Generalized Hodge structures and Mirror Symmetry

The Hodge theory of D-branes

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Overview

Joint work with L. Katzarkov and M. Kontsevich.
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- Will describe (following Kontsevich) how to extract Hodge theoretic invariants from $D$-brane categories.

$$H_{dR}^\bullet(D^b_{\text{sing}}(Y, f))$$
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- Joint work with L. Katzarkov and M. Kontsevich.
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- Will explain how these invariants transform under mirror symmetry.
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Joint work with L. Katzarkov and M. Kontsevich.

Will describe (following Kontsevich) how to extract Hodge theoretic invariants from $D$-brane categories.

Will explain how these invariants transform under mirror symmetry.

Will discuss the structure of the invariants and methods for computation.
Kontsevich’s program

Recall:

Kähler space \( X \)
Kontsevich’s program

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Kähler space $X$
Kontsevich’s program

Recall:

Kähler space $X$ $\rightarrow$ Hodge structure on the de Rham cohomology of $X$
Recall:

\[ H^dR(X, \mathbb{C}) \]

Kontsevich’s program
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Recall:

Kähler space \( X \) \( \implies \) Hodge structure:

\[
H_B^\bullet(X, \mathbb{C}) \cong H_{dR}^\bullet(X, \mathbb{C}) \cong H_{Dol}^\bullet(X, \mathbb{C})
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Kontsevich’s program

Recall:

Kähler space $X$ \quad \rightarrow \quad \text{Hodge structure:}

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de Rham’s theorem
Kontsevich’s program

Recall:

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- de Rham’s theorem
- Hodge’s theorem
- the Kähler condition
Recall:

Kähler space $X$ \[\implies\]

Hodge structure:

$H^\bullet_B(X, \mathbb{C}) \cong H^\bullet_{dR}(X, \mathbb{C}) \cong H^\bullet_{Dol}(X, \mathbb{C})$

$H^\bullet_B(X, \mathbb{Z})$
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Recall:

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Recall:

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$H^B \otimes (X, \mathbb{C}) \cong H_{dR}^\bullet (X, \mathbb{C}) \cong H_{Dol}^\bullet (X, \mathbb{C})$

Want:

generalized (nc) Kähler space $X$

$H^\bullet_B (X, \mathbb{Z}) \oplus H^{p, q}$
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Recall:

Kähler space $\mathcal{X}$

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$$H^*_B(X,\mathbb{C}) \cong H^*_{dR}(X,\mathbb{C}) \cong H^*_{Dol}(X,\mathbb{C})$$

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nc spaces

Definition: (math) [Bondal’90] A nc space $X/\mathbb{C}$ is a small triangulated $\mathbb{C}$-linear category $C_X$ which is:
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\[ \forall E, F \in C_X \rightsquigarrow \text{Hom}_{C_X}(E, F) \in (\text{Compl}/\mathbb{C}) \]

so that $\text{Hom}_{C_X}(E, F[i]) = H^i(\text{Hom}_{C_X}(E, F))$
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- (math) $C_X$ is the category of sheaves on $X$.
- (physics) $C_X$ is the category of $D$-branes in the TQFT $X$. 
Examples

(math) If $X/\mathbb{C}$ is a scheme of finite type, then $X/\mathbb{C}$ is also a nc space with $C_X := \text{Perf}(X)$ - perfect complexes of quasi-coherent sheaves on $X$. 
Examples

(math) If $X/\mathbb{C}$ is a scheme of finite type, then $X/\mathbb{C}$ is also a \textbf{nc} space with $C_X := \text{Perf}(X)$ - perfect complexes of quasi-coherent sheaves on $X$. If $X$ - smooth and quasi-projective, then $C_X$ is quasi-equivalent to $D^b_{\text{qcoh}}(X)$.
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Enhancement: The twisted complexes of Toledo-Tong and Bondal-Kapranov.
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$$X = (M, \mathcal{J}) - \text{gc manifold in the sense of Hitchin, which fits in a generalized Kähler structure } (X, \mathcal{J}_1 = \mathcal{J}, \mathcal{J}_2).$$
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(physics) If $X$ is a topological twist of a $(2,2)$ sigma model, then $X$ is also a nc space. $C_X$ - the category of topological generalized complex branes of Kapustin-Li.
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**Theorem** [Bondal-Van den Bergh’02] $C_X = \text{Perf}(X)$ has a strong split generator: $E \in C_X$, with $\text{Perf}(X) \cong \text{Perf}(\text{RHom}(E, E)^{\text{op}} - \text{mod})$. 

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**Example:** [Beilinson’78] $X = \mathbb{P}^n$, $A = \text{End}(\mathcal{O} \oplus \ldots \oplus \mathcal{O}(n))^{\text{op}}$. 
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$S_X : \mathcal{C}_X \to \mathcal{C}_X$;

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If $X$ - scheme, then $S_X(\bullet) = (\bullet) \otimes K_X[\text{dim } X]$. 

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[Kontsevich’01]: Fix $L \in \text{Pic}(X)$ - ample, and $\gamma \in \Gamma(\text{tot}(L^\times), \wedge^2 T)^{\mathbb{C}^\times}$ - Poisson structure. Get quantized space $X_\gamma/\mathbb{C}((\hbar))$ with a new homogeneous coordinate ring: $f \star g = fg + \hbar \langle \gamma, df \wedge dg \rangle + \ldots$. 

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- the non-commutative \( \mathbb{P}^2 \)'s of [Artin-Tate-Van den Bergh’90];
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- the elliptic projective spaces of [Odesskij-Feigin’98];
- the quantized del Pezzo surfaces of [Artin’96]
proper/smooth nc spaces (II)

**Expect:** If $X/\mathbb{C}$ is a topological twist of a $(2, 2)$ sigma model, corresponding to a compact gc manifold $(M, \mathcal{J})$, then $X$ is a proper and smooth nc space.
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- The notions of **algebraic/proper/smooth** extend to $\mathbb{Z}/2$-graded nc spaces
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**Conjecture [Kontsevich’04]** Suppose $X = (Y, f)$ is a Landau-Ginzburg model with a proper critical locus. Then $(Y, f)$ is a proper and smooth nc space.
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- $(Y, f)$ is the germ of an isolated quasi-homogeneous hypersurface singularity [K.Saito’98].
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**proper/smooth nc spaces (II)**

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- $(Y, f)$ is the Hori-Vafa mirror of a (quantized) del Pezzo surface or a weighted projective space [Auroux-Katzarkov-Orlov’04].
Cohomology (I)

Consider

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$\mathbb{C}(\langle u \rangle)$-module

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Facts:

- $X/\mathbb{C}$ - smooth affine variety, $A = \Gamma(X, \mathcal{O})$, then $HH_{-k}(A) = \Gamma(X, \Omega_X^k)$. The differential $B$ is the algebraic de Rham differential [Hochschild-Kostant-Rosenberg’62].
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**nc Kähler spaces**

**Definition:** For a proper and smooth nc space $X/\mathbb{C}$ the Hodge-to-de Rham spectral sequence collapses at $E_1$ if

$$\dim_{\mathbb{C}}((u)) \left( H_{dR}^{\text{even}}(X) \oplus H_{dR}^{\text{odd}}(X) \right) = \dim_{\mathbb{C}} \left( \oplus_k H_{Dol}^k(X) \right).$$
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A proper and smooth nc space $X/\mathbb{C}$ with a collapsing Hodge-to-de Rham spectral sequence $\leftrightarrow$ substitute for nc Kähler space.
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Residual structures

Suppose $X/\mathbb{C}$ is proper, smooth with degenerating Hodge-to-de Rham ss.
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Definition: A pure generalized (nc) Hodge structure is a triple $(H, \nabla, K^{\text{top}})$, where:
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Generalized Hodge structures

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- \(K^{\text{top}} \subset H|_{\{u \neq 0\}}\) - local subsystem of \(\mathbb{Z}/2\)-graded abelian groups, with \(K^{\text{top}} \otimes \mathbb{C} = H|_{\{u \neq 0\}}\).
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Problem: How can we define the lattice $K^{\text{top}}$? Answer is clear in the almost commutative examples, e.g. for schemes, stacks, gerbes, LG models.
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- \(K^{\text{top}} \subset H_{\{u \neq 0\}}\) - local subsystem of \(\mathbb{Z}/2\)-graded abelian groups, with \(K^{\text{top}} \otimes \mathbb{C} = H_{\{u \neq 0\}}\).

Problem: How can we define the lattice \(K^{\text{top}}\)? Can \(K^{\text{top}}\) be defined entirely in terms of the nc data?
nc Hodge conjecture: If $X/\mathbb{C}$ is a proper and smooth nc space, then

$$\text{im} \left[ K_0(C_X) \xrightarrow{\text{ch}} \Gamma(K^\text{top}) \right] \otimes \mathbb{Q} = \text{Hom}_{\text{ncHS}}(1, H^\bullet_{dR}(X)) \otimes \mathbb{Q}.$$
Definition: A polarization on a nc Hodge structure $(H, \nabla, K^{\text{top}})$ at radius $r \in \mathbb{R}_{>0}$ is the data $(\mathcal{H}, \nabla, K, \psi)$, where:
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Remark: If \((H, \nabla, K^{\text{top}})\) is a nc Hodge structure, then it suffices to specify \(\psi\) on \(H_{\{\mid u \mid <1\}}\).
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Remark: Polarizations appear under different names in the works of Hertling and Sabbah: trTERP structure [Hertling], integrable polarized twistor structure [Sabbah].
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**Conjecture** [Kontsevich’03] For any proper+smooth nc space \(X/\mathbb{C}\) the nc Hodge structure on \(H_{dR}^\bullet(X)\) is polarizable.
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Structure results

**Theorem [Katzarkov-Kontsevich-P’05]** The category of pure nc Hodge structures is semisimple (as a rigid category).
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**Theorem** [Katzarkov-Kontsevich’05] For Landau-Ginzburg models \( X = (Y, f) \) the nc Hodge conjecture follows from the commutative Hodge conjecture.
Fix $X = (Y, f)$ - LG with a proper $\text{crit}(f)$,

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Hodge invariants of LG models

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Note:
The geometric de Rham and Dolbeault cohomology of $(Y, f)$ coincide with the periodic cyclic and Hochschild homology of $C_{(Y,f)}$, [Katzarkov-Kontsevich-P’05].
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Note:
The geometric definition can be used to show that the Hodge-to-de Rham spectrals sequence degenerates, [Barannikov-Kontsevich’97].
Question: How can we compute the nc Hodge structure on $H^\bullet_{dR}(Y, f)$?
Vanishing cocycles

**Question:** How can we compute the nc Hodge structure on $H_{dR}^\bullet((Y, f))$?

**Idea:** Relate to commutative Hodge theory.
Vanishing cocycles

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$\mathcal{Y}$ has the homotopy type of $\mathcal{Y}_0$: If $i_0 : \mathcal{Y}_0 \hookrightarrow \mathcal{Y}$, then there exists $r : \mathcal{Y} \to \mathcal{Y}_0$ - a strict deformation retraction ($r \circ i \simeq id_{\mathcal{Y}_0}$). Specialization to $0$ map: $r_t := r|_{\mathcal{Y}_t} : \mathcal{Y}_t \to \mathcal{Y}_0$. 
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[Deligne’73] Nearby and vanishing cocycles functors:

$$\psi_f, \phi_f: D^{-}(Y, \mathbb{Z}) \rightarrow D^{-}(Y_0, \mathbb{Z})$$
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$\psi_f K^\bullet = Rr_t^* i^*_t K^\bullet$, $\phi_f K^\bullet = \text{cone}(i^* K^\bullet \to \psi_f K^\bullet)$. 

Generalized Hodge structures and Mirror Symmetry – p.22/24
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\[ \ldots \to H^i(Y_0) \to H^i(Y_t) \to H^i(\phi_f \mathbb{C}) \to H^{i+1}(Y_0) \to \ldots \]
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Hence $H_B^i((Y, f); C) = H^{i-1}(\phi_f C)$.

In fact $H^i_{dR}((Y, f); C) = H^{i-1}(\phi_f(\Omega_Y, d + df \wedge \bullet))$ and $H^i_{Dol}((Y, f); C) = H^{i-1}(\phi_f(\Omega_Y, df \wedge \bullet))$, [Sabbah’00].
Limiting Hodge structures

The family $V_\tau = H^\bullet_{dR}((Y, \tau \cdot f))$, $\tau \in \mathbb{C}$ is a variation of nc pure Hodge structures and by the work of Sabbah induces a limiting mixed twistor structure on $H^\bullet_{dR}((Y, f))$ for $\tau \to \infty$. 
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[Sabbah’05, Szabo’05]: the limiting mixed twistor structure on $H^\bullet_{dR}((Y, f))$ for $\tau \to \infty$ is an ordinary MHS which is isomorphic to Steenbrink’s MHS on the vanishing cohomology $H^{\bullet-1}(\phi_f \mathbb{C})$. 
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**Corollary** [Katzarkov-Kontsevich-P’05] For Landau-Ginzburg models $X = (Y, f)$ the MHS on the vanishing cohomology is an invariant of the category $C_X$. 
Corollary [Katzarkov-Kontsevich-P’05] Suppose \((Z, \omega)\) is a symplectic manifold and suppose \(X = (Y, f)\) is the Hori-Vafa mirror. Then the MHS on the vanishing cohomology of \(f\) is a symplectic invariant of \((Z, \omega)\).
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Expect: Mirror symmetry exchanges the nc Hodge structures on cohomology. In the case of varieties this can be tested since the nc pure Hodge structure can be reconstructed from the MHS on the vanishing cohomology.
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Theorem [Gross-Katzarkov’05] Suppose \((Z, \omega)\) is a symplectic manifold underlying a c.i. variety \(M\), \(\dim M \leq 3\) which is either Fano, CY or of general type. Suppose \(X = (Y, f)\) is the Hori-Vafa mirror. Then the \(90^\circ\) rotation of the MHS on the vanishing cohomology of \(f\) reconstructs the pure Hodge structure on \(M\).