EXACT COUNTING OF 4D BPS BLACK HOLES

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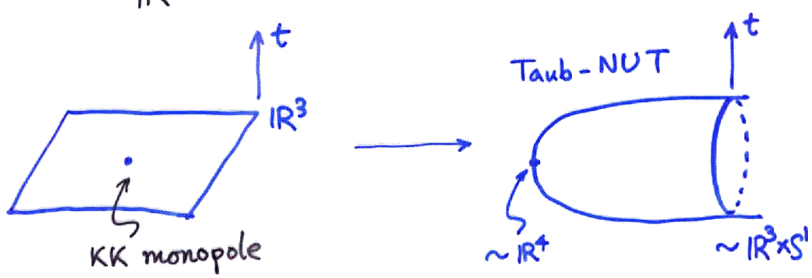
PLAN OF TALK

- 4D/5D Connection
- 1/4 BPS BHs in $\mathcal{N}=4$ string theory
- 1/8 BPS BHs in $\mathcal{N}=8$ string theory
- Comparison to topological strings

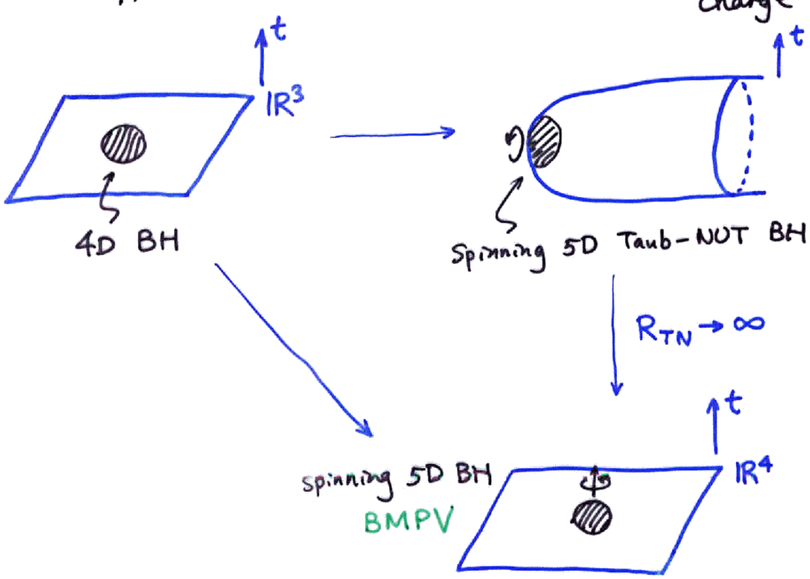
THE 4D/5D CONNECTION

[Gaiotto, Strominger, XY]

$$\mathbb{R}^{1,4} \xrightarrow{KK} \mathbb{R}^{1,3} \times S^1$$



A 4D BH with 1 unit of KK monopole charge?



Type IIA string theory on a CY

BPS BH $\longleftrightarrow$ D0 - D2 - D4 - D6 system

charge: $(q_0, q_A; p^A, p^0)$

D6 wrapped on CY

$\to$ KK monopole charge

(M-theory on $CY \times S^1$)

CONSIDER D0 - D2 - D6 with a single D6-brane,

PROPOSAL:

$$Z_{4D}(p^0 = 1, q_0, q_A) = Z_{5D}\left(q_A, J_L = \frac{q_0}{2}\right)$$

Subtlety: 4D index $\mathrm{Tr}_{BPS}(-)^{2J^3}$

5D index $\mathrm{Tr}_{BPS}(-)^{2J_L^3 + 2J_R^3}$

$J^3 \longleftrightarrow J_R^3$

$\frac{q_0}{2} \longleftrightarrow J_L^3$

$$Z_{5D} = (-)^{q_0} Z_{4D}$$

CLASSICAL ENTROPY

D0-D2-D6

$$S_{4D} = 2\pi\sqrt{p^0 Q^3 - \frac{1}{4}(p^0 q_0)^2},$$

$$Q^3 \equiv (D_{ABC}\, y^A y^B y^C)^2,$$
$$q_A = 3 D_{ABC}\, y^B y^C.$$

$$\Big\downarrow p^0=1$$

$$S_{4D} = 2\pi\sqrt{Q^3 - \frac{1}{4} q_0^2}$$

BMPV

$$S_{5D} = 2\pi\sqrt{Q^3 - J^2}$$

$\mathcal{N}=4$ String theory

- IIA on $K3 \times T^2$

[Shih, Strominger, XY]

A 1/4-BPS BH

$$\underset{(p^0=1)}{\text{D0-D2-D6}} \xrightarrow{\text{lift to 5D}} \text{M2 on } K3\times T^2 \text{ w/ } J_L$$

$\Big\downarrow$ compactify on $S^1 \subset T^2$

IIA: D2 on K3
+ F1 on S^1
+ J_L

$\xleftarrow{\text{T-duality}}$

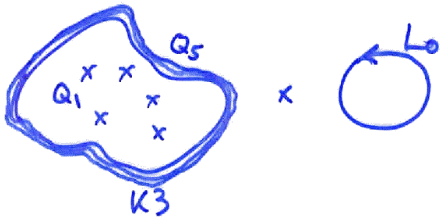
IIB:
D3 on $K3 \times S^1$
+ momentum along S^1
+ J_L

$\Big\downarrow$ U-duality

IIB:
D1-D5 on $K3\times S^1$
+ momentum along S^1
+ J_L

D1-D5 on $K3 \times S^1$

$\longrightarrow \mathrm{Sym}^{Q_1 Q_5}(K3)$ CFT



elliptic genus:

$$\chi_{N=Q_1Q_5}(y,q) = \mathrm{Tr}\,(-)^F y^{F_L} q^{L_0}$$

4D/5D Connection

$$\Rightarrow \quad Z_{4D} = Z_0 \cdot Z_{D1-D5}$$

partition function of D5 on $K3 \times S^1$

$\updownarrow$

winding modes of a heterotic string

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$$Z_{D1-D5} = \sum d_{5D}(L_0, \overset{Q_1Q_5}{N}, J_L)\, e^{2\pi i(L_0\rho + N\sigma + 2J_L\nu)}$$

$$= \prod_{\substack{k\geq 0,\, l>0 \\ m\in\mathbb{Z}}} \left(1 - e^{2\pi i(k\rho + l\sigma + m\nu)}\right)^{c(4kl-m^2)}$$

Dijkgraaf, Moore, Verlinde, Verlinde

$c(4kl-m^2)$: coefficients of elliptic genus of a single K3

$$Z_0 = (e^{\pi i\nu} - e^{-\pi i\nu})^{-2} e^{-2\pi i\rho} \times \prod_{n\geq 1} (1-e^{2\pi i(n\rho+\nu)})^{-2}(1-e^{2\pi i(n\rho-\nu)})^{-2} \times (1-e^{2\pi i n\rho})^{-20}$$

Antoniadis, Gava, Narain, Taylor

$$Z_0 \cdot Z_{D1-D5} = \frac{e^{2\pi i\sigma}}{\Phi(\rho,\sigma,\nu)}$$

shift of Q_1Q_5 by -1 due to anomalous charge

weight 10 modular form of $Sp(2,\mathbb{Z})$

Dijkgraaf, Verlinde, Verlinde

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$\mathcal{N}=4$ string theory has U-duality group

$$SL(2,\mathbb{Z}) \times O(6,22;\mathbb{Z})$$

charge vectors: (ignoring F1 and NS5 charges)

$$q_e = (\underset{\substack{\uparrow\\ D0}}{q_0},\ \underset{\substack{\uparrow\\ D2 \text{ on } K3}}{q_1,\dots,q_{22}},\ \underset{\substack{\uparrow\\ D4 \text{ on } K3}}{p^{23}})$$

$$q_m = (\underset{\substack{\uparrow\\ D6}}{p^0},\ \underset{\substack{\uparrow\\ D4 \text{ on } \alpha\times T^2,\ \alpha\subset K3}}{p^1,\dots,p^{22}},\ \underset{\substack{\uparrow\\ D2 \text{ on } T^2}}{q_{23}})$$

BH degeneracy

$$d_{4D}(q_e,q_m) = d(\underset{\substack{\updownarrow\\ L_0}}{q_e^2},\ \underset{\substack{\updownarrow\\ Q_1Q_5}}{q_m^2},\ \underset{\substack{\updownarrow\\ J_L}}{q_e\cdot q_m})$$

$$Z_{4D} = \sum_{k,l,m} d(k,l,m)\, e^{2\pi i(k\rho + l\sigma + m\nu)}$$

$$= \frac{1}{\Phi(\rho,\sigma,\nu)}$$

8.5

$\mathcal{N}=8$ string theory [Shih, Strominger, XY]

- IIA on T^6

Similar chain of dualities as in $\mathcal{N}=4$ case

$$\Rightarrow \quad Z_{4D} = Z_{D1\text{-}D5}$$

$$= Z_{\mathrm{Sym}^{Q_1Q_5}(T^4)}$$

- $Z_{D1\text{-}D5}$ is given by a modified elliptic genus $\quad \mathrm{Tr}\,(-)^F (J_R^3)^2\, y^{2J_L^3}\, q^{L_0}\, \bar{q}^{\bar{L}_0}$

Maldacena, Moore, Strominger

Correspondingly

$$Z_{4D} = \mathrm{Tr}_{BPS}\, (J^3)^2\, (-)^{2J^3}$$

- Z_0 is absent - part. fn. of D5 on $T^4\times S^1$ trivial

- Need to assume coprime set of $(N=Q_1Q_5,\ L_0,\ J_L)$ to get Z_{4D} that is consistent with U-duality $E_{7,7}$

$$Z_{4D} = \sum \underset{\substack{\uparrow\\ \text{Cremmer - Julia invariant}}}{d(J)}\, q^J$$

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ANSWER: (from 5D)

$$Z_{4D} = \eta(q^4)^{-6} \sum_{m\in\mathbb{Z}} q^{m^2}$$

A 4D DERIVATION?

- degeneracy depends only on $\mathcal{J}$

D4-D0 system

$$\mathcal{J} = 4q_0 D = 4q_0 p^1 p^2 p^3$$

CONSIDER $p^1 = p^2 = p^3 = 1$

$$\mathcal{J} = 4q_0$$

- $\mathcal{J} \equiv 0$ or -1 mod 4

  The other case can be obtained by turning on D2-brane charge

- D0 partition function is then

$$Z(q) = \sum_n d(\mathcal{J} = 4n)\, q^n = \eta(q)^{-6} \sum_{m\in\mathbb{Z}} q^{(m+\frac{1}{2})^2}$$

D4 wrapped on (1,1,1) cycle $P\subset T^6$

$P \subset T^6$ cplx submanifold for generic (tilted) T^6 (algebraic)

$$\begin{array}{ccccc} & & 1 & & \\ & 3 & & 3 & \\ 3 & & 10 & & 3 \\ & 3 & & 3 & \\ & & 1 & & \end{array}$$

$\chi(P) = 6$

$\sigma(P) = -2$

$b_1(P) = 6$

$b_2(P) = 16 = b_2(T^6) + 1$

- P has an extra 2-cycle γ not induced from T^6

  Can turn on gauge field flux without induced D2-brane charge but with induced D0-brane charge

  $\alpha_1, \alpha_2, \alpha_3$ 2-cycles induced from T^6

  $\gamma\cdot\alpha_i = 1, \quad \gamma\cdot\gamma = 1$

  $\beta \equiv 2\gamma - \sum \alpha_i, \quad \beta\cdot\alpha_i = 0, \quad \beta\cdot\beta = -2$

- Freed - Witten anomaly:

$$F = \frac{c_1(P)}{2} + \text{integral}$$

$$c_1(P) = -\sum\alpha_i$$

Allowed fluxes that do not induce D2-charge:

$$F = (m+\tfrac{1}{2})\beta \quad , \quad m \in \mathbb{Z}$$

Induce D0-brane charge

$$\Delta q_0 = -\int \frac{F^2}{2} + \frac{c_2(P)}{24}$$

$$= (m+\tfrac{1}{2})^2 - \tfrac{1}{4}$$

D4-D0 bound state:

D0 either dissolve into F

or be bound to D4 as instantons

$$\Rightarrow \quad Z = \eta(q)^{\overset{\chi(P)}{-6}} \sum_{m\in\mathbb{Z}} q^{(m+\frac{1}{2})^2}$$

agree with 5D!

A little more on D4-D0 on T^6

- D4 wraps on a holomorphic (1,1,1) cycle in T^6 $\Rightarrow$ T^6 a principally polarized Abelian variety

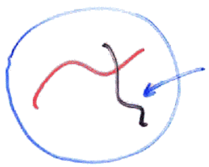
S_3 : genus 3 Riemann surface

$$S_3 \hookrightarrow J(S_3) \simeq T^6$$

D4 is wrapped on divisor $P_{(1,1,1)} = S_3 + S_3 \simeq \mathrm{Sym}^2 S_3$

$S_3 \subset P_{(1,1,1)}$ represents the class γ

- D4 wrapped on (p_1, p_2, p_3)? moduli space $\mathbb{P}^{p_1p_2p_3-1} (\times T^6)$



special locus in moduli space where D0's can dissolve into other kinds of F^- flux...

work in progress

MACROSCOPIC ENTROPY ?

$$S = \int_{S_H^2} \epsilon_{ab}\,\epsilon_{cd} \frac{\partial \mathcal{L}_{eff}}{\partial R_{abcd}}$$

Ward's formula

correction from $R^2 F^{2g-2}$ terms

· Cardoso, de Wit, Mohaupt

related to $\mathcal{F}_{top}(g_{top}, t^A)$ by Legendre transform

· Ooguri, Strominger, Vafa

· Sen

OSV Conjecture :

$$Z_{BH}(p,\phi) = \sum_{q} \Omega(p,q)\, e^{-q\cdot\phi} = |Z_{top}|^2$$

What are these?

Strategy : compute Z_{BH} from the exact degeneracy as given by our indices for $\mathcal{N}=4$ and $\mathcal{N}=8$ BHs, and compare to Z_{top}.

A simplification

We shall restrict to the case

$p^0 = 0$. i.e. no D6-brane charge

- simplifies calculations
- if $p^0 \neq 0$, classical entropy

$$S_{cl} \sim 2\pi \sqrt{p^0 q_A^3}$$

Summation

$$\sum_{q_A} e^{S_{cl}} e^{-q_A \phi^A}$$

"badly" divergent

if $p^0 = 0$, D4-D2-D0

$$S_{cl} \sim 2\pi \sqrt{D\left(q_0 + \tfrac{1}{12} D^{AB} q_A q_B\right)}$$

OSV summation can be regularized.

$\mathcal{N} = 8$

$$\Omega(p^\Lambda; q_\Lambda) = d(J(p^\Lambda; q_\Lambda))$$

$$\sum d(J)\, q^J = \eta(q^4)^{-6} \sum_{m \in \mathbb{Z}} q^{m^2}$$

charges:

electric q_0, q_A ← wrapped D2, KK-monopole, NS5,

magnetic p^0, p^A ← wrapped D4, momenta, F1

$p^0 = 0$

$$J = 4 D_{ABC} p^A p^B p^C \left(q_0 + \tfrac{1}{12} D^{AB} q_A q_B\right)$$

Compute mixed partition sum Z_{BH} (Poisson resummation)

$$Z_{BH}(p, \phi) = \sum_{\phi^\Lambda \to \phi^\Lambda + 2\pi i k^\Lambda} |Z_{top}|^2 \, |g_{top}|^8 \, V_{T6} \times \widetilde{F}\left(\frac{i V_{T6}}{|g_{top}|^2}\right)$$

$$\widetilde{F} = 1 + \mathcal{O}\left(e^{-V_{T6}/|g_{top}|^2}\right)$$

topological string on T^6

$$Z_{top} = e^{F_0}, \quad F_0 = \frac{i}{g_{top}^2} D_{ABC} t^A t^B t^C$$

$$= i D_{ABC} \frac{X^A X^B X^C}{X^0}$$

$$V_{T^6} = D_{ABC} \, \mathrm{Im}\, t^A \, \mathrm{Im}\, t^B \, \mathrm{Im}\, t^C.$$

$\mathcal{N}=4$

$(-)^{2J_L}$

$$\Omega(p,q) = \oint d\rho \, d\sigma \, d\nu \, \frac{e^{\pi i (\rho q_m^2 + \sigma q_e^2 + (2\nu - 1) q_e \cdot q_m)}}{\Phi(\rho, \sigma, \nu)}$$

DVV: calculate by contour integral around poles

- rational quadratic divisors

$Sp(2,\mathbb{Z})$ images of $\nu = 0$

$$\Omega = \begin{pmatrix} \rho & \nu \\ \nu & \sigma \end{pmatrix} \to (A\Omega + B)(C\Omega + D)^{-1}$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2,\mathbb{Z})$$

rqd's are of the form

$$a(\rho\sigma - \nu^2) + k\rho + \ell\sigma + m\nu + c = 0$$

(k, ℓ, m, a, c) integers

$$4ac - 4k\ell + m^2 - 1 = 0.$$

- Exclude $a=0, \rightarrow \nu=0$

$\rho\sigma+\nu-\nu^2=0$ dominates the contribution.

- other rgd's suppressed by $\mathcal{O}(e^{-Q^2})$ non-perturbative

.... Calculate ....

- $Z_{BH}(p,\phi) = \sum_{\phi^\Lambda \to \phi^\Lambda + 2\pi i k^\Lambda} |Z_{top}|^2 \, |g_{top}|^2 e^{-K}$

+ non-pert.

- $Z_{top} = e^{F_0+F_1} = e^{i D_{ABC} \frac{X^A X^B X^C}{X^0}} \eta(t')^{-24}$

- $e^{-K} = \mathrm{Re}(\bar{X}^\Lambda \partial_\Lambda F_{top})$, $F_{top} = F_0 + F_1$

a "quantum corrected" Kähler potential ?

A better (?) way to organize our answers (both $\mathcal{N}=8$ (T^6) and $\mathcal{N}=4$ ($K3\times T^2$))

$$Z_{BH}(p,\phi) = \sum_{\phi \to \phi + 2\pi i k} |Z'_{top}|^2 \sqrt{\det g^{(q)}}$$

- $|Z'_{top}|^2 = \begin{cases} |Z_{top}|^2 (\mathrm{Im}\, t')^{-12} & K3\times T^2 \\ |Z_{top}|^2 e^{4K} & T^6 \end{cases}$

holomorphic anomaly

kind of like the holomorphic anomaly (compare to $T^6/\mathbb{Z}_3\times\mathbb{Z}_3$)

- $g^{(cl)}_{\Lambda\bar{\Sigma}} = \partial_\Lambda \partial_{\bar{\Sigma}} e^{-K^{(cl)}}$
$= \mathrm{Re}(\partial_\Lambda \partial_{\bar{\Sigma}} F_0)$

a natural metric on the space of X^Λ's ?

$g^{(q)} = g^{(cl)} + \delta g$, $\delta g = \begin{cases} 0, & T^6 \\ \text{1-loop}, & K3\times T^2 \end{cases}$

What is the origin of mismatch?

- supersymmetric index

  ?

  Ward's formula from $\mathcal{L}_{eff}$
- The role of holomorphic anomaly?
- Fragmentation?

What's NEXT?

- $\mathcal{N}=2$ orbifold of $\mathcal{N}=4$ models? FHSV
- Precise counting of large $\mathcal{N}=2$ CY BHs?

  quivers? attempt in progress ...