Identifying the Geometry of the MSSM

Brent D. Nelson
Northeastern University

with James Gray, Yang-Hui He & Vishnu Jejjala

Physics Letters **B638** (2006) 253 [hep-th/0511062]
Nuclear Physics **B750** (2006) 1 [hep-th/0604208]

August 31, 2006

A Proposal for a New Approach

We propose to search for unexplained structure in the geometry of the vacuum spaces of supersymmetric theories

$\Rightarrow$ Supersymmetric quantum field theories have scalars $\rightarrow$ a complicated vacuum space of possible field *vevs* $\langle\phi_i\rangle$

- The vacuum manifold, or moduli space $\mathcal{M}$, generally characterized by certain flat directions
- Efforts in the past to understand how these flat directions are “lifted”
- This manifold $\mathcal{M}$ may have special structure that correlates with certain phenomenological properties – but NOT related to gauge invariance or discrete symmetries

Procedure I: Determine the Vacuum Conditions

⇒ So how does one determine the geometry of the vacuum space $\mathcal{M}$?

- Consider a general $N = 1$ supersymmetric system defined by

$$S = \int \mathrm{d}^4 x \left[\int \mathrm{d}^4\theta \; \Phi_i^\dagger e^V \Phi_i + \left(\frac{1}{4g^2} \int \mathrm{d}^2\theta \; \mathrm{Tr} \mathcal{W}_\alpha \mathcal{W}^\alpha + \int \mathrm{d}^2\theta \; W(\Phi) + \text{ h.c.} \right) \right]$$

- The scalar potential can be found from the component form of the above

$$V(\phi_i, \bar{\phi}_i) = \sum_i \left| \frac{\partial W}{\partial \phi_i} \right|^2 + \frac{g^2}{4} \left(\sum_i q_i |\phi_i|^2 \right)^2$$

where ϕ_i is the lowest (scalar) component of superfield Φ_i with charge q_i

- Vacuum configuration is any set of field values $\{\phi_i^0\}$ such that $V(\phi_i^0, \bar{\phi}_i^0) = 0$

⇒ This implies the following relations:

$$\frac{\partial W}{\partial \phi_i} = 0 \quad \text{F-TERMS}; \quad \sum_i q_i |\phi_i|^2 = 0 \quad \text{D-TERMS}$$

⇒ The vacuum moduli space $\mathcal{M}$ is the space of all possible solutions ϕ^0 to these F and D-flatness conditions

Procedure II: Set up an Appropriate Basis

$\Rightarrow$ To every solution of the F-flatness conditions there exists a solution to the D-flatness conditions in the orbit of the complexified gauge group $\mathcal{G}^C$:

$$\boxed{\mathcal{M} = \mathcal{F}//\mathcal{G}^C}$$

where $\mathcal{F}$ is the space of all F-flat field configurations

Procedure II: Set up an Appropriate Basis

⇒ To every solution of the F-flatness conditions there exists a solution to the D-flatness conditions in the orbit of the complexified gauge group $\mathcal{G}^C$:

$$\boxed{\mathcal{M} = \mathcal{F}//\mathcal{G}^C}$$

where $\mathcal{F}$ is the space of all F-flat field configurations

⇒ More practically speaking, the procedure involves the following:

1. Take a theory defined by a superpotential $W = W(\Phi_1, \Phi_2, \ldots, \Phi_n)$
2. Set up a basis of gauge invariant operators (GIOs) $D = \{D_1, D_2, \ldots, D_k\}$
3. Determine the n F-flatness conditions given by $\partial W / \partial \phi_i = 0$
4. Find the set $\tilde{n} \leq n$ of independent relations defined in (3)
5. Use these to eliminate $\tilde{n}$ fields in the GIOs

$$D_k(\phi_1, \ldots, \phi_n) \to D_k(z_i, \ldots, z_n)$$

Procedure III: Find $\mathcal{M}$ as an Algebraic Variety

$\Rightarrow$ The various D_k form the coordinates of $\mathcal{M}$

- These coordinates will NOT (in general) be independent
- Let $\mathrm{Eq}(\mathcal{M})$ be the set of all algebraic relations amongst these D_k

  $\Rightarrow$ $\mathrm{Eq}(\mathcal{M})$ *defines* $\mathcal{M}$ as an algebraic variety

$\Rightarrow$ To *identify* the manifold, we want to know $\mathrm{Eq}(\mathcal{M})$; i.e. want to build the quotient ring explicitly

- The building of the quotient ring is a manifestation of the syzygy problem
- Huge subject in mathematics barely touched by physics
- A generalization of finding divisors for a given polynomial
- `Macaulay 2` and `Singular` can solve this problem using a Groebner bases algorithm; already includes technology for performing ring maps

Attacking the MSSM

$\Rightarrow$ Seven species of chiral superfields $\Rightarrow$ 49 scalar fields ($n = 49$)

$\Rightarrow$ All 991 possible GIOs tabulated below ($k = 991$)

T. Gherghetta, C. Kolda, S. Martin, *Nucl. Phys.*, **B468** (1996)

Operator	Explicit Sum	Index	Number
LH_u	$L_i^\alpha H^\beta \epsilon_{\alpha\beta}$	$i = 1, 2, 3$	3
$H_u H_d$	$H_\alpha (H_d)_\beta \epsilon^{\alpha\beta}$	NA	1
LLe	$L_\alpha^i L_\beta^j e^k \epsilon^{\alpha\beta}$	$i, j = 1, 2, 3;\ k = 1, \ldots, j-1$	9
$LH_d e$	$L_\alpha^i (H_d)_\beta e^j \epsilon^{\alpha\beta}$	$i, j = 1, 2, 3$	9
udd	$u_a^i d_b^j d_c^k \epsilon^{abc}$	$i, j = 1, 2, 3;\ k = 1, \ldots, j-1$	9
QdL	$Q_{a,\alpha}^i d_a^j L_\beta^k \epsilon^{\alpha\beta}$	$i, j, k = 1, 2, 3$	27
QuH_u	$Q_{a,\alpha}^i u_a^j (H_u)_\beta \epsilon^{\alpha\beta}$	$i, j = 1, 2, 3$	9
QdH_d	$Q_{a,\alpha}^i d_a^j (H_d)_\beta \epsilon^{\alpha\beta}$	$i, j = 1, 2, 3$	9
$QQQL$	$Q_{a,\beta}^i Q_{b,\gamma}^j Q_{c,\alpha}^k L_\delta^l \epsilon^{abc} \epsilon^{\beta\gamma} \epsilon^{\alpha\delta}$	$i, j, k, l = 1, 2, 3;\ i \neq k, j \neq k,$ $j < i, (i, j, k) \neq (3, 2, 1)$	24
$QuQd$	$Q_{a,\alpha}^i u_a^j Q_{b,\beta}^k d_b^l \epsilon^{\alpha\beta}$	$i, j, k, l = 1, 2, 3$	81
$QuLe$	$Q_{a,\alpha}^i u_a^j L_\beta^k e^l \epsilon^{\alpha\beta}$	$i, j, k, l = 1, 2, 3$	81
$uude$	$u_a^i u_b^j d_c^k e^l \epsilon^{abc}$	$i, j, k, l = 1, 2, 3;\ j < i$	27
$QQQH_d$	$Q_{a,\beta}^i Q_{b,\gamma}^j Q_{c,\alpha}^k (H_d)_\delta \epsilon^{abc} \epsilon^{\beta\gamma} \epsilon^{\alpha\delta}$	$i, j, k, l = 1, 2, 3; i \neq k, j \neq k,$ $j < i, (i, j, k) \neq (3, 2, 1)$	8
$QuH_d e$	$Q_{a,\alpha}^i u_a^j (H_d)_\beta e^k \epsilon^{\alpha\beta}$	$i, j, k = 1, 2, 3$	27
$dddLL$	$d_a^i d_b^j d_c^k L_\alpha^m L_\beta^n \epsilon^{abc} \epsilon_{ijk} \epsilon^{\alpha\beta}$	$m, n = 1, 2, 3;\ n < m$	3

$i, j, k = 1, 2, 3 \leftrightarrow$ flavor indices, $a, b, c = 1, 2, 3 \leftrightarrow$ color indices, $\alpha, \beta, \gamma = 1, 2 \leftrightarrow SU(2)_L$ indices

Attacking the MSSM

Operator	Explicit Sum	Index	Number
$uuuee$	$u^i_a u^j_b u^k_c e^m e^n \epsilon^{abc}\epsilon_{ijk}$	$m,n=1,2,3;\ n\le m$	6
$QuQue$	$Q^i_{a,\alpha} u^j_a Q^k_{b,\beta} u^m_b e^n \epsilon_{\alpha\beta}$	$i,j,k,m,n=1,2,3;$ as$\{(i,j),(k,m)\}$	108
$QQQQu$	$Q^i_{a,\beta} Q^j_{b,\gamma} Q^k_{c,\alpha} Q^m_{f,\delta} u^n_f \epsilon^{abc}\epsilon^{\beta\gamma}\epsilon^{\alpha\delta}$	$i,j,k,m=1,2,3;\ i\neq m,$ $j\neq m, j<i,$ $(i,j,k)\neq(3,2,1)$	72
$dddLH_d$	$d^i_a d^j_b d^k_c L^m_\alpha (H_d)_\beta \epsilon^{abc}\epsilon_{ijk}\epsilon_{\alpha\beta}$	$m=1,2,3$	3
$uudQdH_u$	$u^i_a u^j_b d^k_c Q^m_{f,\alpha} d^n_f (H_u)_\beta \epsilon^{abc}\epsilon_{\alpha\beta}$	$i,j,k,m=1,2,3;\ j<i$	81
$(QQQ)_4 LLH_u$	$(QQQ)_4^{\alpha\beta\gamma} L^m_\alpha L^n_\beta (H_u)_\gamma$	$m,n=1,2,3;\ n\le m$	6
$(QQQ)_4 LH_uH_d$	$(QQQ)_4^{\alpha\beta\gamma} L^m_\alpha (H_u)_\beta (H_d)_\gamma$	$m=1,2,3$	3
$(QQQ)_4 H_uH_dH_d$	$(QQQ)_4^{\alpha\beta\gamma} (H_u)_\alpha (H_d)_\beta (H_d)_\gamma$	NA	1
$(QQQ)_4 LLLe$	$(QQQ)_4^{\alpha\beta\gamma} L^m_\alpha L^n_\beta L^p_\gamma e^q$	$m,n,p,q=1,2,3;$ $n\le m;\ p\le n$	27
$uudQdQd$	$u^i_a u^j_b d^k_c Q^m_{f,\alpha} d^n_f Q^p_{g,\beta} d^q_g \epsilon^{abc}\epsilon_{\alpha\beta}$	$i,j,k,m,n,p,q=1,2,3;$ $j<i,$ as$\{(m,n),(p,q)\}$	324
$(QQQ)_4 LLH_de$	$(QQQ)_4^{\alpha\beta\gamma} L^m_\alpha L^n_\beta (H_d)_\gamma e^p$	$m,n,p=1,2,3;\ n\le m$	9
$(QQQ)_4 LH_dH_de$	$(QQQ)_4^{\alpha\beta\gamma} L^m_\alpha (H_d)_\beta (H_d)_\gamma e^n$	$m,n=1,2,3$	9
$(QQQ)_4 H_dH_dH_de$	$(QQQ)_4^{\alpha\beta\gamma} (H_d)_\alpha (H_d)_\beta (H_d)_\gamma e^m$	$m=1,2,3$	3

In the above we defined $[(QQQ)_4]_{\alpha\beta\gamma} = Q^i_{a,\alpha} Q^j_{b,\beta} Q^k_{c,\gamma} \epsilon^{abc}\epsilon^{ijk}$

$\Rightarrow$ The reason the problem has languished for a decade...

Attacking the MSSM

⇒ Superpotential we would ultimately like to study is given by

$$\begin{aligned} W_{\mathrm{MSSM}} &= \lambda^0 H_u H_d + \lambda^1_{ij} Q_i H_u u_j + \lambda^2_{ij} Q_i H_d d_j + \lambda^3_{ij} L_i H_d e_j \\ &= \lambda^0 \sum_{\alpha,\beta} H_u^\alpha H_d^\beta \epsilon_{\alpha\beta} + \sum_{i,j} \lambda^1_{ij} \sum_{\alpha,\beta,a} Q^i_{a,\alpha} (H_u)_\beta u^j_a \epsilon_{\alpha\beta} \\ &\quad + \sum_{i,j} \lambda^2_{ij} \sum_{\alpha,\beta,a} Q^i_{a,\alpha} (H_d)_\beta d^j_a \epsilon_{\alpha\beta} + \sum_{i,j} \lambda^3_{ij} \sum_{\alpha,\beta} L^i_\alpha (H_d)_\beta e^j \epsilon_{\alpha\beta} \end{aligned}$$

⇒ The matrices λ_{ij} are flavor mixing matrices

- In explicit computations they are randomly generated matrices
- Dimensionality of some coefficients suppressed (irrelevant for topology)

Attacking the MSSM

⇒ Superpotential we would ultimately like to study is given by

$$\begin{aligned} W_{\rm MSSM} &= \lambda^0 H_u H_d + \lambda^1_{ij} Q_i H_u u_j + \lambda^2_{ij} Q_i H_d d_j + \lambda^3_{ij} L_i H_d e_j \\ &= \lambda^0 \sum_{\alpha,\beta} H_u^\alpha H_d^\beta \epsilon_{\alpha\beta} + \sum_{i,j} \lambda^1_{ij} \sum_{\alpha,\beta,a} Q^i_{a,\alpha} (H_u)_\beta u^j_a \epsilon_{\alpha\beta} \\ &\quad + \sum_{i,j} \lambda^2_{ij} \sum_{\alpha,\beta,a} Q^i_{a,\alpha} (H_d)_\beta d^j_a \epsilon_{\alpha\beta} + \sum_{i,j} \lambda^3_{ij} \sum_{\alpha,\beta} L^i_\alpha (H_d)_\beta e^j \epsilon_{\alpha\beta} \end{aligned}$$

⇒ The matrices λ_{ij} are flavor mixing matrices

- In explicit computations they are randomly generated matrices
- Dimensionality of some coefficients suppressed (irrelevant for topology)

⇒ Quotient space far too large and complicated for current methods

- Largest success thus far involved 25 GIOs
- Computational load scales rapidly with $\dim(\mathcal{M})$ for computing topological information

One Generation MSSM

⇒ Drop all flavor indices ($i = j = k = 1$) so now $n = 7$

⇒ There are now only 9 GIOs (one of each variety)

$$LH_u,\ H_uH_d,\ QdL,\ QuH_u,\ QdH_d,\ LH_de,\ QuQd,\ QuLe,\ QuH_de$$

⇒ Simplified superpotential

$$\begin{aligned} W_0 &= \lambda^0 \sum_{\alpha,\beta} H_u^\alpha H_d^\beta \epsilon_{\alpha\beta} + \lambda^1 \sum_{\alpha,\beta,a} Q_{a,\alpha}(H_u)_\beta u_a \epsilon^{\alpha\beta} \\ &\quad + \lambda^2 \sum_{\alpha,\beta,a} Q_{a,\alpha}(H_d)_\beta d_a \epsilon^{\alpha\beta} + \lambda^3 \sum_{\alpha,\beta} L_\alpha (H_d)_\beta e \epsilon^{\alpha\beta} \end{aligned}$$

⇒ Computation of vacuum manifold $\mathcal{M}$ for various deformations

$W_0 + ?$	$\dim(\mathcal{M})$	$\mathcal{M}$	$W_0 + ?$	$\dim(\mathcal{M})$	$\mathcal{M}$
0	1	$\mathbb{C}$	$QuQd$	1	$\mathbb{C}$
LH_u	0	point	$QuLe$	1	$\mathbb{C}$
QdL	0	point	QuH_de	1	$\mathbb{C}$

MSSM Electroweak Sector I

⇒ Set vevs for u_L^i, u_R^i, d_L^i, d_R^i to zero by hand

⇒ This leaves $n = 13$ scalar fields and $k = 22$ GIOs

Operator	Explicit Sum	Index	Number
LH_u	$L_i^\alpha H^\beta \epsilon_{\alpha\beta}$	$i = 1, 2, 3$	3
$H_u H_d$	$H_\alpha (H_d)_\beta \epsilon^{\alpha\beta}$	NA	1
LLe	$L_\alpha^i L_\beta^j e^k \epsilon^{\alpha\beta}$	$i, j = 1, 2, 3;\ k = 1, \ldots, j-1$	9
$LH_d e$	$L_\alpha^i (H_d)_\beta \epsilon^{\alpha\beta} e^j$	$i, j = 1, 2, 3$	9

$$W_0 = \lambda^0 H_u H_d + \lambda_{ij}^3 L_i H_d e_j = \lambda^0 \sum_{\alpha,\beta} H_u^\alpha H_d^\beta \epsilon_{\alpha\beta} + \sum_{i,j} \lambda_{ij}^3 \sum_{\alpha,\beta} L_\alpha^i (H_d)_\beta e^j \epsilon_{\alpha\beta}$$

⇒ Computation of vacuum manifold $\mathcal{M}$ for various deformations

$W_0+?$	$\dim(\mathcal{M})$	$\mathcal{M}$	$W_0+?$	$\dim(\mathcal{M})$	$\mathcal{M}$
0	5	cone over $(\mathbb{CP}^8 \vert 6 \vert 2^6)$	LLe	0	point
LH_u	1	$\mathbb{C}$	$LLe + LH_u$	0	point

⇒ Affine cone over base manifold $\mathcal{B}$ with $\dim(\mathcal{B}) = 4$ formed by non-complete intersection of six quadratics in $\mathbb{CP}^8$

MSSM Electroweak Sector II

⇒ Next logical choice of deformation is dimension four terms which lift the Higgs directions:

$$W_1 = W_0 + \lambda'(H_u^\alpha H_d^\beta \epsilon_{\alpha\beta})^2 + \lambda''_{ij}(L^i H_u^\alpha)(L^j H_d^\beta)\epsilon_{\alpha\beta}$$

- We find that $\dim(\mathcal{M}) = 3$....*interesting!*
- The manifold $\mathcal{M}$ is an affine cone over a compact, two-dimensional base $\mathcal{B}$
- This base is the non-complete degree 4 intersection of 6 quadrics in $\mathbb{CP}^5$ as a projective variety

⇒ Consider the simplest geometrical information about this surface, the Hodge diamond

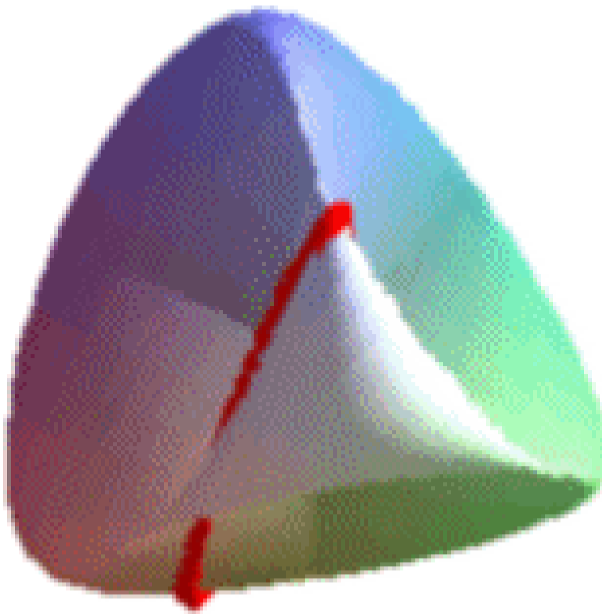
$$h^{p,q}(B) = \begin{matrix} & & h^{0,0} & & \\ & h^{0,1} & & h^{0,1} & \\ h^{0,2} & & h^{1,1} & & h^{0,2} \\ & h^{0,1} & & h^{0,1} & \\ & & h^{0,0} & & \end{matrix} \longrightarrow \begin{matrix} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 1 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{matrix}$$

⇒ No explanation for the simplicity of this structure from field theory

MSSM Electroweak Sector III

- This manifold turns out to be one of the simplest you can imagine: the Veronese surface embedding $\mathbb{CP}^2$ in $\mathbb{CP}^5$

Giuseppe Veronese



The Veronese Surface

Interpretation...and Future Directions

$\Rightarrow$ Ultimate goal: provide a guide-book of "target" geometries for top-down explicit string constructions

Interpretation...and Future Directions

$\Rightarrow$ Ultimate goal: provide a guide-book of “target” geometries for top-down explicit string constructions

$\Rightarrow$ Short-term goal: A new principle for low-energy phenomenology?

- Any special geometry of the vacuum moduli space $\mathcal{M}$ should be regarded as fundamental
- Any deformation of the gauge theory should be restricted to those which enhance/preserve the features of $\mathcal{M}$
- Divide theories into “conjugacy classes” on the basis of their common geometrical structres
- Guide to bottom-up model building akin to “naturalness” or fine-tuning