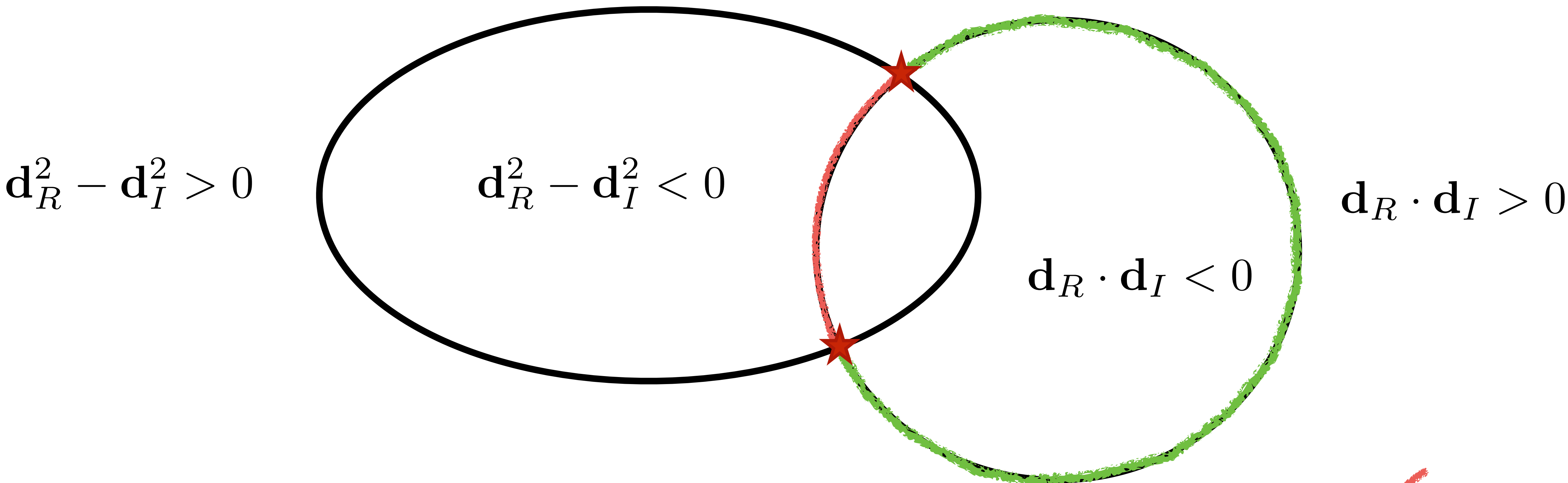


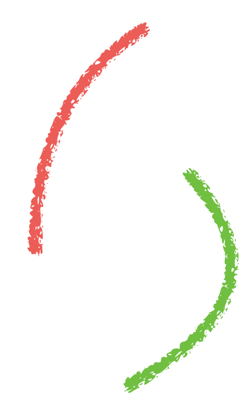
- 2d bulk Fermi arcs!

V. Kozii and L. Fu, arXiv:1708.05841

Let’s have a closer look: arcs

$$E(\mathbf{k}) = \pm\sqrt{\mathbf{d}_{\mathrm{R}}(\mathbf{k})^2 - \mathbf{d}_{\mathrm{I}}(\mathbf{k})^2 + 2i\mathbf{d}_{\mathrm{R}}(\mathbf{k})\cdot\mathbf{d}_{\mathrm{I}}(\mathbf{k})}$$





- Fermi arcs $\mathrm{Re}[E] = 0$ when $\mathbf{d}_{\mathrm{R}} \cdot \mathbf{d}_{\mathrm{I}} = 0$ and $\mathbf{d}_R^2 - \mathbf{d}_I^2 < 0$
- i-Fermi arcs $\mathrm{Im}[E] = 0$ when $\mathbf{d}_{\mathrm{R}} \cdot \mathbf{d}_{\mathrm{I}} = 0$ and $\mathbf{d}_R^2 - \mathbf{d}_I^2 > 0$
- Irremovable degeneracies; generic (d-1)-dimensional open nodal surfaces/arcs

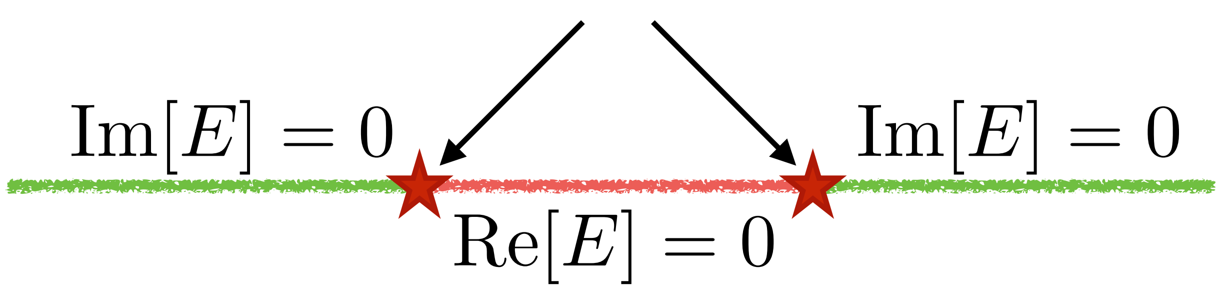
Splitting Weyl/Dirac points

- Minimal 2d model

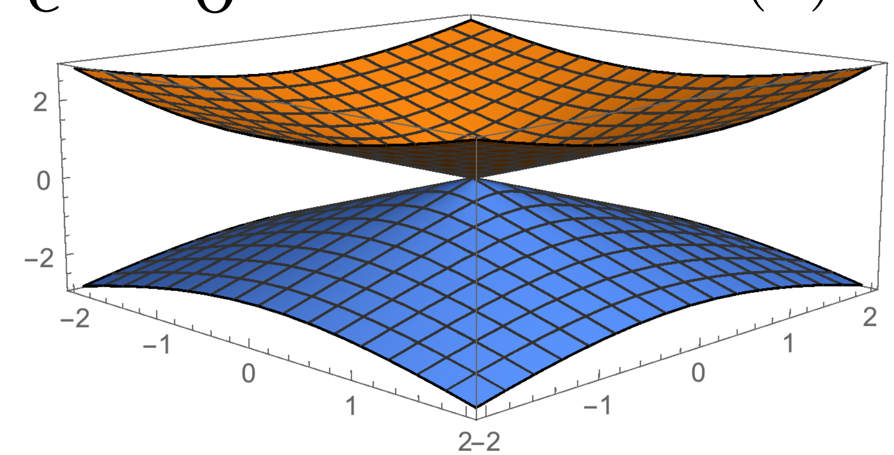
$$H = k_x\sigma_x + k_y\sigma_y + i\epsilon\sigma_x$$

$$\Rightarrow \quad E = \pm\sqrt{k_x^2 + k_y^2 - \epsilon^2 + 2i\epsilon k_x}$$

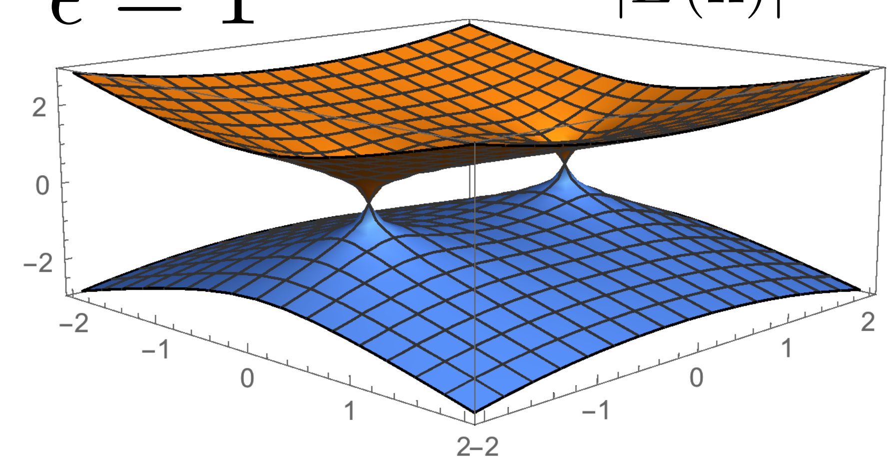
- EPs at $k_x = 0, k_y = \pm\epsilon$



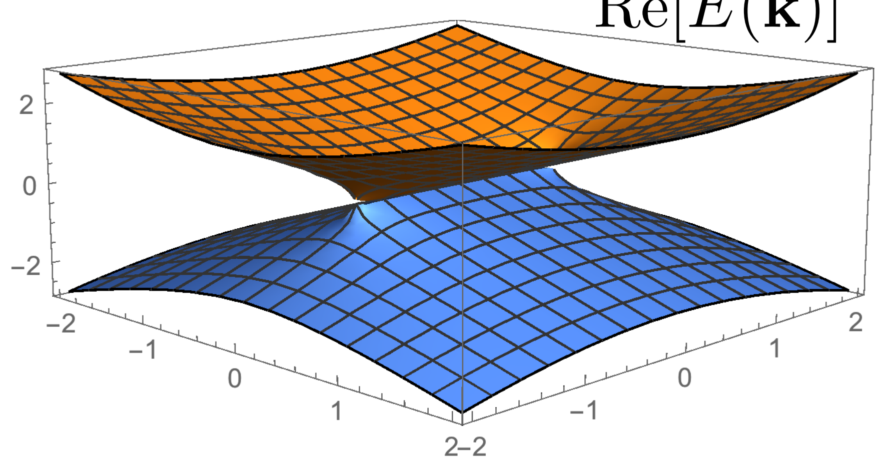
$\epsilon = 0$ $E(\mathbf{k})$



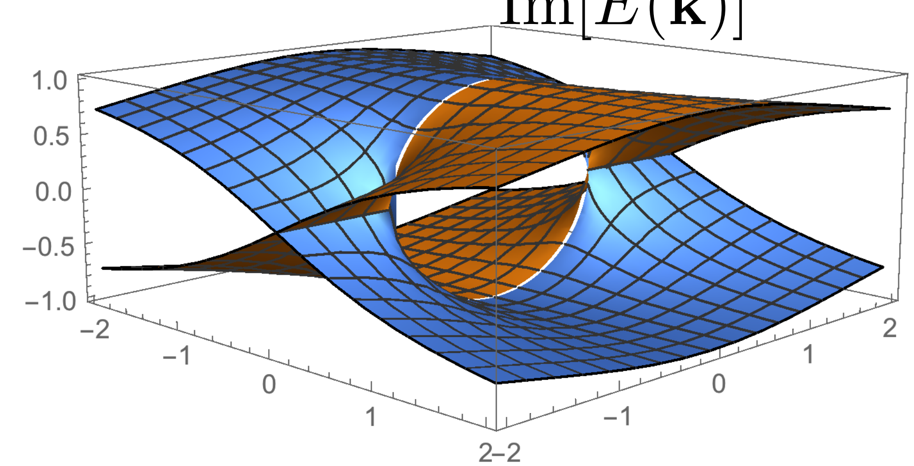
$\epsilon = 1$ $|E(\mathbf{k})|$



$\mathrm{Re}[E(\mathbf{k})]$



$\mathrm{Im}[E(\mathbf{k})]$

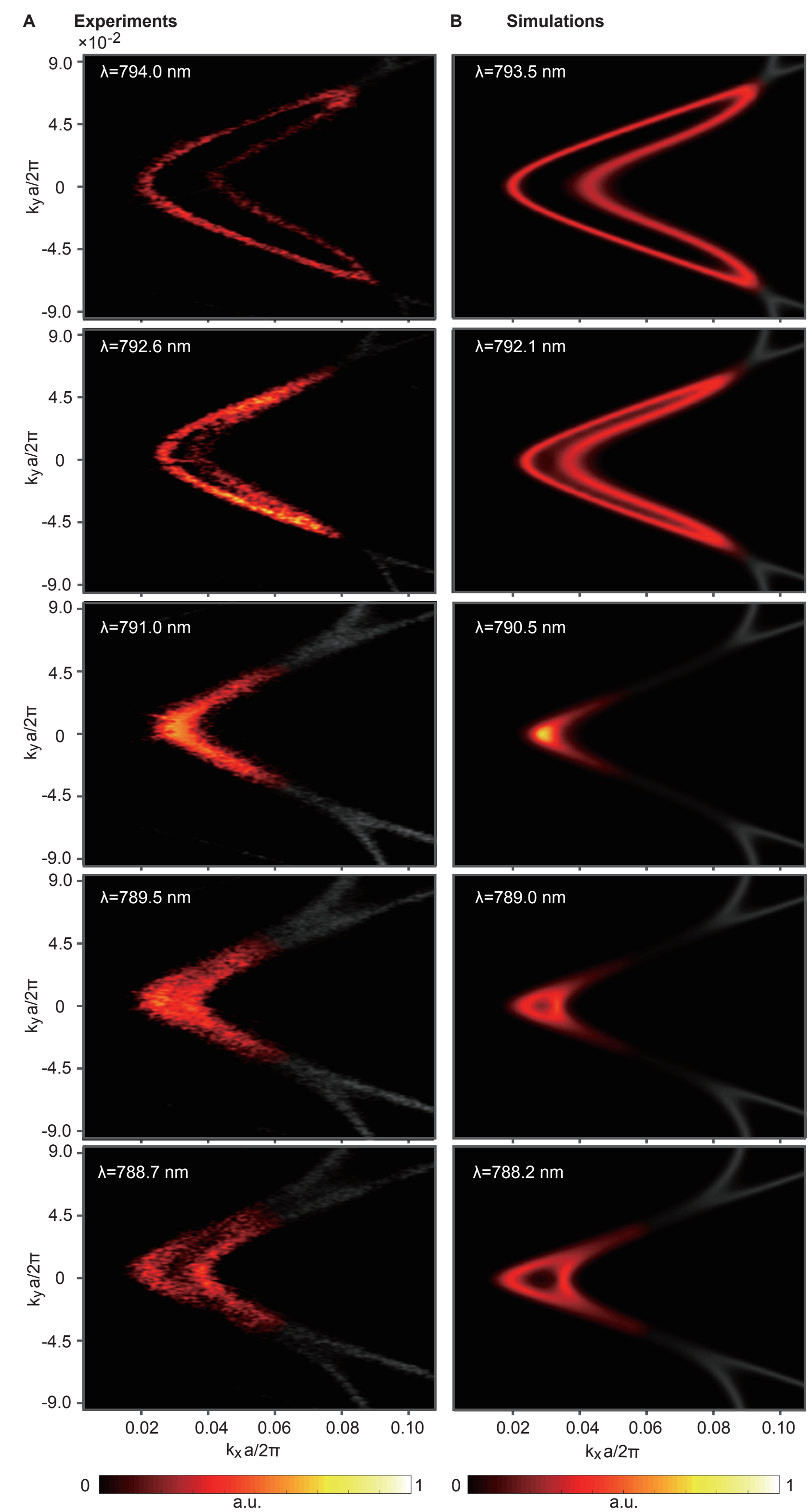


Experimental observation of 2d bulk Fermi arcs

- Fermi arcs observed in photonic crystal slabs with losses

  H.Zhou, et. al. Science p. eaap9859 (2018)

- These experiments directly measure $\mathrm{Re}[E(\mathbf{k})]$!



- Light scattering, iso-frequency contours vs. theoretical band structure

Material junctions?

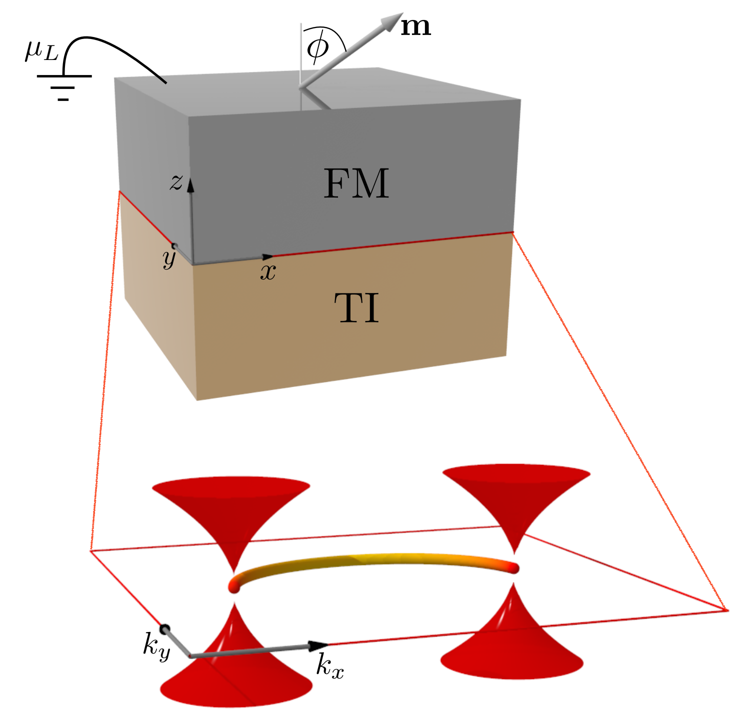
E.J. Bergholtz and J.C. Budich,
Phys. Rev. Research 1, 012003 (2019)

- Example: 3d Topological insulator coupled to a ferromagnetic lead

$$H_{\mathrm{NH}} = H + \Sigma_L^r(\omega = 0)$$

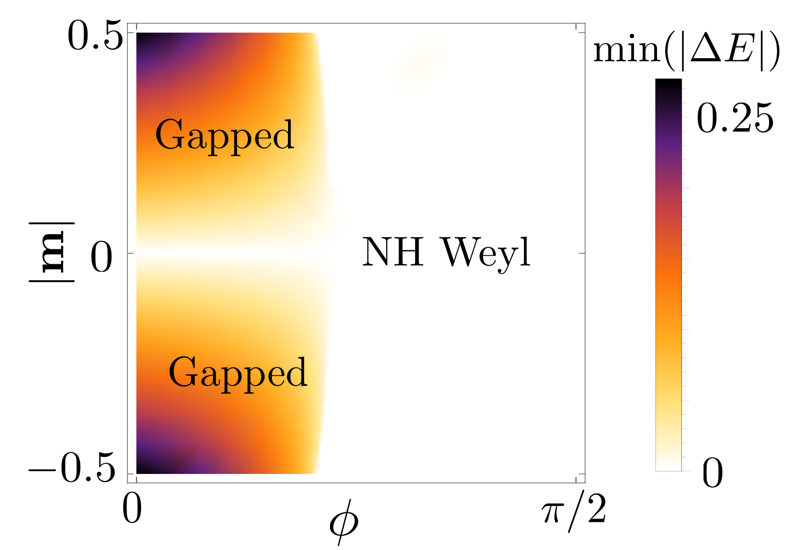
  - Surface theory + lead self energy:

$$\tilde{H} = \lambda(k_y\sigma_x - k_x\sigma_y) + \Sigma_L^r(0) - B\sigma_z \equiv \epsilon_0 + \mathbf{d}\cdot\boldsymbol{\sigma}$$



- Symmetry protected state promoted to a generic topological phase!

    - Sufficiently generic coupling needed



Symmetries in non-Hermitian systems

- Specifically non-Hermitian symmetries

D. Bernard and A. LeClair, arXiv:cond-mat/0110649 (2001)
S. Lieu, Phys. Rev. B 98, 115135 (2018)
K. Kawabata et al. 2018, 2019
H. Zhou, J.Y. Lee, 2019, …

- Example $H = qH^\dagger q^{-1}, \quad q^\dagger q^{-1} = qq^\dagger = \mathbb{I}.$ "Pseudo hermiticity"

  - For 2-band models, pick $q = \sigma_x$

$$d_x, d_0 \in \mathbb{R}, \qquad d_y, d_z \in i\mathbb{R}.$$

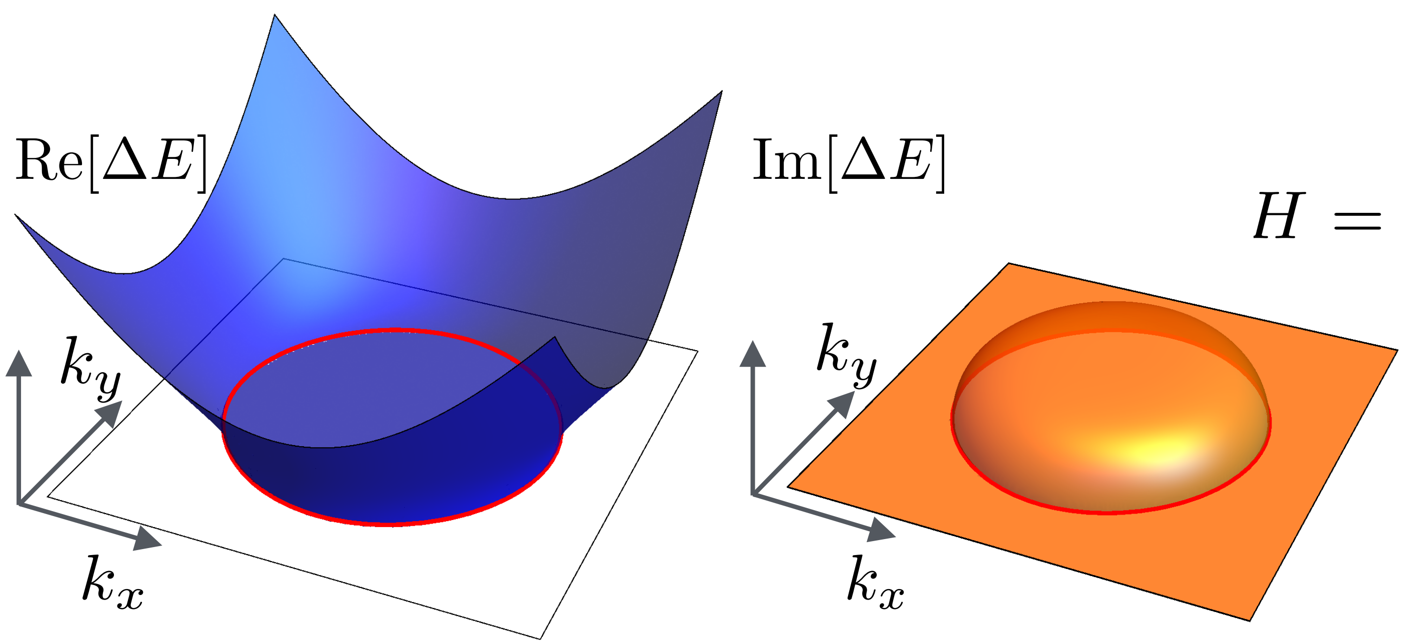
  - Trivial in the Hermitian limit
  - Generally $\mathbf{d}_R \cdot \mathbf{d}_I = 0 \Rightarrow E_\pm = \pm\sqrt{d_R^2 - d_I^2}, \qquad (d_0 = 0)$

Purely real or imaginary!

- $\mathcal{PT}$ symmetric systems, popular in optics, work analogously

Symmetry protected nodal non-Hermitian phases

- Generically one equation less…
  - Exceptional (d-1)-dimensional surfaces
  - d-dimensional open “Fermi volumes”
- 2d example



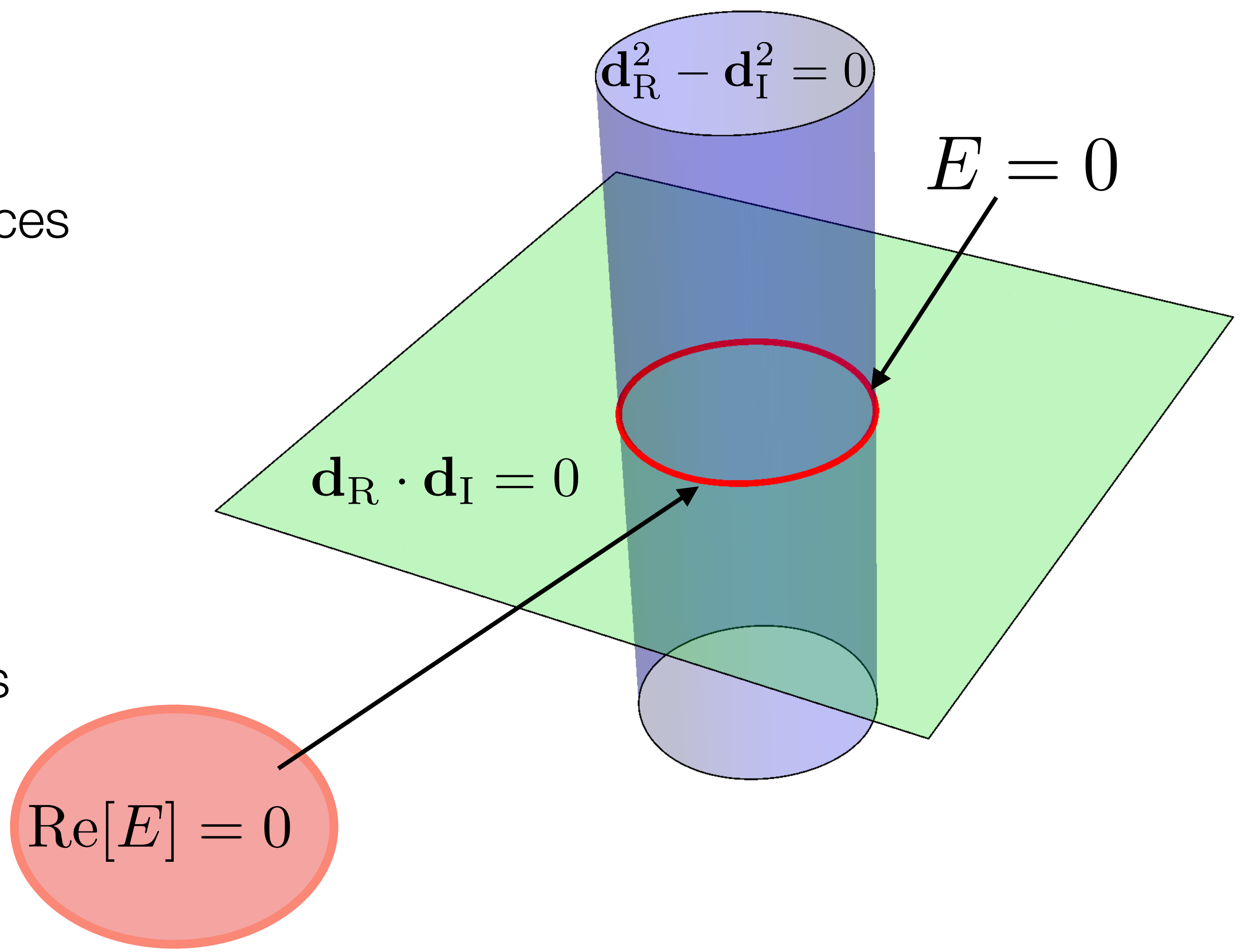
$$H = (2 - \cos k_x - \cos k_y)\sigma_x + i\sigma_z/4$$

J.C. Budich, J. Carlström, F.K. Kunst and E.J. Bergholtz, in arXiv:1810.00914
+ several subsequent postings…

3d: generic exceptional rings,...

- Remember: $$E(\mathbf{k}) = \pm\sqrt{\mathbf{d}_{\mathrm{R}}(\mathbf{k})^2 - \mathbf{d}_{\mathrm{I}}(\mathbf{k})^2 + 2i\mathbf{d}_{\mathrm{R}}(\mathbf{k})\cdot\mathbf{d}_{\mathrm{I}}(\mathbf{k})}$$
- E=0 solutions form exceptional rings Y. Xu, S.-T. Wang, and L.-M. Duan, PRL 118, 045701 (2017)
- Think about this geometrically
  - Intersections between 2d surfaces

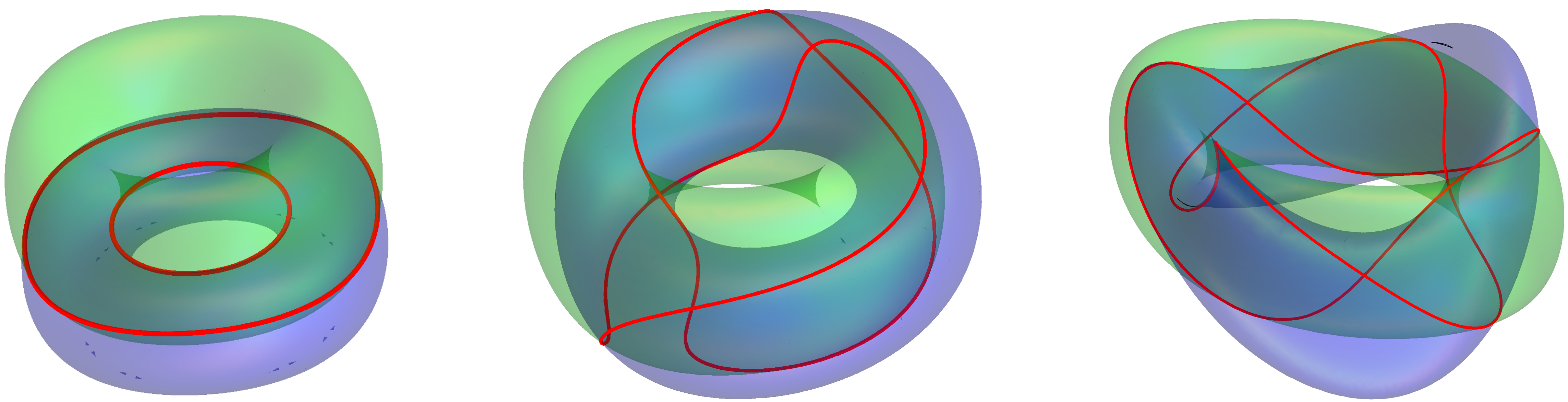
    J. Carlström and E.J. Bergholtz, Phys. Rev. A 98, 042114 (2018)
- Leads to unusual open Fermi surfaces
  - Terminated by exceptional lines



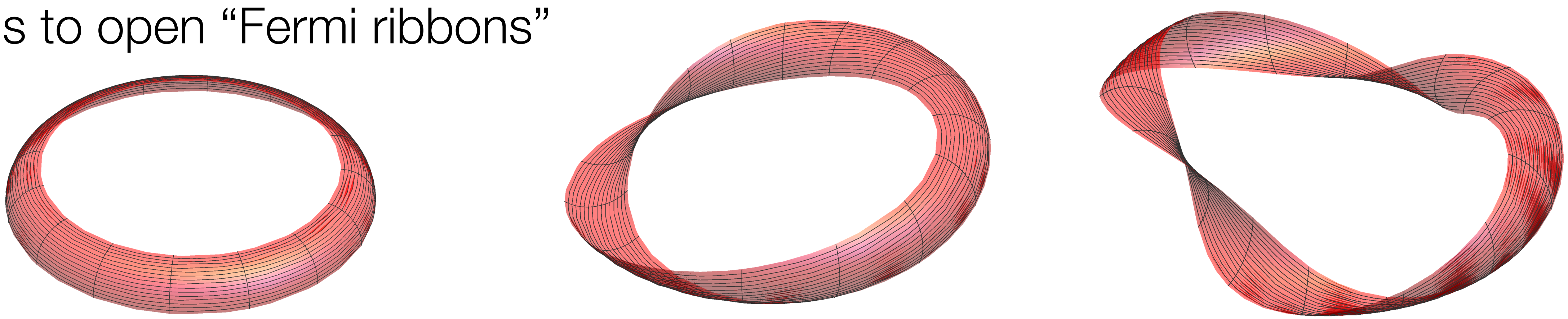
Exceptional links and twisted "Fermi Ribbons"

- Exceptional links generated as generic intersections between more general 2d closed surfaces

J. Carlström and E.J. Bergholtz, Phys. Rev. A 98, 042114 (2018)



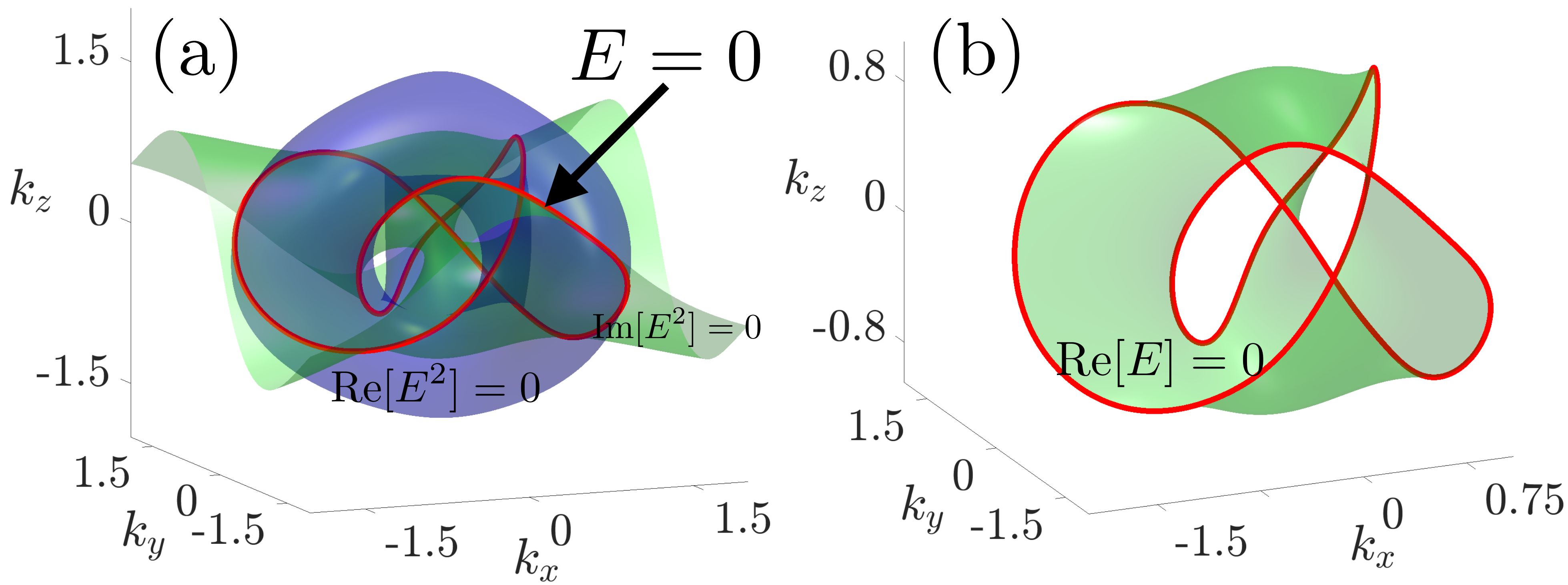
- Leads to open "Fermi ribbons"



  - Seifert surfaces, orientable

Generalization: Knotted non-Hermitian metals

J. Carlström, M. Stålhammar, J.C. Budich and E.J. Bergholtz, Phys. Rev. B 99, 161115 (2019)



"Fermi-Seifert surfaces"

- Two notions of topology combined — a unique NH possibility
  - Hermitian generic line-like nodes occur in D=4, but in D>3 all knots are trivial!
- Short-range tight-binding models — nearest neighbour for a nodal link and next nearest for a trefoli knot
- Boundary states, hyperbolic knots etc in followup works…

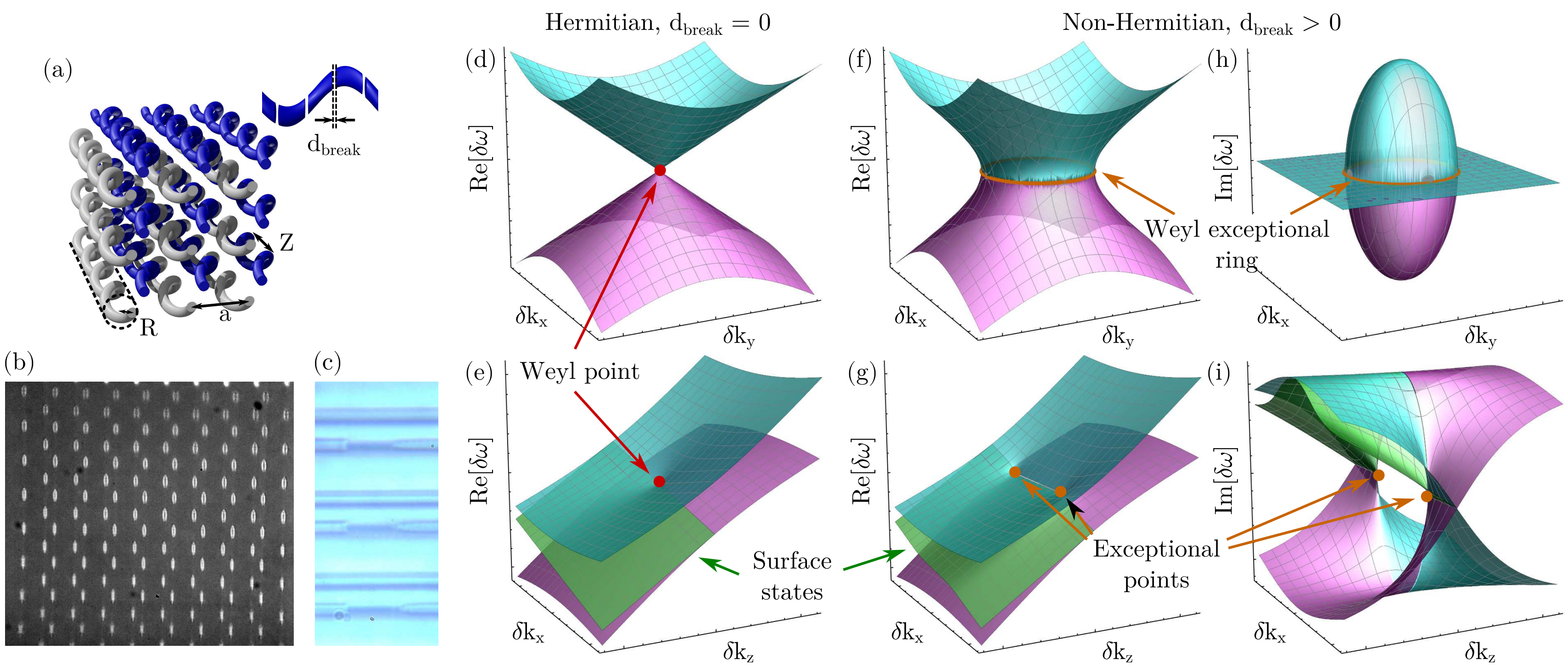
C.H. Lee et. al., arXiv:1812.02011

M. Stålhammar, et. al., SciPost Phys. 7, 019 (2019)

Exceptional rings: Experiments

- Realised with coupled waveguides

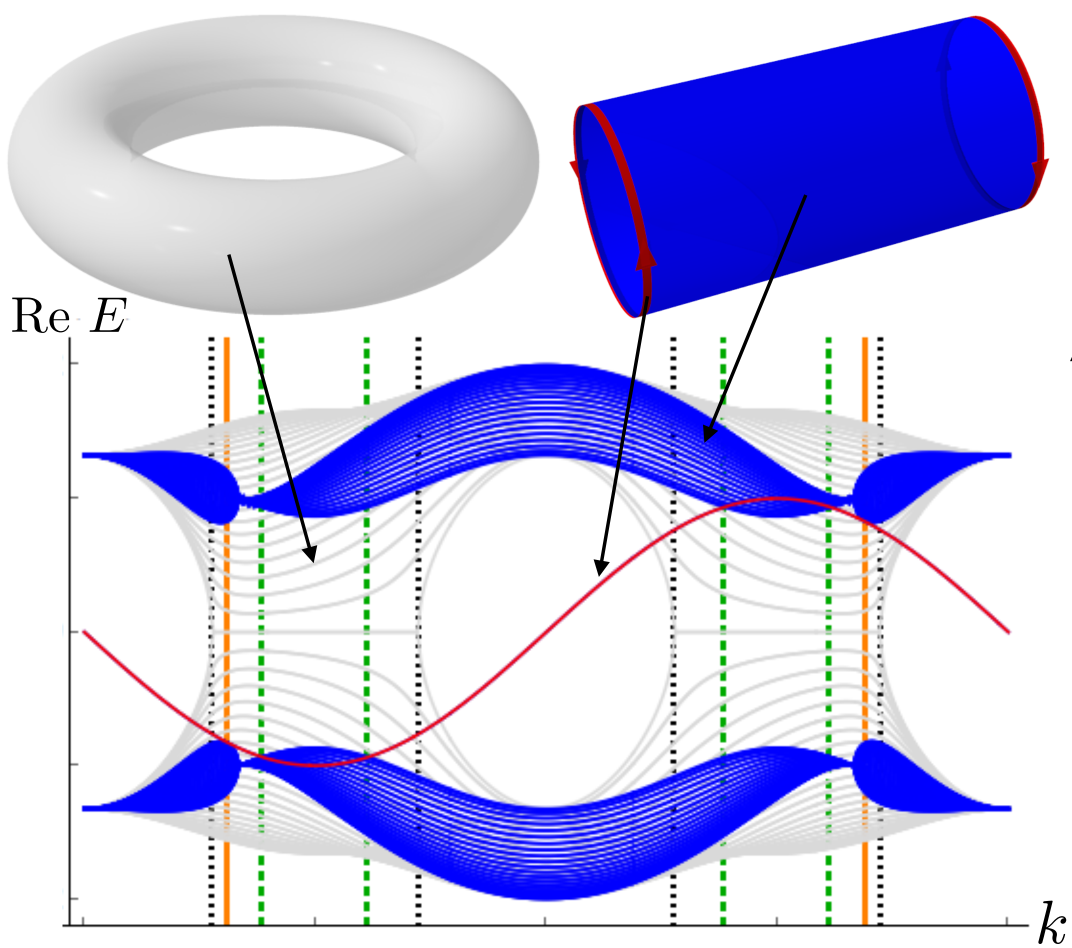
  Cerjan et. al.
  Nature Photonics 13, 623 (2019)



- Links and knots to come…

- Exotic new bulk physics — but how about boundary states?

Biorthogonal bulk-boundary correspondence



F. K. Kunst, E. Edvardsson, J. C. Budich, and E. J. Bergholtz,
Phys. Rev. Lett. 121, 026808 (2018)

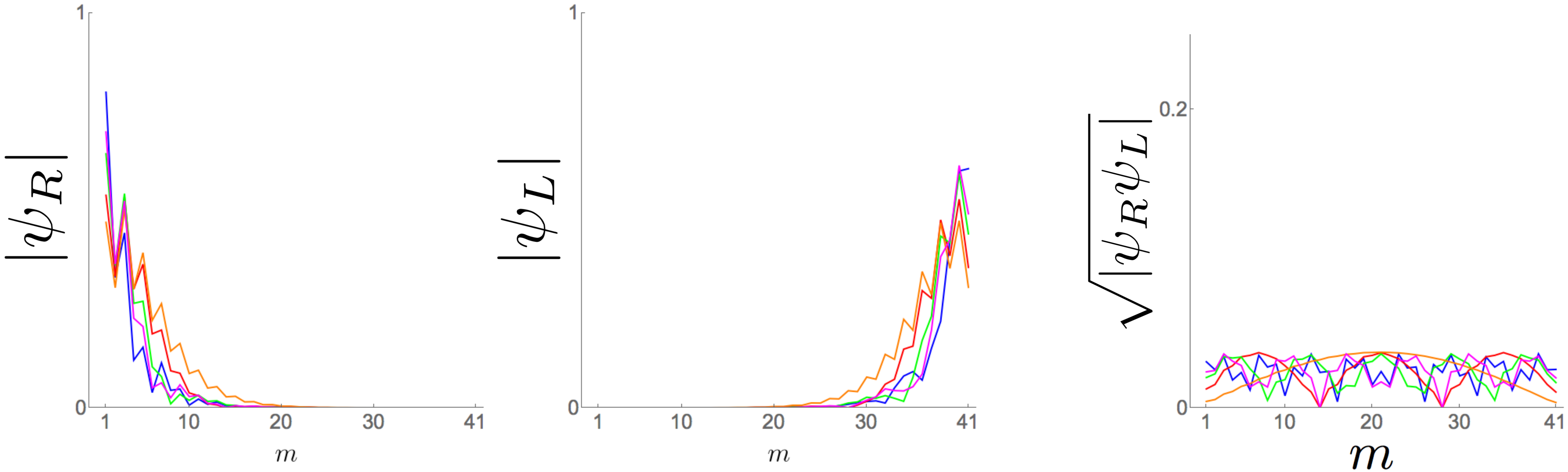
Alternative approach :

S. Yao, F. Song, and Z. Wang,
Phys. Rev. Lett. 121, 136802 (2018)

- Take home: Open and closed boundary conditions give dramatically different physics — but cases can be understood and are experimentally relevant!

Non-Hermitian skin effect

- At the heart of the problem



- Left and right eigenstates pile up at the boundaries — exceptional physics?

- But their “product” does not pile up…

Biorthogonal quantum mechanics

Brody, J Phys. A: Math. Theor. 47, 035305 (2013)

- By definition we have

$$H|u_n^R\rangle = E_n|u_n^R\rangle \quad \text{and} \quad H^\dagger|u_n^L\rangle = E_n^*|u_n^L\rangle$$

- Away from (but arbitrarily close to) exceptional points one can get a *complete orthonormal basis* by normalizing as

$$\langle u_n^L|u_m^R\rangle = \delta_{nm} \; \langle u_n^L|u_n^R\rangle = \delta_{nm}$$

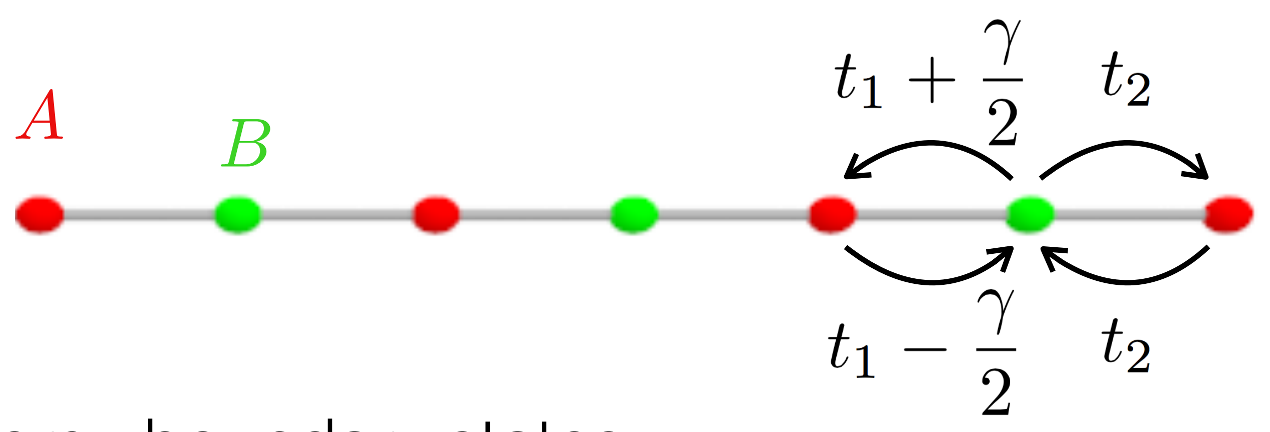
  - Consistent with

$$\sum_n \Pi_n = \mathbb{1} \quad \text{with} \quad \Pi_n = |u_n^R\rangle \langle u_n^L|$$

and $E_n = \langle u_n^L|H|u_n^R\rangle$

- This provides the "product"…

Application: non-Hermitian SSH chain



(odd length)

- Exact zero energy boundary states:

$$|\psi_R\rangle = \mathcal{N}_R \sum_{m=1}^{M} r_R^m c_{A,m}^\dagger |0\rangle \qquad |\psi_L\rangle = \mathcal{N}_L \sum_{m=1}^{M} r_L^m c_{A,m}^\dagger |0\rangle$$

$$r_R = -\frac{t_1 - \frac{\gamma}{2}}{t_2} \neq r_L = -\frac{t_1 + \frac{\gamma}{2}}{t_2}$$

- Observation: when $|r_L^* r_R| = 1$ we have zero energy biorthogonal bulk state!

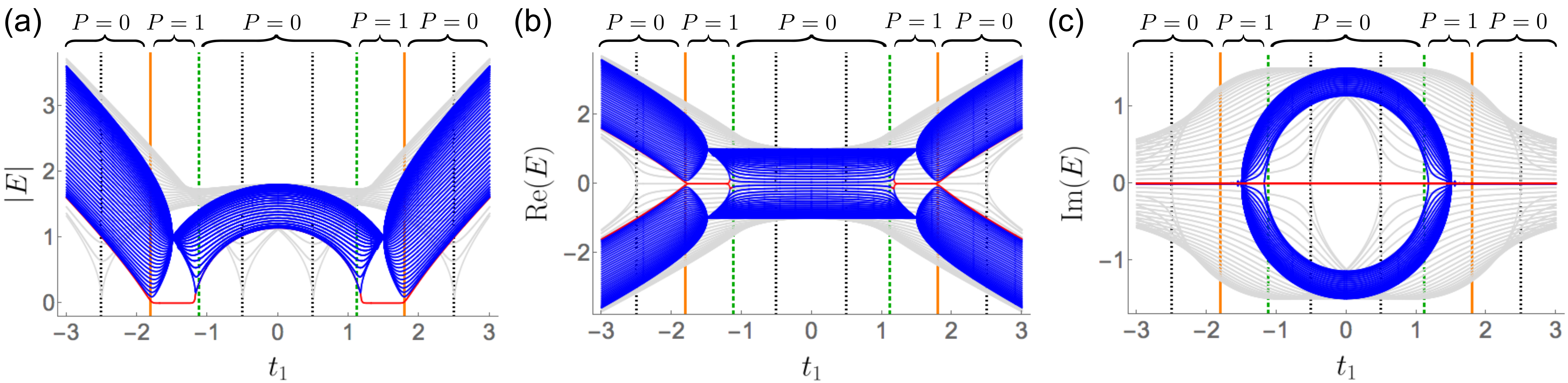
$$\langle \Pi_m \rangle \equiv \langle \psi_L | \Pi_m | \psi_R \rangle \sim (r_L^* r_R)^m$$ (now with $\Pi_m = |e_{A,m}\rangle\langle e_{A,m}| + |e_{B,m}\rangle\langle e_{B,m}|$)

- Phase transitions and changes in zero-modes at $t_1 = \pm\sqrt{\frac{\gamma^2}{4} + t_2^2}, \pm\sqrt{\frac{\gamma^2}{4} - t_2^2}$?

Biorthogonal polarisation and boundary modes

- We construct a "biorthogonal polarisation", P, which is quantised and jumps precisely when $|r_L^* r_R| = 1$

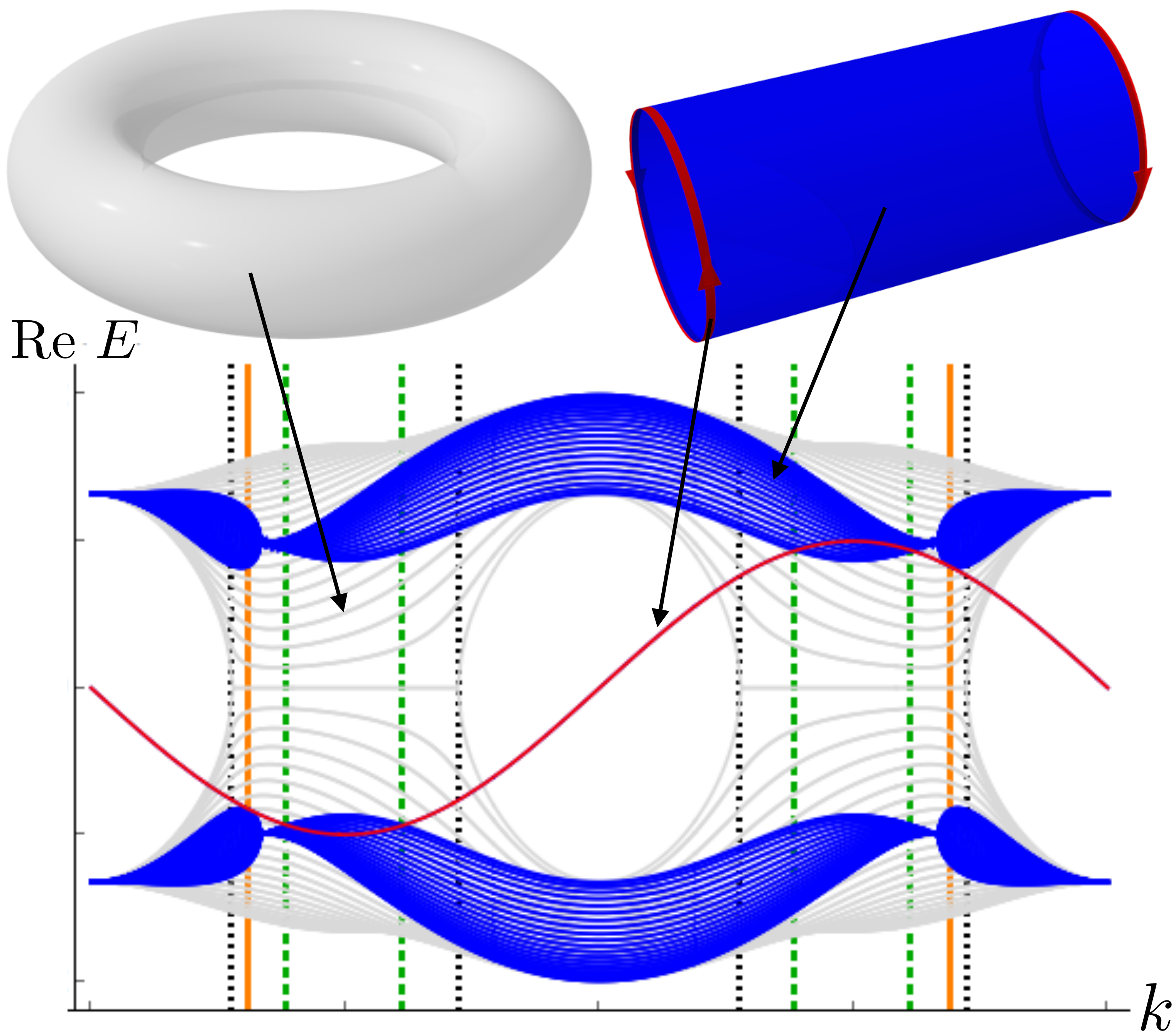
$$P \equiv 1 - \lim_{M\to\infty} \left\langle \psi_L \left| \frac{\sum_m m\,\Pi_m}{M} \right| \psi_R \right\rangle$$



(even length spectra)

- Predicts the correct phase transitions — strikingly different from Bloch band invariants!
  - Generalises directly to more complicated systems

NH Chern insulators…



Why does it work?

- Spectrum from *left and right* eigenvectors

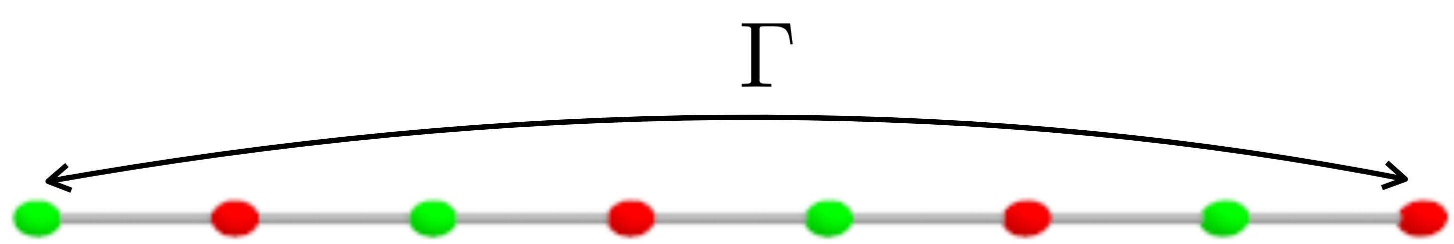
$$E_n = \langle u_n^L | H | u_n^R \rangle$$

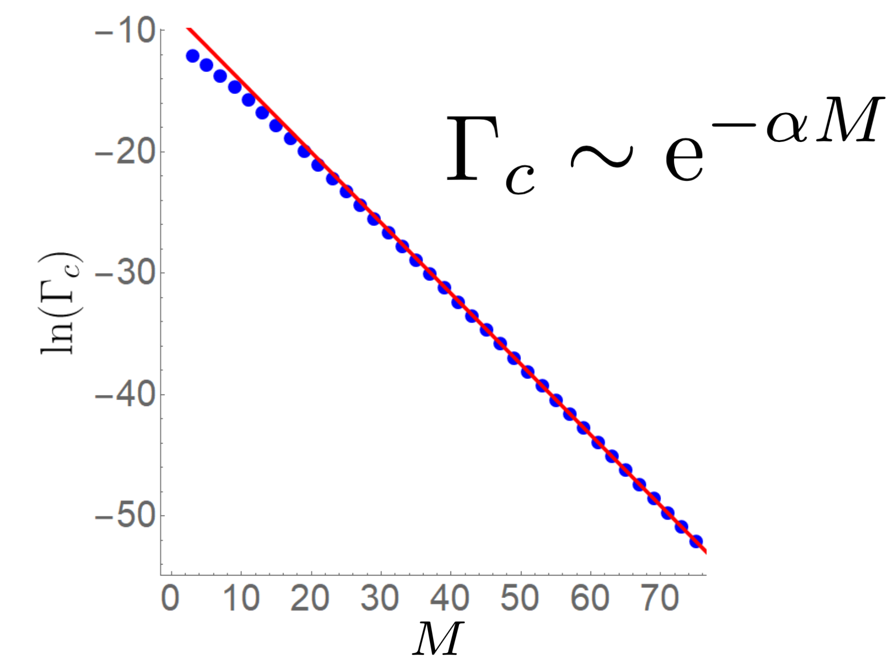
  - Extended biorthogonal states play the same role as extended states in Hermitian models where the distinction between right and left is gone

Periodic vs. open boundary conditions

Inspired by Xiong, Journal of Physics Communications 2, 035043 (2018)

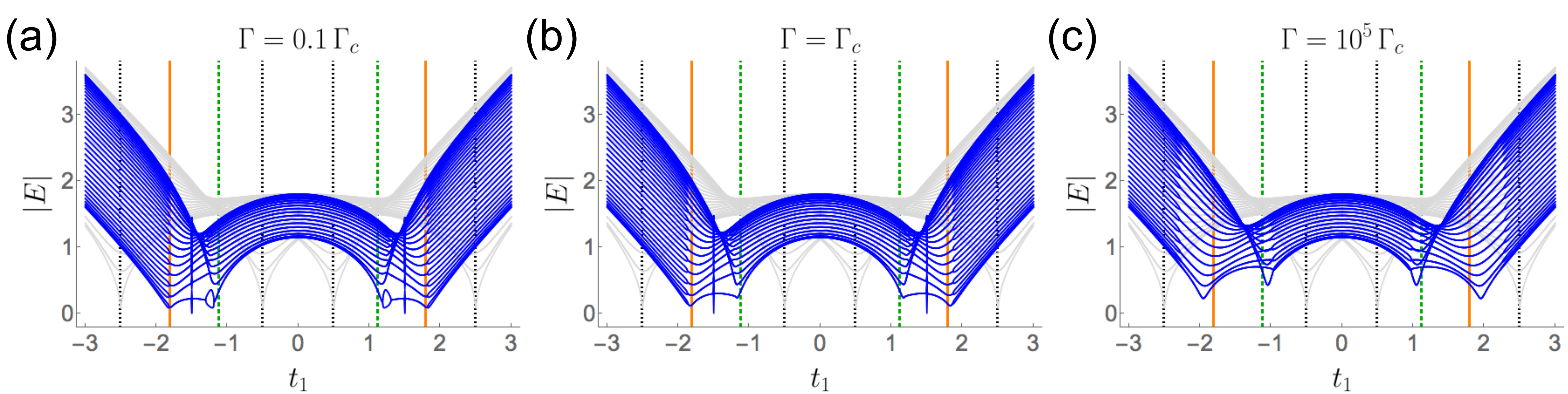
- Effect of coupling the ends





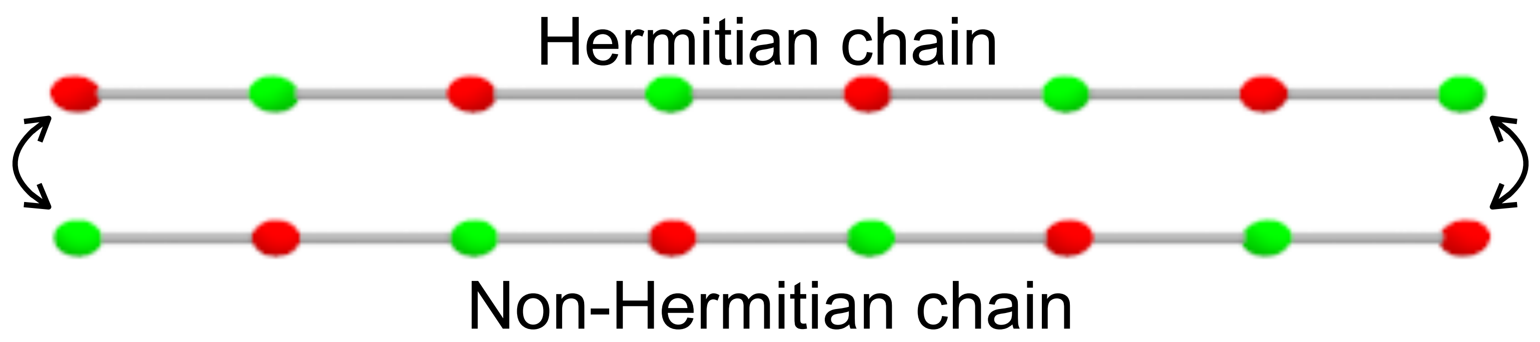
- Crossover at exponentially small Γ

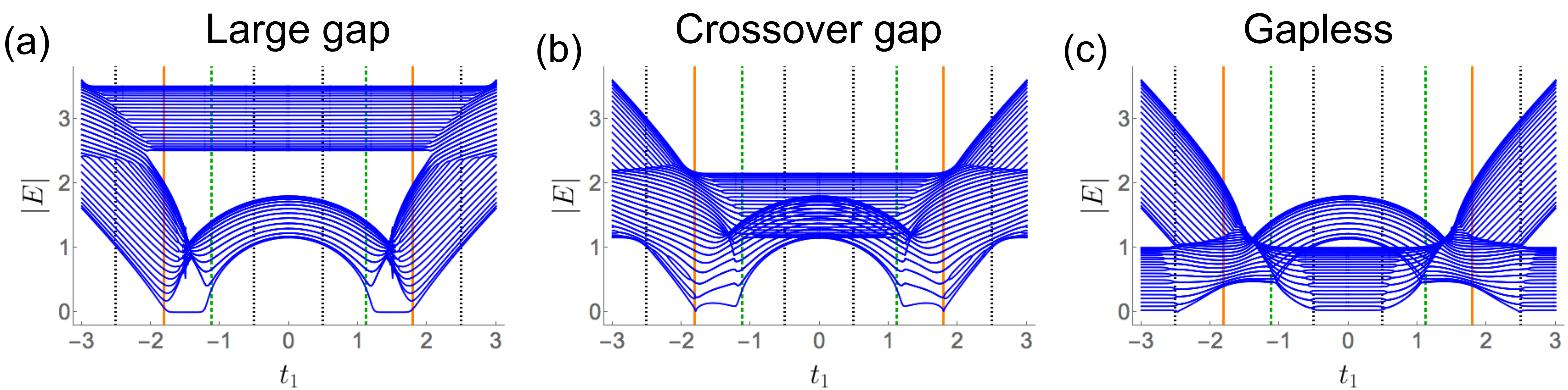
- Intuitive from the perspective of the skin effect — related to the proximity to EPs



Domain walls

- Physical mechanism: coupling ends via a Hermitian domain



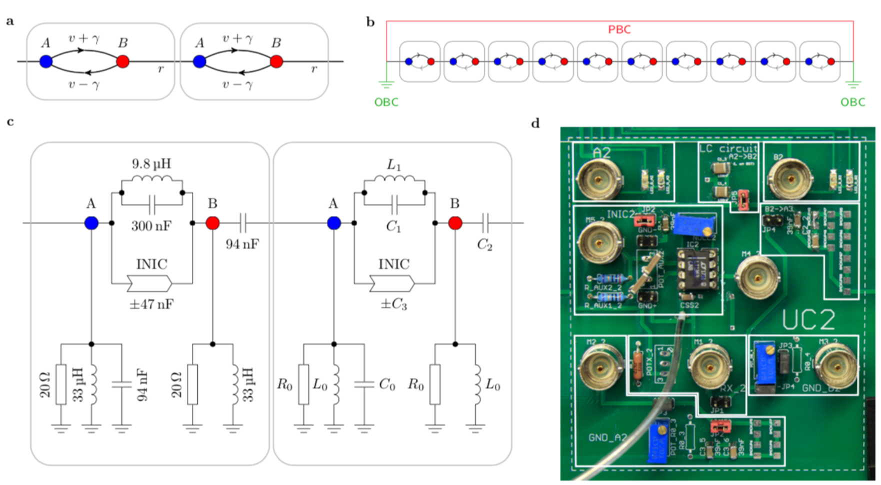


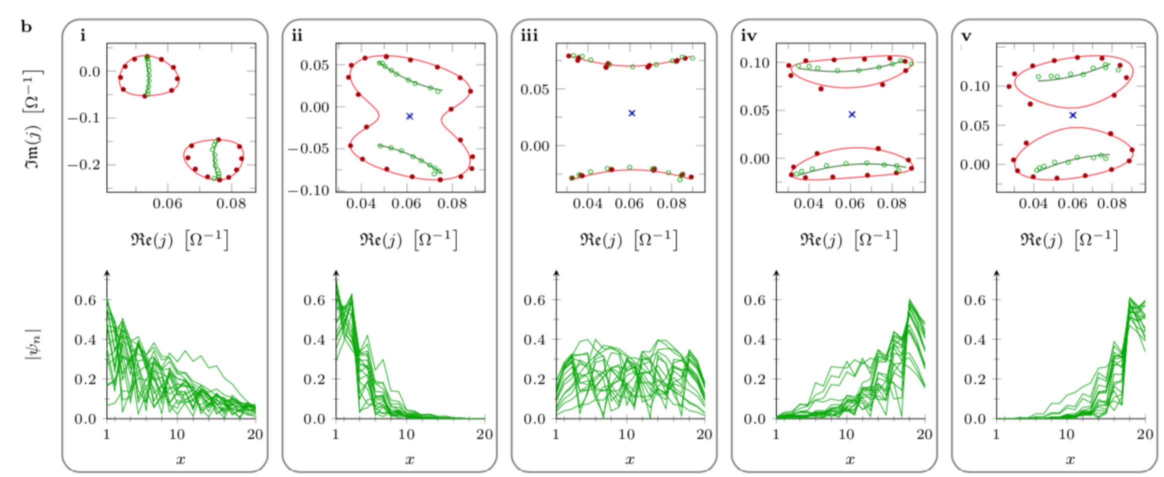
- Both periodic and open system physics can be realised depending on the strength of the effective coupling!
  - Also tuneable geometrically and/or by Wannier function engineering

Experiments!

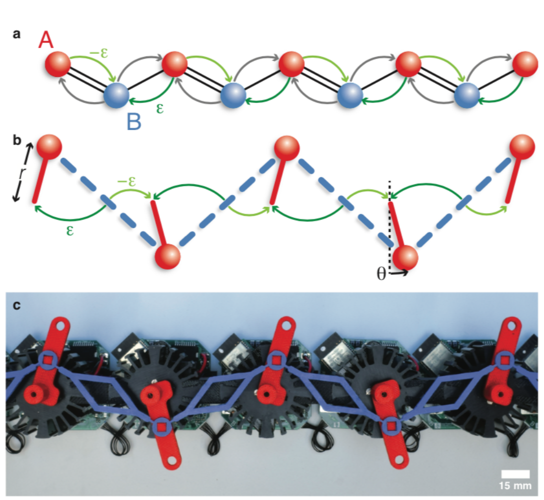
- Very recent experimental studies of the bulk-boundary correspondence

Topolectric circuits: Helbig et al., arXiv:1907.11562.

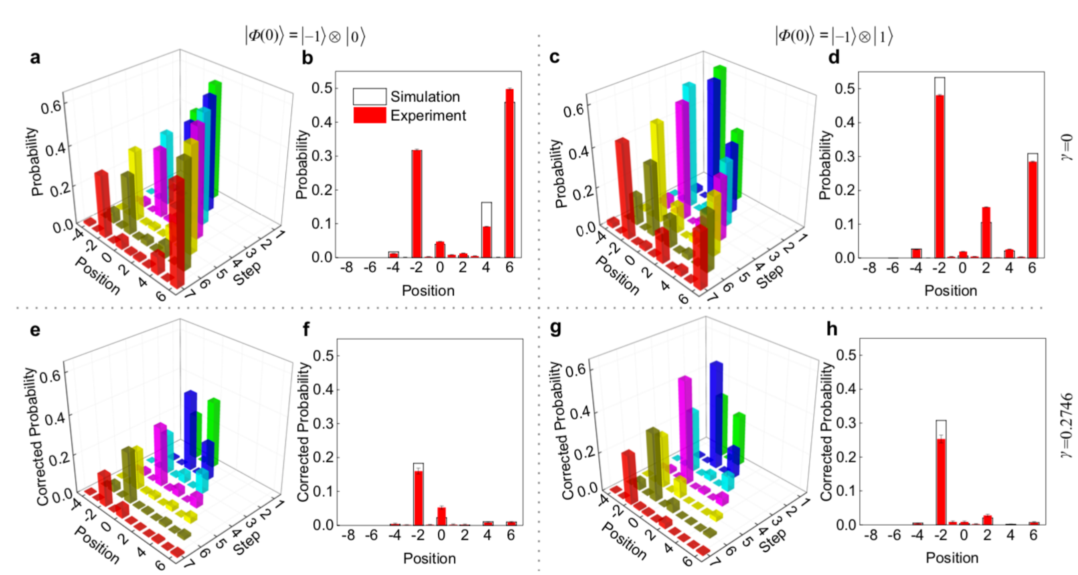




Mechanical/robotic system:
Ghatak et al., arXiv:1907.11619.

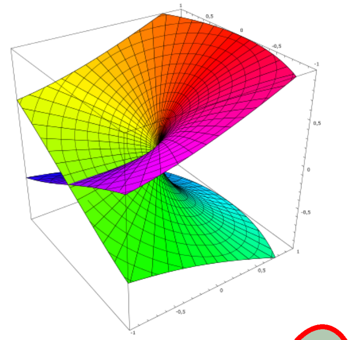


Quantum walks: Xiao et al., arXiv:1907.12566

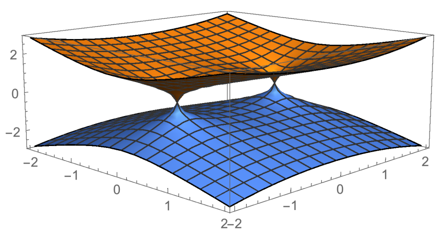


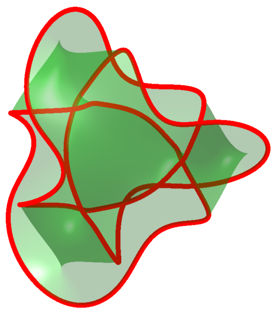
Summary

- **Exceptional Topology of Non-Hermitian Systems**

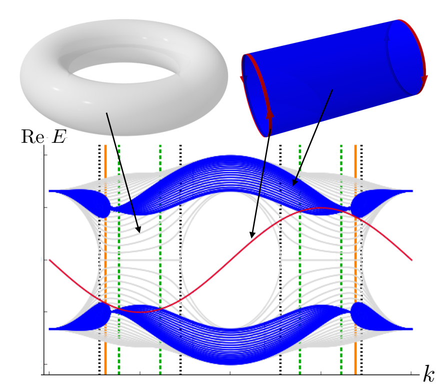


- Exceptional degeneracies, square roots





- Gapless nodal phases theoretically more abundant and conceptually richer than in the Hermitian realm



- Open and closed boundary conditions give very different physics — but cases can be understood and are experimentally relevant!

- **Intriguing and relevant!**

See e.g. M.A. Bandres and M. Segev, Physics 11, 96 (2018) and V. M. Martinez Alvarez, J. E. Barrios Vargas, M. Berdakin, and L. E. F. Foa Torres, Eur. Phys. J. Spec. Top. (2018)

Review together with Flore Kunst and Jan Budich to appear soon…