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# Counting BPS states in $\mathcal{N}=1$ $d=4$ SCFT's

based on hep-th/0510060

see also:
- Lin, Maldacena hep-th/0509235
- Kinney, Maldacena, Minwalla, Raju hep-th/0510251

Outline:

I. Group theory → BPS-multiplets, → index

II. Construction of field theories on $S^3 \times \mathbb{R}$

III. Ungauged theories

IV. Gauge theories & chiral ring

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## I. Group theory

• **Algebra:** $SU(2,2|1)$

in a notation appropriate for radial quantization → $S^3 \times \mathbb{R}$

Hamiltonian: $H$

$SO(4) = SU(2)_L \times SU(2)_R$: $J_i$, $\tilde{J}_i$

$U(1)_R$: $R$

Conformal generators: $K_{\alpha\dot{\beta}}$, $K^{+}_{\alpha\dot{\beta}}$

SUSY: $Q_\alpha$, $Q^{+}_\alpha$ ← $SU(2)_L$-index

$S_{\dot{\alpha}}$, $S^{+}_{\dot{\alpha}}$ ← $SU(2)_R$-index

| | $H$ | $R$ | $SU(2)_L$ | $SU(2)_R$ |
|---|---|---|---|---|
| $K_{\alpha\dot{\beta}}$ | $-1$ | $0$ | $2$ | $2$ |
| $Q_\alpha$ | $-\frac{1}{2}$ | $-1$ | $2$ | $1$ |
| $S_{\dot{\alpha}}$ | $-\frac{1}{2}$ | $1$ | $1$ | $2$ |

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• BPS - bounds :

from the anti-commutation relation

$$\{Q_\alpha, Q_\beta^\dagger\} = \delta_\alpha^\beta (H - \tfrac{3}{2}R) - 4\, \sigma^{i\beta}{}_\alpha J_i$$

* Chiral primaries : (sum over $\alpha = \beta$)

$\rightsquigarrow E - \frac{3}{2} r \geq 0$

Saturate : $E = \frac{3}{2} r$

$\rightsquigarrow Q_\alpha |\phi\rangle = Q_\alpha^\dagger |\phi\rangle = J_i |\phi\rangle = 0$

also : $(H - \frac{3}{2}R) S_{\dot\alpha} |\phi\rangle = -2\, S_{\dot\alpha} |\phi\rangle$

$\rightsquigarrow S_{\dot\alpha} |\phi\rangle = K_{\alpha\dot\beta} |\phi\rangle = 0$

i.e. $|\phi\rangle$ is chiral and primary.

$|\phi\rangle$ preserves 2+4 out of 4+4 supersymmetries $\rightsquigarrow \frac{1}{2}$ BPS state.

(can also be a $\frac{3}{4}$ or $\frac{1}{1}$ BPS state) depending on bounds involving $S_{\dot\alpha}$

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* Semi-long multiplet ($\alpha = \beta = 1$)

$\rightsquigarrow E - \frac{3}{2} r - 2 j_3 \geq 0$

Saturate : $E = \frac{3}{2} r + 2 j_3$

$\rightsquigarrow Q_1 |\phi\rangle = Q_1^\dagger |\phi\rangle = 0$

also : $(H - \frac{3}{2}R - 2J_3)\, Q_2^\dagger |\phi\rangle = -2\, Q_2^\dagger |\phi\rangle$

$\rightsquigarrow Q_2^\dagger |\phi\rangle = 0$

and : $(H - \frac{3}{2}R - 2J_3)\, S_{\dot\alpha} |\phi\rangle = -2\, S_{\dot\alpha} |\phi\rangle$

$\rightsquigarrow S_{\dot\alpha} |\phi\rangle = 0$

$|\phi\rangle$ is not a primary, but $Q_2 |\phi\rangle$ is.

$|\phi\rangle$ is a $\frac{1}{4}$ BPS state. (can also be $\frac{1}{2}$ or $\frac{3}{4}$ BPS)

* Similar bounds from

$$\{S_{\dot\alpha}, S_{\dot\beta}^\dagger\} = \delta_{\dot\alpha}^{\dot\beta} (H + \tfrac{3}{2}R) - 4\, \sigma^{i\dot\beta}{}_{\dot\alpha} \tilde{J}_i$$

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## • The SU(2|1) subalgebra

Those BPS-bounds involve only a $SU(2|1)_L$ subalgebra of the full superconformal algebra.

The subalgebra is generated by

$H - \frac{3}{2}R$ (quantum numbers $\Delta = E - \frac{3}{2}r$)

$j_i$

$Q_\alpha$ , $Q_\alpha^\dagger$

This subalgebra commutes with $SU(2)_R \times U(1)$ generated by

$\tilde{j}_i$

$H - \frac{1}{2}R$

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There are 3 kinds of representations of $SU(2|1)$:

* long representations : $\Delta > 2j$ (with $\Delta = E - \frac{3}{2}r$)

$Q\downarrow$

| | |
|---|---|
| $(\Delta, j)$ | $(\Delta, 0)$ |
| $(\Delta+1, j-\frac{1}{2})$ $(\Delta+1, j+\frac{1}{2})$ | $(\Delta+1, \frac{1}{2})$ |
| $(\Delta+2, j)$ | $(\Delta+2, 0)$ |

the long representations have as many Bosons as Fermions

$\rightarrow \mathrm{tr}\,(-1)^F = 0$

* short representations : $\Delta = 2j > 0$

$Q\downarrow$

$(\Delta = 2j, j)$ ← BPS-states.

$(\Delta+1 = 2j+1, j-\frac{1}{2})$ ← primaries

the short representations have one more Boson than Fermions or vice versa

$\rightarrow \mathrm{tr}\,(-1)^F = (-1)^{2(j+\tilde{j})}$

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* trivial representation: $\Delta = j = 0$

$\rightarrow \mathrm{tr}(-1)^F = (-1)^{2\tilde{j}}$

→ There is an index

$$\underbrace{\mathrm{tr}(-1)^F e^{-\beta(H-\frac{1}{2}R)} e^{\gamma \tilde{j}}}_{\text{commute with } SU(2|1)_L}$$

The $e^{-\beta(H-\frac{1}{2}R)}$ is needed as a regulator:

$$H - \tfrac{1}{2}R \geq H - \tfrac{1}{3}H = \tfrac{2}{3}H .$$

One could also define the index

$$\underbrace{\mathrm{tr}(-1)^F e^{-\beta(H+\frac{1}{2}R)} e^{\gamma j}}_{\text{commute with } SU(2|1)_R}$$

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- The index is a topological quantity and can be calculated in some weakly coupled UV regime that preserves a $SU(2|1)_L \times SU(2)_R \times U(1)$ symmetry.
- The index counts BPS states with a sign.
- Kinney, Maldacena, Minwalla & Raju proved that this is the most general index (topological quantity depending only on the $SU(2,2|1)$ group theory)

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## II. Construction of field theories on $S^3 \times \mathbb{R}$

- $\mathcal{N}=1$ SCFT's are usually defined through a UV theory, that flows to a conformal fixed point in the IR.

- For our purposes the UV-theory needs to have a $SU(2|1)_L \times SU(2)_R \times U(1)$ symmetry.

- The space-time symmetries are generated by the Killing vector fields on $S^3 \times \mathbb{R}$

$$H \sim \partial_t$$
$$\partial_i \sim \sigma_i^{(L)}$$
$$\tilde{\partial}_i \sim \sigma_i^{(R)}$$

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- The R-symmetry is generated by a constant scalar field on $S^3 \times \mathbb{R}$

- The Supersymmetries are generated by Killing spinors. $\zeta$.

* Choose a frame:

$$e^0 = R_1\, dt$$
$$e^i = R_2\, \sigma^i_{(R)}$$

↑ right-invariant 1-forms

* The Killing spinor is chiral

$$\gamma^5 \zeta = \zeta$$

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* The killing spinor is an $SU(2)_R$ singlet

$$\sigma_i^{(R)} \zeta = 0$$

⇝ The killing spinor is an $SU(2)_L$ doublet:

$$\left(\sigma_i^{(L)} + \frac{1}{4} \varepsilon_{ijk} \gamma^{jk}\right) \zeta = \frac{1}{4} \varepsilon_{ijk} \gamma^{jk} \zeta$$

$\frac{1}{4} \varepsilon_{ijk} \gamma^{jk} = \begin{pmatrix} i\sigma^i_{(P)} & 0 \\ 0 & i\sigma^i_{(P)} \end{pmatrix}$ ← Pauli matrix

* The killing spinor has charge $\frac{1}{2}$ under the Hamiltonian

$$\partial_t \zeta = -\frac{i}{2} \zeta$$

$$\rightarrow \quad \zeta = e^{-\frac{i}{2}t} \begin{pmatrix} \hat{\zeta} \\ 0 \end{pmatrix}$$

($\hat{\zeta}$: constant.)

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## • Chiral multiplets and the WZ-model.

The chiral multiplet contains a cx. scalar $\phi$, a chiral fermion $\psi$ and a cx. aux. field $F$

$$(\phi, \psi, F)$$

R-charge: $q$, $q-1$, $q-2$

Susy transformations:

$$\delta_\zeta \phi = \tilde{\zeta} \psi$$

$$\delta_\zeta \psi = \left(\partial_t + \frac{3iq}{2}\right) \phi \, \gamma^0 \zeta^{\Theta} - 2 \sigma_i^{(R)} \phi \, \gamma^i \zeta^{\Theta} + \zeta F$$

$$\delta_\zeta F = -i \bar{\zeta} \gamma^0 \left(\partial_t + i \frac{3q-5}{2}\right) \psi + 2i \bar{\zeta} \gamma^i \sigma_i^{(R)} \psi$$

($\tilde{\zeta}$: transpose; $\zeta^{\Theta}$: cx. conjugate; $\bar{\zeta}$: hermitean conjugate)

The SUSY transformations close appropriately to reproduce

$$\{Q_\alpha, Q^\dagger_\beta\} = \delta_\alpha^\beta \left(H - \frac{3}{2} R\right) - 4 \sigma^{i\beta}{}_\alpha J_i$$

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The Wess-Zumino Lagrangian is then

$$\mathcal{L}_0 = \left(\partial_t - i\frac{3q-2}{2}\right)\phi^\dagger\left(\partial_t + i\frac{3q-2}{2}\right)\phi - 4\,\sigma_i^{(L)}\phi^\dagger\sigma_i^{(L)}\phi - \phi^\dagger\phi$$
$$+ i\bar{\psi}\gamma^0\left(\partial_t + i\frac{3q-2}{2}\right)\psi - 2i\bar{\psi}\gamma^i\left(\sigma_i^{(L)} + \frac{1}{8}\varepsilon_{ijk}\gamma^{jk}\right)\psi$$
$$+ F^\dagger F$$

$$\mathcal{L}_W = W'(\phi)\,F - \frac{1}{2}W''(\phi)\,\tilde{\psi}\psi + \text{h.c.}$$

with

$$W(\phi) = c\,\phi^{\frac{2}{q}} \qquad \text{(R-charge 2)}$$

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- **Vector multiplets**

$(A_0, A_i, \lambda, D)$ in the adj. of the gauge group (Anti-hermitean matrices)

$$\delta_\xi A_0 = 2\,\mathrm{Re}(\bar{\xi}\gamma_0\lambda)$$
$$\delta_\xi A_i = -\mathrm{Re}(\bar{\xi}\gamma_i\lambda)$$
$$\delta_\xi \lambda = -2i\,F_{0i}\gamma^{0i}\xi + 2i\,F_{ij}\gamma^{ij}\xi + D\xi$$
$$\delta_\xi D = \mathrm{Im}\left(\bar{\xi}\gamma^0\left(\mathcal{D}_0 - \frac{3i}{2}\right)\lambda\right) - 2\,\mathrm{Im}(\bar{\xi}\gamma^i\mathcal{D}_i\lambda)$$

($\mathcal{D}_0$, $\mathcal{D}_i$: gauge covariant derivative)

Matter:

$$\delta_\xi\phi = \tilde{\xi}\psi$$
$$\delta_\xi\psi = \left(\mathcal{D}_0 + \frac{3iq}{2}\right)\phi\,\gamma^0\xi^{\ominus} - 2\mathcal{D}_i\phi\,\gamma^i\xi^{\ominus} + \xi F$$
$$\delta_\xi F = -i\bar{\xi}\gamma^0\left(\mathcal{D}_0 + i\frac{3q-5}{2}\right)\psi + 2i\bar{\xi}\gamma^i\mathcal{D}_i\psi - 2(\bar{\xi}\lambda^{\ominus})\phi$$

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Lagrangian :

$$\mathcal{L}_g = \frac{1}{g^2}\Big(4\,\mathrm{tr}\,F_{0i}F_{0i} - 8\,\mathrm{tr}\,F_{ij}F_{ij} + i\,\mathrm{tr}\,\bar{\lambda}\gamma^0 D_0\lambda - 2i\,\mathrm{tr}\,\bar{\lambda}\gamma^i\big(D_i + \tfrac{1}{8}\varepsilon_{ijk}\gamma^{jk}\big)\lambda + \mathrm{tr}\,D^2\Big)$$

← covariantized Lagrangians from flat space

$$\mathcal{L}_\Theta = \Theta\,\varepsilon_{ijk}F_{0i}F_{jk}$$

$$\mathcal{L}_{FI} = \kappa\,\mathrm{tr}(D - A_0)$$

New! Modifies Gauss law constraint!

$$\mathcal{L}_\phi = \Big(D_0 - i\frac{3q-2}{2}\Big)\phi^\dagger\Big(D_0 + i\frac{3q-2}{2}\Big)\phi - 4D_i\phi^\dagger D_i\phi - \phi^\dagger\phi + i\bar{\psi}\gamma^0\Big(D_0 + i\frac{3q-2}{2}\Big)\psi - 2i\bar{\psi}\gamma^i\big(D_i + \tfrac{1}{8}\varepsilon_{ijk}\gamma^{jk}\big)\psi + F^\dagger F + 2i\,\phi^\dagger D\phi - 2i\,\phi^\dagger\tilde{\lambda}\psi + 2\bar{\psi}\lambda^0\phi$$

$$\mathcal{L}_W = W'(\phi)F - \frac{1}{2}W''(\phi)\tilde{\psi}\psi + h.c.$$

~~~ ← New terms

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## III. Chiral primaries in ungauged theories

Chiral primaries are the ground states of the twisted theory with a Hamiltonian

$$H' = H - \frac{3}{2}R$$

In the Lagrangian this is achieved by replacing

$$\partial_t\phi \to \partial_t\phi - \frac{3q}{2}i\phi$$

$$\partial_t\psi \to \partial_t\psi - \frac{3(q-1)}{2}i\psi$$

$$\mathcal{L}_0' = (\partial_t + i)\phi^\dagger(\partial_t - i)\phi \underbrace{- 4\sigma_i^{(L)}\phi^\dagger\sigma_i^{(L)}\phi - \phi^\dagger\phi}_{\text{P.E.}} + i\bar{\psi}\gamma^0(\partial_t - i)\psi - 2i\bar{\psi}\gamma^i\sigma_i^{(L)}\psi + F^\dagger F$$
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Lowest modes of the free theory satisfy

$$\sigma_i^{(R)} \phi = 0$$
$$\sigma_i^{(R)} \psi = 0$$

Reduction to Quantum mechanics

$$L_0 = (\partial_t + i)\phi^\dagger (\partial_t - i)\phi - \phi^\dagger \phi + i\bar{\psi}\gamma^0(\partial_t - i)\psi + F^\dagger F$$

(is supersymmetric!)

→ Harmonic oscillator in a frame that is rotating at the frequency of the oscillator. $(W=0)$

→ zero energy mode. $(W=0)$

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Canonical formalism

$$p = (\partial_t + i\phi^\dagger)$$
$$\pi = -i\psi^\dagger$$

Canonical commutation relations:

$$\{\psi_\alpha, \psi_\beta^\dagger\} = \delta_\alpha^\beta$$
$$[\phi, p] = -i$$
$$[\phi^\dagger, p^\dagger] = -i$$

sign dictated by the anticommutation relation of the fermions.

Hamiltonian

$$H = (p - i\phi^\dagger)(p^\dagger + i\phi) + \psi^\dagger\psi + |W'(\phi)|^2 - W''(\phi)\psi_1\psi_2 + (W''(\phi))^\dagger \psi_1^\dagger \psi_2^\dagger$$

Supercharges:

$$Q_\alpha = -(p - i\phi^\dagger)\psi_\alpha + i(W'(\phi))^\dagger \epsilon_{\alpha\beta}\psi_\beta^\dagger$$

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## The free theory

$$H = 2a_2^\dagger a_2 + b_1^\dagger b_1 + b_2^\dagger b_2$$

where

$$a_1 = \frac{1}{\sqrt{2}}(p + i\phi^\dagger)$$
$$a_2 = \frac{1}{\sqrt{2}}(p^\dagger + i\phi)$$
$$b_1 = \psi_1$$
$$b_2 = \psi_2$$

are annihilation operators.

The chiral primaries are created by $a_1^\dagger$, which has R-charge $\frac{2}{3}$.

$$\rightarrow Z_x = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

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## The WZ model

Classical BPS equations:

$$\partial_t \phi = 0$$
$$W'(\phi) = 0$$

deform $W(\phi) \rightarrow$ n-1 solutions.

But cannot resolve R-charges.

Better: Chiral primaries satisfy

$$\partial_i |\phi\rangle = 0$$
$$Q_\alpha |\phi\rangle = 0$$

Find those states modulo $Q_\alpha^\dagger$ - exact states.

$\rightarrow$ chiral ring!

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Those elements of the chiral ring should be in 1-1 correspondence with the ground states of the twisted Hamiltonian.

The supersymmetry generators are

$$Q_1^+ = -\sqrt{2}\, a_2 b_1^+ - i\, W'\left(\frac{a_2 - a_1^+}{\sqrt{2}\, i}\right) b_2$$

$$Q_2^+ = -\sqrt{2}\, a_2 b_2^+ + i\, W'\left(\frac{a_2 - a_1^+}{\sqrt{2}\, i}\right) b_1$$

$J_i$ - invariant states have the form

$|m_1, m_2, +, +\rangle \qquad |m_1, m_2, -, -\rangle$

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States of the form

$|m_1, 0, -, -\rangle$

are $Q_\alpha^+$ - closed.

States of the form

$$W'\left(\frac{a_2 - a_1^+}{\sqrt{2}\, i}\right) |m_1, 0, -, -\rangle \sim |m_1 + n - 1, 0, -, -\rangle$$

are $Q_\alpha^+$ - exact

$\rightsquigarrow$ $|m_1, 0, -, -\rangle \qquad m_1 = 0, \ldots, n-2$

span the chiral ring.

To show that this is all, note that the states $|m_1, 0, -, -\rangle$ are already the chiral primaries of the free theory (Perturb the free theory)

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## IV. Gauge theories & chiral ring

- It is hard to reduce a twisted gauge theory to a gauged supersymmetric quantum mechanics.
- Instead use the chiral ring:

$\partial_i |\phi\rangle = 0$

$Q_\alpha^\dagger |\phi\rangle = 0$

modulo $Q_\alpha^\dagger$-exact states.

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Expand fields in spherical harmonics.

| | | |
|---|---|---|
| $\phi$ | $(j, j)$ | |
| $\psi$ | $(j+\frac{1}{2}, j)$ | $(j, j+\frac{1}{2})$ |
| $\lambda$ | $(j+\frac{1}{2}, j)$ | $(j, j+\frac{1}{2})$ |
| $A$ | $(j+1, j)$ | $(j, j+1)$ |

$\partial_i$ - invariant:

| | |
|---|---|
| $\phi$ | $(0, 0)$ |
| $\psi$ | $(0, \frac{1}{2})$ |
| $\lambda$ | $(0, \frac{1}{2})$ |
| $A$ | $(0, 1)$ |

$Q_\alpha^\dagger$ - invariant:

| | |
|---|---|
| $\phi$ | $(0, 0)$ |
| $\lambda$ | $(0, \frac{1}{2})$ |

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Relations:

$\{\lambda_\alpha, \lambda_\beta\} = 0$
$[\lambda_\alpha, \phi] = 0$
$[\phi, \phi] = 0$

} c.c. relations + transl. inv.

$F^\dagger = W'(\phi) = 0$
$\lambda_\alpha^{(R)} \phi^{(R)} = 0$

} Variations of elementary fields + conjugates.

Possibly more relations

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• Seiberg duality for SU(2) SYM with 3 flavors.

For SU(2) the fund. and anti fund. representations are the same.

→ 6 fund. matter multiplets.
SU(6) flavor symmetry

Count the number of gauge inv. "matter" chiral primaries:

use
$$n_{singlet} = \int_G [dg] \prod_i \chi_{R_i}(g)$$
$$\sum_{n=0}^{\infty} x^{\frac{n}{2}} \chi_{sym^n(R)}(g) = \exp\left(\sum_{\ell=1}^{\infty} \frac{x^{\frac{\ell}{2}}}{\ell} \chi_R(g^\ell)\right)$$

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$$Z(x) = \int_{SU(2)} [dg] \exp\left(6 \sum_{l=1}^{\infty} \frac{x^{l/2}}{l} \chi_f(g^l)\right)$$

Eigenvalue basis

$$Z(x) = \frac{2}{\pi} \int_{\varphi=0}^{\pi} \sin^2\varphi \, d\varphi \exp\left(12 \sum_{l=1}^{\infty} \frac{x^{l/2}}{l} \cos(l\varphi)\right)$$

$$= \frac{2}{\pi} \int_{\varphi=0}^{\pi} \frac{\sin^2\varphi \, d\varphi}{(1 - 2\sqrt{x}\cos\varphi + x)^6}$$

$$= \frac{1 + 6x + 6x^2 + x^3}{(1-x)^9}$$

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- The Seiberg dual is a SU(1) gauge theory, i.e. has trivial gauge group
- i.e. the Seiberg dual is a Wess-Zumino model with dual Meson matter fields $M^{ij}$ in the antisymmetric representation of the SU(6) flavor group.
- The superpotential is

$$W = \epsilon_{i_1 i_2 \cdots i_6} M^{i_1 i_2} M^{i_3 i_4} M^{i_5 i_6}$$

and imposes that the antisymmetric product of two mesons is vanishing.

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- Represent a dual meson by the Young tableaux

- The n-th excited level is in the symmetric product of dual mesons.
- The columns have an even height in the sym. product.
- Furthermore the superpotential constraint implies that no column is higher than 3!

→ $\mathrm{sym}^n(\square)$ mod $(dW=0)$ =

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This representation has dimension

$$d(n) = \frac{(n+5)!\,(n+4)!}{5!\,4!\,(n+1)!\,n!}$$

And the partition function is

$$Z(x) = \sum_{n=0}^{\infty} d(n)\,x^n = \frac{1+6x+6x^2+x^3}{(1-x)^9}$$

→ Agreement!

However, what happens to chiral primaries that involve gauginos?

i.e. $S = \mathrm{tr}\,\lambda_1\lambda_2$

The relation

$$S^2 = 0$$

is not enough!

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## What about the index?

Choose $H - \frac{3}{2}R - 2J_3$ as the twisted Hamiltonian.

$$\{Q_1, Q_1^\dagger\} = H - \frac{3}{2}R - 2J_3$$

→ $\frac{1}{4}$ BPS-states are in 1-1 correspondence with $Q_1^\dagger$-cohomology classes.

→ The index is the Euler-character of the $Q_1^\dagger$-cohomology.

Kinney, Maldacena, Minwalla & Raju calculated the index for $\mathcal{N}=4$ SYM in the large $N$ limit.

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## Conclusions & Outlook.

- Defined an index to count BPS states.
- Derived $\mathcal{N}=1$ supersymmetric Lagrangians on $S^3 \times \mathbb{R}$
  - → Allows Born-Oppenheimer approximation to describe BPS-States of SCFTs
- Checked Seiberg duality
- Would be good to calculate the index for theories which are not $\mathcal{N}=4$ SYM.
- Understand the extra chiral ring relation.
- Look at $\mathcal{N}=1$ large $N$ gauge theories in the spirit of Berenstein.