Non-Abelian topological orders in superconducting states

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Outline

1. Non-Abelian anyons in topological superconductors
   • chiral $p$-wave superconductor
   • non-Abelian anyon in $s$-wave SC

2. Gapless Topological Superconductors
What is topological superconductor?

**Topological superconductors**

**Bulk:**
gapped state with non-zero topological #

**Boundary:**
gapless state with Majorana condition
The gapless boundary state = Majorana fermion

Majorana Fermion

Dirac fermion with Majorana condition

1. Dirac Hamiltonian

\[ \mathcal{H}(k) = \sigma \cdot k, \quad \text{or} \quad \mathcal{H}(k_x) = c k_x \]

2. Majorana condition

\[ \Psi = C \Psi^* \]

particle = antiparticle

For the gapless boundary states, their Hamiltonians are naturally given by the Dirac Hamiltonians
How about the Majorana condition?

The Majorana condition is imposed by superconductivity.

\[ \Psi(x) = \begin{pmatrix} \psi_{\uparrow}(x) \\ \psi_{\downarrow}(x) \\ \psi^\dagger_{\uparrow}(x) \\ \psi^\dagger_{\downarrow}(x) \end{pmatrix} \]

\[ \Psi(x) = C\Psi^*(x), \quad C = \begin{pmatrix} 0 & 1_{2 \times 2} \\ 1_{2 \times 2} & 0 \end{pmatrix} \]

Majorana condition

[Wilczek, Nature (09)]
different bulk topological #
= different Majorana fermions

<table>
<thead>
<tr>
<th>2+1D time-reversal breaking SC</th>
<th>2+1D time-reversal invariant SC</th>
<th>3+1D time-reversal invariant SC</th>
</tr>
</thead>
<tbody>
<tr>
<td>1\textsuperscript{st} Chern # (TKNN82, Kohmoto85)</td>
<td>Z\textsubscript{2} number (Kane-Mele 06, Qi et al (08))</td>
<td>3D winding # (Schnyder et al (08))</td>
</tr>
<tr>
<td>1+1D chiral edge mode</td>
<td>1+1D helical edge mode</td>
<td>2+1D helical surface fermion</td>
</tr>
<tr>
<td>![Sr\textsubscript{2}RuO\textsubscript{4}]</td>
<td>![Noncentosymmetric SC (MS-Fujimoto(09))]</td>
<td>![\textsuperscript{3}He B]</td>
</tr>
</tbody>
</table>
A representative example of topological SC: Chiral p-wave SC in 2+1 dimensions

BdG Hamiltonian

\[ \mathcal{H} = \sum_k \epsilon(k) c_k^\dagger c_k + \frac{1}{2} \sum_k \left[ \Delta(k) c_k^\dagger c_{-k} + \text{h.c} \right] \]

\[ = \frac{1}{2} \sum_k \begin{pmatrix} c_k^\dagger, c_{-k} \end{pmatrix} \mathcal{H}(k) \begin{pmatrix} c_k \\ c_{-k}^\dagger \end{pmatrix} + \text{const.} \]

with

\[ \mathcal{H}(k) = \begin{pmatrix} \epsilon(k) & \Delta(k) \\ \Delta(k)^* & -\epsilon(k) \end{pmatrix} \]

\[ \epsilon(k) = -2t_x \cos k_x - 2t_y \cos k_y - \mu \]

\[ \Delta(k) \sim d(k_x + i k_y) \]

chiral p-wave
Topological number = 1st Chern number

\[ A_i(k) = i \sum_{a \in \text{filled}} \langle u_a(k) | \frac{\partial}{\partial k_i} | u_a(k) \rangle \]

\[ \nu_{\text{Ch}} = \frac{1}{2\pi} \int d^2k \left[ \partial_{k_x} A_y(k) - \partial_{k_y} A_x(k) \right] \]

\[ = -\frac{1}{2} \sum_{\Delta(k_0) = 0} \text{sgn}(\epsilon(k_0) \cdot \text{sgn}[\det(\partial_i R^j(k_0)))] \]

MS (09)

\[ (\Delta(k) = R^1(k) + iR^2(k)) \]
**Edge state**

\[ \mu = -1, \ d = 0.5 \]

**Fermi surface**

**Spectrum**

2 gapless edge modes (left-moving, right-moving, on different sides on boundaries)

*Majorana fermion*

\[ t_x = 1, \ t_y = 0.2 \]
\[ t_x = t_y = 1 \]

\[ \nu_{\text{Ch}} = 0 \]
\[ \nu_{\text{Ch}} = 1 \]

**Bulk-edge correspondence**
In the second case, there also exist a Majorana zero mode in a vortex

\[ \gamma_0^\dagger = \gamma_0 \]

We need a pair of the zero modes to define creation op.

\[ \gamma^\dagger = \frac{\gamma_0^{(1)} + i\gamma_0^{(2)}}{\sqrt{2}} \quad \{\gamma^\dagger, \gamma\} = 1 \]

non-Abelian anyon
topological quantum computer
For spin-triplet SCs (or odd parity SCs), there exists a simple criterion for topological phases.

If the number of TRIMs enclosed by the Fermi surface is odd, the spin-triplet SC is (strongly) topological.

2D spinless SC

\[ \Delta(k) = k_x + i k_y \]

[Sato (09), Sato (10), Fu-Berg (10)]
3D time-reversal invariant spin-triplet SC

Even

Odd

\[ d_x(k) = k_x, \quad d_y(k) = k_y, \quad d_z(k) = k_z \]
With proper topology of the Fermi surface, spin-triplet SCs (or odd-parity SCs) naturally become topological.

Is it possible to realize non-Abelian anyon in $s$-wave superconducting state?

Yes!

A) MS, Physics Letters B535, 126 (03), Fu-Kane PRL (08)

B) MS-Takahashi-Fujimoto, Phys. Rev. Lett. 103, 020401 (09);
MS-Takahashi-Fujimoto, Phys. Rev. B82, 134521 (10) (Editor’s suggestion),
J. Sau et al, PRL (10), J. Alicea PRB (10)

Key point: Spin-orbit interaction
Majorna fermion in **spin-singlet SC**

\[ \mathcal{H} = \begin{pmatrix}
-i\sigma_i \partial_i & \Phi^* \\
\Phi & -i\sigma_i \partial_i
\end{pmatrix} \quad \Phi = \Phi_0 f(r)e^{i\theta} \]

① 2+1 dim Dirac fermion + s-wave Cooper pair

Zero mode in a vortex \[\text{[Jackiw-Rossi (81), Callan-Harvey(85)]}\]

With Majorana condition, non-Abelian anyon is realized \[\text{[MS (03)]}\]
**On the surface of topological insulator**

- \( \text{Bi}_{1-x}\text{Sb}_x \)  

- \( \text{Bi}_2\text{Se}_3 \)  
  Hsieh et al., Nature (2009)

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**Dirac fermion + s-wave SC**

**S-wave SC**

**Topological insulator**

**Spin-orbit interaction**  
=> topological insulator
2nd scheme of Majorana fermion in spin-singlet SC

② 2+1 dim s-wave SC with **Rashba spin-orbit interaction**

\[
\mathcal{H}(k) = \begin{pmatrix}
\epsilon_k - \hbar \sigma_z + \mathbf{g}_k \cdot \mathbf{\sigma} \\
-i\psi_s \sigma_y \\
-i\psi_s \sigma_y \\
-\epsilon_k + \hbar \sigma_z + \mathbf{g}_k \cdot \mathbf{\sigma}^* 
\end{pmatrix}
\]

\[
\mathcal{H}^D(k) = D\mathcal{H}(k)D^\dagger, \quad D = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i\sigma_y \\
\sigma_y & 1 \end{pmatrix}
\]

\[
\mathcal{H}^D(k) = \begin{pmatrix}
\psi_s - \hbar \sigma_z \\
-\epsilon_k \sigma_y \\
-\epsilon_k \sigma_y + ig_k \cdot \sigma \sigma_y \\
-\psi_s + \hbar \sigma_z 
\end{pmatrix}
\]

**Rashba SO**

p-wave gap is induced by **Rashba SO int.**
Gapless edge states

For $h > \sqrt{\psi_s^2 + \mu^2}$

**a single chiral gapless edge state appears like p-wave SC!**

**Chern number**

- nonzero Chern number $Q = 1$
  - topologically equivalent to spinless chiral p-wave SC

[MS, Takahashi, Fujimoto PRL(09)]
\[ h > \sqrt{\psi_s^2 + \mu^2} \] strong magnetic field is needed

How to suppress orbital depairing effect

a) s-wave superfluid of cold atoms with laser generated Rashba SO coupling

[Sato-Takahashi-Fujimoto PRL(09)]

b) semiconductor-superconductor interface

[J.Sau et al. PRL(10)
J. Alicea, PRB(10)]

c) semiconductor nanowire on superconductors ....

[R. M. Lutchyn et al. PRL(10), Y. Oreg et al, PRL(10), ...]
Summary (part 1)

With proper topology of Fermi surfaces, topological SCs are naturally realized in spin-triplet (odd-parity) SCs.

But with SO interaction, spin-singlet SCs can be topological as well.
Gapless topological phase in superconductors

MS-Fujimoto, Phys. Rev. Lett. 105, 217001 (10)
Sasaki, Kriener, Segawa, Yada, Tanaka, MS, Ando, Phys. Rev. Lett. 107, 217001 (11)
Motivation

We usually suppose full-gapped bulk spectrum for topological SCs. However, unconventional SCs often support bulk nodes in the gap function.

Can we use such nodal SCs to realize Majorana fermion?

Yes!
We find two classes of topological SCs with gap nodes.

1. 2D time-reversal breaking topological nodal SCs
   MS-Fujimoto, Phys. Rev. Lett. 105, 217001 (10)

2. 3D time-reversal invariant topological nodal SCs
   Sasaki, Kriener, Segawa, Yada, Tanaka, MS, Ando, Phys. Rev. Lett. 107, 217001 (11)

- They support non-zero bulk topological # defined in the entire space of the BZ.
- Existence of Majorana fermions on the boundary

“Strong” topological SC
2D Time-reversal breaking topological nodal SC

Model: 2d d-wave superconductor with Rashba SO int

\[ \mathcal{H}(\mathbf{k}) = \begin{pmatrix} \epsilon_k - h\sigma_z + g_k \cdot \sigma & i\Delta_k \sigma_y \\ -i\Delta_k \sigma_y & -\epsilon_k + h\sigma_z + g_k \cdot \sigma \end{pmatrix} \]

Zeeman

Rashba SO

dx^2-y^2-wave gap function

\[ \Delta_k = \Delta_0 (\cos k_x - \cos k_y) \]

Dxy-wave gap function

\[ \Delta_k = \Delta_0 \sin k_x \sin k_y \]
To understand what happens, we use the dual transformation again

$$
\mathcal{H}(k) = \begin{pmatrix}
\epsilon_k - \hbar \sigma_z + \mathbf{g}_k \cdot \sigma & i \Delta_k \sigma_y \\
-i \Delta_k \sigma_y & -\epsilon_k + \hbar \sigma_z + \mathbf{g}_k \cdot \sigma
\end{pmatrix}
$$

$$
\mathcal{H}^D(k) = D \mathcal{H}(k) D^\dagger, \quad D = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & i \sigma_y \\
\sigma_y & 1
\end{pmatrix}
$$

$$
\mathcal{H}^D(k) = \begin{pmatrix}
\Delta_k - \hbar \sigma_z & -i\epsilon_k \sigma_y - i \mathbf{g}_k \cdot \sigma \sigma_y \\
i \epsilon_k \sigma_y + i \mathbf{g}_k \sigma_y \sigma & -\Delta_k + \hbar \sigma_z
\end{pmatrix}
$$

p-wave gap is induced by Rashba SO int.
Edge state

\[(\hbar^2 > \mu^2)\]

dx²-y² – wave gap function

\[\text{dx}_y \text{dy} \text{– wave gap function}\]
There also exist a Majorana zero mode in a vortex

\[
\gamma = \int \left[ u_\uparrow \psi_\uparrow \dagger + u_\downarrow \psi_\downarrow \dagger + u_\uparrow^* \psi_\uparrow + u_\downarrow^* \psi_\downarrow \right]
\]

\[
u_\uparrow = ie^{i \frac{n-1}{2} \theta} f(r), \quad u_\downarrow = -ie^{i \frac{n+1}{2} \theta} f(r)
\]

\[
f(r) = \sqrt{\frac{h}{\pi \lambda r}} e^{-\frac{h}{2\lambda} r}
\]

Zero mode satisfies Majorana condition! \( \gamma^\dagger = \gamma \)

Non-Abelian anyon
The Majorana zero mode is stable against nodal excitations

From the particle-hole symmetry, zero modes become massive in pair.

At least one Majorana zero mode survives
The nodal excitations may change the finite size effect

Long-range tunneling between two edges due to the nodal excitations
Topological phase in 2D TRB nodal SCs is characterized by the parity of the Chern number \((-1)^{\nu_{Ch}}\)

\[
\begin{align*}
(-1)^{\nu_{Ch}} &= -1 & (-1)^{\nu_{Ch}} &= 1 \\
\text{There exist an odd number of gapless Majorana fermions} & + \text{nodal excitation} & \text{There exist an even number of gapless Majorana fermions} & + \text{nodal excitation} \\
\text{Topologically stable Majorana fermion} & & \text{No Majorana fermion survives} 
\end{align*}
\]

[MS, Fujimoto PRL (10)]
3D time-reversal invariant topological nodal SC

[Sasaki, Kriener, Segawa, Yada, Tanaka, MS, Ando PRL (11)]

\[ \text{Cu}_x \text{Bi}_2 \text{Se}_3 \]

Superconducting only for \( 0.10 \leq x \leq 0.30 \)

Hor et al., PRL (2010)

Recent measurement of tunneling conductance shows a pronounced zero-bias conductance peak

Evidence of Majorana fermion on the surface

[Sasaki, Kriener, Segawa, Yada, Tanaka, MS, Ando PRL (11)]
Proposed gap functions \cite{(10)}

<table>
<thead>
<tr>
<th></th>
<th>gap type</th>
<th>parity</th>
<th>energy gap structure</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta_1$</td>
<td>$-\Delta \uparrow \downarrow = \Delta \uparrow \downarrow = \Delta \uparrow \downarrow = -\Delta \uparrow \downarrow = \Delta \uparrow \downarrow = \Delta \uparrow \downarrow = \Delta \uparrow \downarrow$</td>
<td>even</td>
<td>full gap</td>
</tr>
<tr>
<td>$\Delta_2$</td>
<td>$-\Delta \uparrow \downarrow = \Delta \uparrow \downarrow = \Delta \uparrow \downarrow = -\Delta \uparrow \downarrow = \Delta \uparrow \downarrow = \Delta \uparrow \downarrow = \Delta \uparrow \downarrow$</td>
<td>odd</td>
<td>full gap</td>
</tr>
<tr>
<td>$\Delta_3$</td>
<td>$\Delta \uparrow \downarrow = \Delta \uparrow \downarrow = -\Delta \uparrow \downarrow = \Delta \uparrow \downarrow = \Delta \uparrow \downarrow = \Delta \uparrow \downarrow = \Delta \uparrow \downarrow$</td>
<td>odd</td>
<td>point node</td>
</tr>
<tr>
<td>$\Delta_4$</td>
<td>$\Delta \uparrow \downarrow = \Delta \uparrow \downarrow = -\Delta \uparrow \downarrow = \Delta \uparrow \downarrow = \Delta \uparrow \downarrow = \Delta \uparrow \downarrow = \Delta \uparrow \downarrow$</td>
<td>odd</td>
<td>point node</td>
</tr>
</tbody>
</table>

- $\Delta_2$ is full gapped and topological \cite{(10), (10)}
- $\Delta_3$ and $\Delta_4$ are nodal but topological \cite{(10)}

The both cases can be consistent with the experimental result for tunneling conductance.
Surface state of nodal topological SC

\[ \Delta_4 \ (111) \] surface

Deformed Majorana fermion

However, to determine the actual gap function, we need a further theoretical investigation on the tunneling conductance.

[Sasaki et al. PRL (11)]
Topological phase in 3D TRI nodal SCs is characterized by the parity of 3d winding number (= mod 2 winding number) \((-1)^{\nu_w}\).

Full gapped SC [Schnyder et al (08)]

<table>
<thead>
<tr>
<th>Time-reversal breaking SC</th>
<th>class D</th>
<th>2 dim</th>
<th>Chern #</th>
<th>Parity of Chern #</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time-reversal invariant SC</td>
<td>class DIII</td>
<td>2 dim</td>
<td>(\mathbb{Z}_2 # \mathbb{Z}_2)</td>
<td>(\mathbb{Z}_2 # \mathbb{Z}_2)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3 dim</td>
<td>3d winding #</td>
<td>(\mathbb{Z}_2)</td>
</tr>
</tbody>
</table>

Nodal SC

[MS-Fujimoto (10), STSYTSA (11)]
Summary

With SO interaction, various superconductors become topological superconductors

1. Majorana fermion in spin singlet SCs

2. Majorana fermion in nodal SC
Reference

• Non-Abelian statistics of axion strings, by MS, Phys. Lett. B575, 126(2003),

• Topological Phases of Noncentrosymmetric Superconductors: Edge States, Majorana Fermions, and the Non-Abelian statistics, by MS, S. Fujimoto, PRB79, 094504 (2009),

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• Non-Abelian Topological Orders and Majorna Fermions in Spin-Singlet Superconductors, by MS, Y. Takahashi, S.Fujimoto, PRB 82, 134521 (2010) (Editor’s suggestion)

• Existence of Majorana fermions and topological order in nodal superconductors with spin-orbit interactions in external magnetic field, PRL105,217001 (2010)

• Anomalous Andreev bound state in Noncentrosymmetric superconductors, by Y. Tanaka, Mizuno, T. Yokoyama, K. Yada, MS, PRL105, 097002 (2010)

• Surface density of states and topological edge states in noncentrosymmetric superconductors by K. Yada, MS, Y. Tanaka, T. Yokoyama, PRB83, 064505 (2011)

• Topology of Andreev bound state with flat dispersion, MS, Y. Tanaka, K. Yada, T. Yokoyama, PRB 83, 224511 (2011)

Topological \# in nodal SC

Formally, the bulk topological \# in nodal SCs can be defined after removing the gap node by perturbation.

\[ \varepsilon(k) \]

However, sometimes, the resultant bulk topological \# depends on the perturbation.

\[ \Delta_k \rightarrow \Delta_k + i\epsilon\Delta_k' \]

(e.g.) \[ d_{xy} \rightarrow d_{xy} + i\epsilon d_{x^2-y^2}, \quad d_{xy} \rightarrow d_{xy} + i\epsilon s \]
However, the parity of the topological # does not have the ambiguity.

- Because of the particle-hole symmetry, nodes are paired in the momentum space.
- Each node may give an ambiguity, but the total ambiguity of the topological # should be even since nodes are paired.
- We have a unique value of the parity of the topological #