Generalized Hodge structures and Mirror Symmetry

The Hodge theory of D-branes

Tony Pantev

University of Pennsylvania

Overview

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- Will describe (following Kontsevich) how to extract Hodge theoretic invariants from D-brane categories.
- Will explain how these invariants transform under mirror symmetry.
- Will discuss the structure of the invariants and methods for computation.

Kontsevich's program

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Kähler space X

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$\forall\ E, F \in C_X \rightsquigarrow \underline{\mathrm{Hom}}_{C_X}(E, F) \in (\mathsf{Compl}/\mathbb{C})$
so that $\mathrm{Hom}_{C_X}(E, F[i]) = H^i(\underline{\mathrm{Hom}}_{C_X}(E, F))$

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(physics) C_X is the category of D-branes in the TQFT X.

Examples

- (math) If $X/\mathbb{C}$ is a scheme of finite type, then $X/\mathbb{C}$ is also a **nc** space with $C_X := \mathrm{Perf}(X)$ - perfect complexes of quasi-coherent sheaves on X.

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  Enhancement: The twisted complexes of Toledo-Tong and Bondal-Kapranov.

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$X = (M, \mathscr{J})$ - gc manifold in the sense of Hitchin, which fits in a generalized Kähler structure $(X, \mathscr{J}_1 = \mathscr{J}, \mathscr{J}_2)$.

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- (physics) If X is a topological twist of a $(2,2)$ sigma model, then X is also a **nc** space. C_X - the category of topological generalized complex branes of Kapustin-Li.

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Example: [Beilinson'78] $X = \mathbb{P}^n$, $A = \mathrm{End}(\mathcal{O} \oplus \ldots \oplus \mathcal{O}(n))^{\mathrm{op}}$.

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If X - scheme, then $S_X(\bullet) = (\bullet) \otimes K_X[\dim X]$.

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[Kontsevich'01]: Fix $L \in \mathrm{Pic}(X)$ - ample, and $\boldsymbol{\gamma} \in \Gamma(\mathrm{tot}(L^\times), \wedge^2 T)^{\mathbb{C}^\times}$ - Poisson structure. Get quantized space $X_{\boldsymbol{\gamma}}/\mathbb{C}((\hbar))$ with a new homogeneous coordinate ring: $f \star g = fg + \hbar\langle \boldsymbol{\gamma}, df \wedge dg\rangle + \ldots$.

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  - the quantized del Pezzo surfaces of [Artin'96]

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Answer is clear in the almost commutative examples, e.g. for schemes, stacks, gerbes, LG models.

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Problem: How can we define the lattice K^{top}?

Can K^{top} be defined entirely in terms of the **nc** data?

nc Hodge conjecture

nc Hodge conjecture: If $X/\mathbb{C}$ is a proper and smooth **nc** space, then

$$\mathrm{im}\left[K_0(C_X) \xrightarrow{\mathrm{ch}} \Gamma(K^{\mathrm{top}})\right] \otimes \mathbb{Q} = \mathrm{Hom}_{\textbf{ncHS}}(\mathbf{1}, H^{\bullet}_{dR}(X)) \otimes \mathbb{Q}.$$

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Remark: Polarizations appear under different names in the works of Hertling and Sabbah: trTERP structure [Hertling], integrable polarized twistor structure [Sabbah].

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Structure results

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Theorem [Katzarkov-Kontsevich'05] *For Landau-Ginzburg models $X = (\boldsymbol{Y}, \mathsf{f})$ the **nc** Hodge conjecture follows from the commutative Hodge conjecture.*

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Fix $X = (\boldsymbol{Y}, \mathbf{f})$ - LG with a proper $\mathrm{crit}(f)$,
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Note:

The geometric de Rham and Dolbeault cohomology of $(\boldsymbol{Y}, \mathbf{f})$ coincide with the periodic cyclic and Hochschild homology of $C_{(\boldsymbol{Y}, \mathbf{f})}$, [Katzarkov-Kontsevich-P'05].

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The geometric definition can be used to show that the Hodge-to-de Rham spectrals sequence degenerates, [Barannikov-Kontsevich'97].

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Idea: Relate to commutative Hodge theory.

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Hence $H^i_B((\boldsymbol{Y}, \mathsf{f}); \mathbb{C}) = H^{i-1}(\phi_{\mathsf{f}}\mathbb{C})$.

In fact $H^i_{dR}((\boldsymbol{Y}, \mathsf{f}); \mathbb{C}) = H^{i-1}(\phi_{\mathsf{f}}(\Omega_{\boldsymbol{Y}}, d + d\mathsf{f} \wedge \bullet))$ and $H^i_{Dol}((\boldsymbol{Y}, \mathsf{f}); \mathbb{C}) = H^{i-1}(\phi_{\mathsf{f}}(\Omega_{\boldsymbol{Y}}, d\mathsf{f} \wedge \bullet))$, [Sabbah'00].

Limiting Hodge structures

The family $V_\tau = H^\bullet_{dR}((\boldsymbol{Y}, \tau \cdot \mathsf{f}))$, $\tau \in \mathbb{C}$ is a variation of **nc** pure Hodge structures and by the work of Sabbah induces a limiting mixed twistor structure on $H^\bullet_{dR}((\boldsymbol{Y}, \mathsf{f}))$ for $\tau \to \infty$.

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[Sabbah'05, Szabo'05]: the limiting mixed twistor structure on $H^\bullet_{dR}((\boldsymbol{Y}, \mathbf{f}))$ for $\tau \to \infty$ is an ordinary MHS which is isomorphic to Steenbrink's MHS on the vanishing cohomology $H^{\bullet-1}(\phi_{\mathbf{f}}\mathbb{C})$.

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Corollary [Katzarkov-Kontsevich-P'05] *For Landau-Ginzburg models $X = (\boldsymbol{Y}, \mathbf{f})$ the MHS on the vanishing cohomology is an invariant of the category C_X.*

Mirror symmetry

Corollary **[Katzarkov-Kontsevich-P'05]** *Suppose $(\boldsymbol{Z}, \boldsymbol{\omega})$ is a symplectic manifold and suppose $X = (\boldsymbol{Y}, \mathbf{f})$ is the Hori-Vafa mirror. Then the MHS on the vanishing cohomology of $\mathbf{f}$ is a symplectic invariant of $(\boldsymbol{Z}, \boldsymbol{\omega})$.*

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Expect: Mirror symmetry exchanges the **nc** Hodge structures on cohomology. In the case of varieties this can be tested since the **nc** pure Hodge structure can be reconstructed from the MHS on the vanishing cohomology.

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Theorem [Gross-Katzarkov'05] *Suppose* $(\boldsymbol{Z}, \boldsymbol{\omega})$ *is a symplectic manifold underlying a c.i. variety* M, $\dim M \leq 3$ *which is either Fano, CY or of general type. Suppose* $X = (\boldsymbol{Y}, \mathsf{f})$ *is the Hori-Vafa mirror. Then the* 90^o *rotation of the MHS on the vanishing cohomology of* f *reconstructs the pure Hodge structure on* M.