Compactifications on Generalized Geometries

Jan Louis
Universität Hamburg

based on collaboration with:

Mariana Graña, Sebastien Gurrieri, Andrei Micu, Daniel Waldram

`hep-th/0211102`, `hep-th/0505264`

KITP, September 2005

Introduction

String Theory: strings moving in 10-dimensional space-time background
contact with "our world": compactification

$\Rightarrow$ choose space-time background: $M_4 \times Y_6$

M_4: 4 dim. space-time
Y_6: compact manifold $\Rightarrow$ determines amount of supersymmetry
$\Rightarrow$ fruitful interplay supersymmetry $\leftrightarrow$ geometry

Particle Physics needs spontaneously broken $N = 1$ supersymmetry in M_4

- traditionally
  - Y_6: compact Calabi-Yau threefold
  - employ non-perturbative effects to spontaneously break supersymmetry
- recently: slight variation on the theme (Brane-World Scenarios)
  - include space-time filling D-branes and orientfold-planes
  - spontaneous supersymmetry breaking via
    background fluxes, generalized geometries (& non-perturbative effects)

purpose of this talk:

- ➪ discuss mirror symmetry in the presence of background fluxes
- ➪ discuss string compactifications on manifolds with $SU(3)$-structure
  - compute Kähler potential and superpotential
  - uncover the role of torsion of Y_6 in supergravity

outline of talk:

1. warm up: Calabi-Yau compactifications of type II
2. background fluxes & mirror symmetry
3. manifolds with $SU(3)$ structure
4. compute Kähler potential and superpotential and discuss mirror symmetry
5. conclusions/open problems

Type II string theories in $D = 10$: (32 supercharges)

massless spectrum:

	IIA	IIB
NS:	$G_{MN},\ H_3 = dB_2,\ \Phi$	
RR:	$F_2 = dC_1,\ F_4 = dC_3$	$l,\ F_3 = dC_2,\ F_5^* = dC_4$
NSR	$\Psi_M^{1,2},\ \lambda^{1,2}$	$\Psi_M^{1,2},\ \lambda^{1,2}$

$F_p = p$-form field strength, $C_{p-1} = (p-1)$-form gauge potential

Compactification

choose space-time background: $M_4 \times Y_6$

and decompose fields into eigenfunctions of Δ_6

$$\Delta_{10}\Phi = (\Delta_4 + \Delta_6)\Phi = (\Delta_4 + m^2)\Phi = 0$$

⇒ massless $d = 4$ spectrum = zero modes of Δ_6 = harmonic forms in $H^{(p,q)}(Y_6)$

Calabi-Yau threefolds Y

- Ricci-flat Kähler manifold with holonomy $SU(3)$
  $\Rightarrow$ globally defined spinor exists on Y $\Rightarrow$ supercharges in $d = 4$ exist

- number of massless scalar fields are determined by

  Hodge numbers: $h^{p,q} = \mathrm{dim} H^{p,q}(Y)$

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{1,2} & & h^{1,2} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

- special role played by deformations (moduli) of Calabi-Yau metric

$$\mathcal{M} = \mathcal{M}_{\Omega}^{2h^{(1,2)}} \times \mathcal{M}_{J}^{2h^{(1,1)}} \qquad \text{(geometrical moduli space)}$$

$\mathcal{M}_{\Omega}^{2h^{(1,2)}}$: deformations of complex structure/holomorphic 3-form Ω,
$\mathcal{M}_{J}^{2h^{(1,1)}}$: deformations of complexified Kähler form $B + iJ$

Mirror Symmetry

conjecture:

for ‘every’ Y there exists a mirror manifold $\tilde{Y}$ with

$$h^{1,1}(Y) = h^{1,2}(\tilde{Y}) \ , \qquad h^{1,2}(Y) = h^{1,1}(\tilde{Y})$$

manifestation in string theory:

IIA in background $\mathcal{M}_4 \times Y$ $\equiv$ IIB in background $\mathcal{M}_4 \times \tilde{Y}$

implies:

$$\mathcal{M}_\Omega(Y) = \mathcal{M}_J(\tilde{Y}) \ , \qquad \mathcal{M}_J(Y) = \mathcal{M}_\Omega(\tilde{Y})$$

Low energy effective action of massless modes: $N=2$ supergravity

$$\mathcal{S} = \int_{M_4} \tfrac{1}{2}R - g_{ab}(z)\,\partial_\mu z^a \partial^\mu z^b - V(z) + \dots$$

- scalar manifold: $N=2$ constraint: $\mathcal{M} = \mathcal{M}_{\mathrm{SK}} \times \mathcal{M}_{\mathrm{QK}}$

$$\mathrm{IIA}: \qquad \mathcal{M}_{\mathrm{SK}} = \mathcal{M}_J^{2h^{(1,1)}}\ , \qquad \mathcal{M}_{\mathrm{QK}}^{4(h^{(1,2)}+1)} \supset \mathcal{M}_\Omega^{2h^{(1,2)}}$$

$$\mathrm{IIB}: \qquad \mathcal{M}_{\mathrm{SK}} = \mathcal{M}_\Omega^{2h^{(1,2)}}\ , \qquad \mathcal{M}_{\mathrm{QK}}^{4(h^{(1,1)}+1)} \supset \mathcal{M}_J^{2h^{(1,1)}}$$

- Kähler potentials

$$e^{-K_J} = \int_{Y_6} J \wedge J \wedge J\ , \qquad e^{-K_\Omega} = \int_{Y_6} \Omega \wedge \bar{\Omega}$$

Both geometries are special Kähler geometries [Strominger,Candelas,de la Ossa]

(K determined by a holomorphic prepotential F)

Background fluxes

[Polchinski,Strominger,...]

$$\text{allow} \qquad \int_{\gamma_p^I \in Y} F_p = e_I \neq 0 \qquad \text{keeping} \qquad dF_p = 0 = d^\dagger F_p$$

$$\Rightarrow F_p \;=\; e_I\, \omega_p^I\,, \qquad \omega_p \in H^p(Y)$$

$$e_I = \text{const.} = \begin{cases} \text{quantized in string theory} \\ \text{continuous in low energy approximation} \end{cases}$$

consistency: tadpole cancellation condition

properties:

- large Y: e_I small perturbation such that light spectrum does not change
- low energy supergravity ⇒ gauged/massive supergravity
- potential generated ⇒ vacuum degeneracy (partially) lifted
- supersymmetry spontaneously broken

Fluxes in IIB on Y

[Michelson; Taylor,Vafa; Mayr; Dall'Agata; Micu,JL; ...]

IIB on Y: turn on three-form flux for $G_3 \equiv F_3 - \tau H_3$

$$\text{electric flux :} \qquad e_I(\tau) = e_I^{RR} - \tau e_I^{NS} = \int_{\gamma_I} G_3 \ ,$$

$$\text{magnetic flux :} \qquad m^I(\tau) = m^{IRR} - \tau m^{INS} = \int_{\gamma^{*I}} G_3$$

- electric fluxes e_I: gauged supergravity (gauged transl. isometry of hypermult.)

$$\partial_\mu z \to D_\mu z = \partial_\mu z + k_I(z) A_\mu^I \ , \qquad k_I = e_I = \int_{\gamma_I} G_3$$

- magnetic fluxes m^I: B_2, C_2 become massive

[Micu,JL; Dall'Agata,D'Auria,Sommovigo,Vaula]

both cases: potential $V(z,\tau)$ induced which depends on [Gukov,Taylor,Vafa,Witten]

$$W = \int_Y \Omega \wedge G_3$$

Mirror symmetry in the presence of fluxes

[Gukov,Vafa,Witten; Gurrieri,Micu,Waldram,JL; Fidanza,Graña,Minasien,Tomasiello; ...]

➪ RR-flux: IIB: $e = \int_\gamma F_3$, $m = \int_{\gamma^*} F_3$

IIA: $\tilde{e} = \int_{\gamma_4} F_4$, $\tilde{m} = \int_{\gamma_2} F_2$

mirror symmetry:

$$H^{(1,2)}(Y) \Longleftrightarrow H^{(1,1)}(\tilde{Y})$$

effective actions obey:

$$\mathcal{L}^{IIB}(Y, e, m) \equiv \mathcal{L}^{IIA}(\tilde{Y}, \tilde{e}, \tilde{m}) , \qquad e = \tilde{e}, \quad m = \tilde{m}$$

➪ NS-flux:

no obvious mirror symmetry since flux of H_3 is along $H^3(Y)$ on both sides

⇒ NS F_4 (electric) and F_2 (magnetic) are missing

⇒ can only come from metric/geometry

⇒ compactify on different manifold $\widehat{Y}$ [Vafa]

Discovering $\widehat{Y}$

first recall derivation of Calabi-Yau condition: [Candelas,Horowitz,Strominger,Witten]

Lorentz group on space-time background $\mathcal{M}_{10} = R_{1,3} \times Y_6$ decomposes

$$SO(1,9) \to SO(1,3) \times SO(6)$$

spinor decompose accordingly:

$$\mathbf{16} \to (\mathbf{2}, \mathbf{4}) \oplus (\bar{\mathbf{2}}, \bar{\mathbf{4}})$$

impose two conditions:

1. demand that supercharge Q exist ⇒ structure group of Y_6 has to be reduced

$$SO(6) \to SU(3) \quad \text{s.t.} \quad \mathbf{4} \to \mathbf{3} + \mathbf{1}$$

   globally defined spinor η exists ⇒ Y_6 has $SU(3)$-structure

2. background preserves supersymmetry

$$\delta\Psi_M = \nabla_M \eta + (\gamma \cdot F)_M = 0$$

   ⇒ for $F = 0$: $\nabla_M \eta = 0$ ⇒ Y_6 is Calabi-Yau manifold

Generalizations:

insist on 1. (existence of Q) but relax 2. [Gauntlett, ..., review by Graña]

(i) $F \neq 0$ and $\nabla\eta \neq 0$ such that $\delta\Psi_M = 0$

corresponds to supersymmetric background with non-trivial flux

(ii) $F \neq 0$ and/or $\nabla\eta \neq 0$ but $\delta\Psi_M \neq 0$

corresponds to spontaneously broken supersymmetry

here: study $\nabla\eta \neq 0$: distinguish two cases

(a) compactify on manifold with $SU(3)$ structure

(which is not Calabi-Yau)

(b) be slightly more general:

choose different spinors η_1, η_2 for the two gravitini $\Psi_M^{1,2}$

each spinor defines an $SU(3)$ structure – together an $SU(3) \times SU(3)$ structure

Manifolds with $SU(3)$ structure

[Gray, Hervella, Hitchin, Salamon, Chiossi, ...]

existence of invariant spinor η implies existence of two invariant tensors: J, Ω

- two-form

$$J_{mn} = \eta^\dagger \gamma_{[m}\gamma_{n]}\eta \ , \qquad dJ \neq 0$$

  $\Rightarrow$ almost complex structure

$$I_m{}^n = J_{mp} g^{pn} \ , \qquad I^2 = -1 \ , \qquad N(I) \neq 0$$

  J is $(1,1)$-form with respect to I

- $(3,0)$-form

$$\Omega_{mnp} = \eta^\dagger \gamma_{[m}\gamma_n\gamma_{p]}\eta \ , \qquad d\Omega \neq 0$$

- Fierz implies $(\eta^\dagger\eta = 1)$

$$J \wedge J \wedge J = \tfrac{i}{2}\Omega \wedge \bar{\Omega} \ , \qquad J \wedge \Omega = 0$$

$SU(3) \times SU(3)$ structure: pair $J^{1,2}, \Omega^{1,2}$

Further properties:

η obeys

$$\nabla^{(T)}\eta \;\equiv\; (\nabla^{(LC)} + T_0)\,\eta \;=\; 0\;, \qquad T_0 : \text{intrinsic (con)-torsion}$$

manifolds with $SU(3)$-structure can be divided into different classes
– characterized by $SU(3)$-representation T_0 carries [Chiossi, Salamon]

$$T = T^g \oplus T^0 \in \Lambda_1 \otimes so(6) = \Lambda_1 \otimes su(3) \oplus \Lambda_1 \otimes su(3)^\perp$$

T^0 decomposes as

$$T^0 \in \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4 \oplus \mathcal{W}_5 \sim (\mathbf{3} \oplus \bar{\mathbf{3}}) \otimes (\mathbf{1} \oplus \mathbf{3} \oplus \bar{\mathbf{3}})$$

where

component	interpretation	$SU(3)$-representation
$\mathcal{W}_1$	$J \wedge d\Omega$ or $\Omega \wedge dJ$	$\mathbf{1} \oplus \mathbf{1}$
$\mathcal{W}_2$	$(d\Omega)_0^{2,2}$	$\mathbf{8} \oplus \mathbf{8}$
$\mathcal{W}_3$	$(dJ)_0^{2,1} + (dJ)_0^{1,2}$	$\mathbf{6} \oplus \bar{\mathbf{6}}$
$\mathcal{W}_4$	$J \wedge dJ$	$\mathbf{3} \oplus \bar{\mathbf{3}}$
$\mathcal{W}_5$	$d\Omega^{3,1}$	$\mathbf{3} \oplus \bar{\mathbf{3}}$

Mirrors of Calabi-Yau & electric NS 3-form flux [Gurrieri,Micu,Waldram,JL]

➪ Result: $\widehat{Y}$ is "half-flat" manifold which obey [Hitchin, Chiossi,Salamon]

$$\mathcal{W}_4 = \mathcal{W}_5 = \mathrm{Im}\mathcal{W}_1 = \mathrm{Im}\mathcal{W}_2 = 0 \ , \qquad \Leftrightarrow \qquad d(\mathrm{Im}\Omega) \ = \ 0 \ = \ d\,(J \wedge J)$$

'missing' NS 4-form:

$$F_4 \sim \ d(\mathrm{Re}\Omega) = e^i_{NS}\, \omega^i_4$$

➪ "Proof":

- go to SYZ limit and perform mirror map explicitly
- match type IIB $N = 1$ domain-wall solution of [Behrndt, Cardoso, Lüst] with type IIA solution [Hitchin; Mayer,Mohaupt]
- compute low energy effective action for type IIA compactified on $\widehat{Y}$ [GMLW, Grana,Waldram,JL]

➪ Puzzle: mirror of magnetic fluxes

requires manifolds with $SU(3) \times SU(3)$ structure

Effective action for compactifications on generalized geometries

➪ Questions:

1. how to do it?
   - zero and light modes on $\widehat{Y}$
   - relation between $\widehat{Y}$ and Y
2. what is role of other classes (not half-flat) of manifolds?
3. what is the supergravity meaning of the torsion?

➪ “Answers”

1. rewrite ten-dimensional supergravity in $N=2$-form and then truncate
2. can also be used as compactifications – respecting mirror symmetry
3. determines potential – kinetic terms unchanged

Rewrite ten-dimensional supergravity in $N = 2$-form

[de Wit,Nicolai]

- ➪ consider space-time background with Lorentz group $SO(1,3) \times SU(3)$
  but do not do Kaluza-Klein reduction
- ➪ decompose ten-dimensional fields accordingly
- ➪ truncate all $\mathbf{3} + \bar{\mathbf{3}}$ ⇒ only two gravitinos in gravitational multiplets
  ⇒ ten-dimensional fields can be arranged in $N = 2$ multiplets of $SO(1,3)$

multiplet	$SU(3)$ rep.	field content
gravity multiplet	$\mathbf{1}$	$(g_{\mu\nu}, C_\mu, \psi_\mu)$
tensor multiplet	$\mathbf{1}$	$(B_{\mu\nu}, \Phi, C_{mnp}, \lambda)$
vector multiplets	$\mathbf{8} + \mathbf{1}$	$(C_{\mu np}, g_{mn}, B_{mn}, \psi_m)$
hypermultiplets	$\mathbf{6}$	$(g_{mn}, C_{mnp}, \psi_m)$

insert into ten-dimensional action but do not integrate over internal manifolds

$$
\begin{aligned}
S_{\mathsf{NS}} &= \int d^{10}x\sqrt{g}\,e^{-2\Phi}\left[R+4(\partial\Phi)^2-\tfrac{1}{12}H^2\right] \\
&= \int d^{10}x\sqrt{g^{(4)}}\Big[R^{(4)}-2(\partial\Phi^{(4)})^2-\tfrac{1}{12}e^{-4\Phi^{(4)}}H^2_{(4)} \\
&\qquad -\tfrac{1}{4}g^{mp}g^{nq}(\partial_\mu g_{mn}\partial^\mu g_{pq}+\partial_\mu B_{mn}\partial^\mu B_{pq})+\ldots\Big]
\end{aligned}
$$

$$
\text{where} \qquad g^{(4)}_{\mu\nu}=e^{-2\Phi_4}g_{\mu\nu}\ , \qquad \Phi^{(4)}=\Phi-\tfrac{1}{4}\ln\det g_{mn}
$$

interpret last term as metric on the space of metric/B-field deformations $\mathcal{M}$

<u>find:</u> $\mathcal{M}$ is product of special geometries [Hitchin]

$$\mathcal{M}=\mathcal{M}_J\times\mathcal{M}_\Omega$$

with

$$e^{-K_J}=J\wedge J\wedge J\ , \qquad e^{-K_\Omega}=\Omega\wedge\bar{\Omega}$$

Compute the (Killing pre-) potential

rewrite supersymmetry transformation of gravitinos $\psi_{A\,\mu}$ in '$N = 2$-form'

$$\delta\psi_{A\,\mu} = D_\mu\varepsilon_A + i\gamma_\mu S_{AB}\varepsilon^B + \dots\,, \qquad S_{AB} = \tfrac{i}{2}\, e^{\frac{1}{2}K_V}\,\vec{\sigma}_{AB}\vec{P}\,, \qquad \vec{P} = \text{Killing prepotentials}$$

➪ IIA

$$P^1 + iP^2 \;=\; e^{\frac{1}{2}K_\Omega + \Phi^{(4)}}\left[e^{-(B+iJ)} \wedge d\Omega\right]\,, \qquad P^3 \;=\; e^{2\Phi^{(4)}}\left[e^{-(B+iJ)} \wedge F_{\mathsf{A}}\right]$$

➪ IIB

$$F \equiv \textstyle\sum_{\text{RR-forms}} F^{\text{RR}}$$

$$P^1 + iP^2 = e^{\frac{1}{2}K_J + \Phi^{(4)}}\left[\Omega \wedge d e^{-(B+iJ)}\right]\,, \qquad P^3 = e^{2\Phi^{(4)}}\left[\Omega \wedge F_{\mathsf{B}}\right]$$

Remarks:

- 'electric' mirror symmetry: $e^{-(B+iJ)} \leftrightarrow \Omega\,, \qquad F_{\mathsf{A}} \leftrightarrow F_{\mathsf{B}}$
- 'magnetic' mirror symmetry: 'half-flat' $SU(3)\times SU(3)$ manifolds

  introduce all odd forms $\Omega_1 + \Omega_3 + \Omega_5$
- F, D-terms of $N = 1$

Kaluza-Klein reduction

problem: distinction between heavy and light

idea: keep finite subspace of modes such that $N = 2$ is preserved

$\Rightarrow$ insist on special Kähler geometry on subspace

$\Rightarrow$ space of even/odd forms non-degenerate

result: K and $\vec{P}$ descend to subspace

$$e^{-K_J} = \int_{Y_6} J \wedge J \wedge J \,, \qquad e^{-K_\Omega} = \int_{Y_6} \Omega \wedge \bar{\Omega}$$

$$\text{IIA:} \quad P^1 + iP^2 = e^{\frac{1}{2}K_\Omega + \Phi^{(4)}} \int_{Y_6} e^{-(B+iJ)} \wedge d\Omega \,, \qquad P^3 = e^{2\Phi^{(4)}} \int_{Y_6} e^{-(B+iJ)} \wedge F_{\mathsf{A}}$$

$$\text{IIB:} \quad P^1 - iP^2 = e^{\frac{1}{2}K_J + \Phi^{(4)}} \int_{Y_6} \Omega \wedge d e^{-(B+iJ)} \,, \qquad P^3 = -e^{2\Phi^{(4)}} \int_{Y_6} \Omega \wedge F_{\mathsf{B}}$$

Conclusions/open problems

- half-flat manifolds provide missing electric NS-fluxes
  ⇒ generalization of notion of mirror symmetry including fluxes
- compactifications on manifolds with $SU(3)$ structure
  compatible with $N = 2$ supergravity
  - Kähler potential is "unchanged"
  - scalar potential depends on the torsion
  - "electric" mirror symmetry intact
- mirror of magnetic fluxes – "magnetic" mirror symmetry
  ⇒ manifolds of $SU(3) \times SU(3)$ structure and/or non-geometric background [Hull, Shelton,Taylor,Wecht]
  ⇒ relation with non-commutative geometry [Bouwknegt,Mathai,Rosenberg]
- deformation theory/moduli space of manifolds with $SU(3)$ structure
- relation with (mirror of) Calabi-Yau [Berglund,Mayr]
- mirror of ridged Calabi-Yau
- include warped space-time [Giddings,Maharana]