MIRROR SYMMETRY AND REALITY

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based on works with

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and discussions / collaborations with

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Ragnar Buchweitz
Ilka Brunner
Kazuo Hosomichi
Misha Kapranov
Mike Hopkins
Ronald Douglas ·· & many (other) physicist

closed-oriented

... so far

open-oriented

Homological Mirror Symmetry

$D^b(\mathrm{Coh}\, M) \leftrightarrow \mathrm{Fuk}(\tilde{M})$

$\vdots \qquad \vdots$

unoriented

Spaces with involutions

real geometry

$\mathbb{C}$ with anti-involutions

Clifford algebra

$\vdots$

heterotic

Spaces with G-bundles

"Flux"

Generalized Geometry

Orientifold

data : (X, τ) manifold with involution
(2d QFT)

gauge the QFT by $\mathbb{Z}_2 = \{1, \tau \cdot \Omega\}$

$\Omega : (t, \sigma) \to (t, -\sigma)$

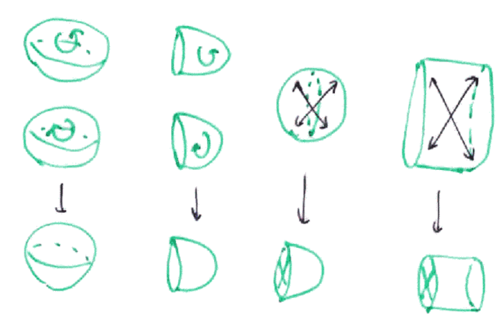
for σ-model : sum over maps

oriented double cover of Σ

$$\begin{array}{ccc} \tilde{\Sigma} & \longrightarrow & X \\ \Omega \downarrow & \circlearrowleft & \downarrow \tau \\ \tilde{\Sigma} & \longrightarrow & X \end{array}$$

$\tilde{\Sigma}$

Σ



D-branes = boundary conditions for open string

data: $(\mathbf{W}, \mathbf{E})$

$W = \bigcup_i W_i \subset X$ submanifolds

$E = \bigcup_i E_i \xrightarrow{\mathbb{C}^{n_i}} W_i$ hermitian vector bundle with connection

For $\tau\Omega$-orientifold, we need

$$\begin{array}{ccc} \boldsymbol{\gamma} : E^* & \longrightarrow & E \\ \downarrow & & \downarrow \\ W & \xrightarrow{\tau} & W \end{array}$$

(metric, connection)-invariant

invariant $(\tau W_i = W_{\tau(i)})$

$\gamma_x : E_x^* \to E_{\tau x}$

$\gamma_{\tau x}^T = c_x \gamma_x \ : \ c_x c_{\tau x} = 1$

$(c_x = \pm 1$ on $W^\tau = X^\tau \cap W)$

on $X \times \mathbb{R}^4$:

$\mathcal{N} = 1$ SUSY $\Rightarrow$ A type or B-type (for Type IIA or Type IIB)

X: CY^3 assumed here

A $\tau : X \to X$ antiholomorphic $(\overset{\text{isometry}}{\Longrightarrow}$ antisymplectic$)$

$W \subset X$ (Special) Lagrangian

$E \to W$ flat

Light spectrum:

Closed : $H_+^{1,1}(X) \Rightarrow \overset{U(1)}{\text{vectors}}$

$H_-^{1,1}(X) \oplus H_+^3(X) \Rightarrow \overset{\text{neutral}}{\text{chirals}}$

open : $\bigoplus_{i,j} HF\big((W_i E_i)(W_j E_j)\big)^{(\tau,\gamma)} \Rightarrow$ vectors, charged chirals

interactions

Certain "holomorphic" couplings are computable

"GW-inv" $\quad \tilde{\Sigma} \xrightarrow{\text{holo}} X$, $\ \circlearrowleft_\Omega \quad \circlearrowleft_\tau$

$\Sigma =$  $\Rightarrow$ superpotential terms

B $\quad \tau : X \to X$ holomorphic

$W \subset X$ complex submfd } (stable)
$E \to W$ holomorphic } coherent sheaf

Light spectrum $\left(\tau^*\Omega = \pm\Omega \left\langle \begin{matrix} 09,\ 05 \\ 07,\ 03 \end{matrix} \right. \right)$

Closed : $H^{2,1}_{\mp}(X) \Rightarrow U(1)$ vectors

$H^{2,1}_{\pm}(X) \oplus H^{1,1}(X) \Rightarrow$ neutral chirals

open : $\left(\bigoplus_{i,j} \bigoplus_{p} \mathrm{Ext}^p(E_i, E_j) \right)^{(\tau,\gamma)} \Rightarrow$ vectors
charged chirals

holomorphic couplings

$\longleftrightarrow$ deformation theory

Mirror Symmetry : A $\leftrightarrow$ B

Include **Tachyons** in D-brane data

before Orientifold :

IIB data: $(E, \mathbb{T})$ $\quad E = E_0 \oplus E_1$ $\mathbb{Z}_2$ graded b'dle over X

$\mathbb{T} = \begin{pmatrix} 0 & T \\ T' & 0 \end{pmatrix}$ odd endomorphism of E

If $\mathbb{T} : E \to E$ isom $\Rightarrow$ "Nothing"

$W \sim \mathrm{Zero}(\mathbb{T})$

$E \cong E \oplus F, \ F_0 \cong F_1 \ \to K(X)$

IIA data: $(E, \mathbb{T})$ $\quad E$ (ungraded) b'dle over X

$\mathbb{T} \in \mathrm{End}(E)$ hermitian

If $\mathbb{T} : E \to E$ isom with certain eigenvalues $(+1, -1)$
it is regarded as "Nothing"

topological classification $\to K^{-1}(X)$

With orientifold

IIB $(E, \mathbb{T}, \gamma)$ $\quad E = E_0 \oplus E_1$, $\mathbb{T} = \begin{pmatrix} 0 & T \\ T^T & 0 \end{pmatrix}$

$$\gamma : E^* \to \tau^* E$$

$$\gamma \mathbb{T}^T \gamma^{-1} = \tau^* \mathbb{T}$$

4 cases

- γ preserves E_0, E_1
  - $\gamma^T = \gamma \to \underline{KR(X)}$
  - $\gamma^T = -\gamma \to \underline{KR^{-4}(X)}$
- $\gamma : E_0 \leftrightarrow E_1$
  - $\gamma^T = \gamma \to \underline{KR^{-6}(X)}$
  - $\gamma^T = -\gamma \to \underline{KR^{-2}(X)}$

IIA $(E, \mathbb{T}, \gamma)$ $\quad \gamma : E^* \to \tau^* E$

- $\gamma \mathbb{T}^T \gamma^{-1} = \tau^* \mathbb{T}$
  - $\gamma^T = \gamma \to \underline{KR^{-7}(X)}$
  - $\gamma^T = -\gamma \to \underline{KR^{-3}(X)}$
- $\gamma \mathbb{T}^T \gamma^{-1} = -\tau^* \mathbb{T}$
  - $\gamma^T = \gamma \to \underline{KR^{-1}(X)}$
  - $\gamma^T = -\gamma \to \underline{KR^{-5}(X)}$

$\underline{KR^{-p}(X)}$ Atiyah's KR-theory

$KR(X)$ Grothendieck group of Real vector bundle

$$\begin{array}{ccc} E & \xrightarrow{\sigma} & E \\ \downarrow & & \downarrow \\ X & \xrightarrow{\tau} & X \end{array} \quad \text{anti linear}$$

E : hermitian $\Longleftrightarrow$

$$\begin{array}{ccc} E^* & \xrightarrow{\gamma} & E \\ \downarrow & & \downarrow \\ X & \xrightarrow{\tau} & X \end{array} \quad \text{linear}, \ \gamma_{\tau x}^T = \gamma_x$$

$$KR^{-p}(X) = KR(X \times B^p / \partial(X \times B^p))$$

$$\cong [X, \mathcal{F}^p(H_{\mathbb{C}})]^{\mathbb{Z}_2} \qquad \text{Atiyah-Singer}$$

Space of graded $Cliff_p$ linear Self adjoint Fredholm operators

T-duality

H

Type II $\quad X^{10-n}\times T^n \xleftrightarrow{\;T\;} X^{10-n}\times \tilde{T}^n$ $\quad$ n even: IIA → IIA, IIB → IIB; n odd: IIA ↔ IIB

map of D-brane charge

Poincaré bdle

$$\begin{tikzcd} & K^{-i}(X\times T^n\times \tilde{T}^n) \arrow[dr, "\mathrm{ind}\,\not{\partial}"] & \\ K^{-i}(X^{10-n}\times T^n) \arrow[ur, "\otimes P"] \arrow[rr, "T"] & & K^{-i-n}(X\times \tilde{T}^n) \end{tikzcd}$$

Type II Orientifolds

$$\begin{tikzcd} & KR^{-i}(X\times T^n\times \tilde{T}^n) \arrow[dr, "\mathrm{ind}\,\not{\partial}"] & \\ KR^{-i}(X\times \underset{+\cdots+}{T^n}) \arrow[ur, "\otimes P"] \arrow[rr, "T"] & & KR^{-i-n}(X\times \underset{-\cdots-}{\tilde{T}^n}) \end{tikzcd}$$

Strominger-Yau-Zaslow: mirror symmetry of CY^3 is T-duality on SLAG T^3 fibrations

D-brane charge on Type I $(\tau=\mathrm{id}) \to KO(X)=KR(X)$

$$KO(\mathbb{R}^4\times X_{cy}) \xrightarrow[\text{mirror}]{T^3} KR^{-3}(\mathbb{R}^4\times \tilde{X}_{cy})$$

We want the analogue of HMS Kontsevich

$$\begin{tikzcd} D^b(\mathrm{Coh}\,X) \arrow[r, "\sim"] \arrow[d] & \mathrm{Fuk}(\tilde{X}) \arrow[d] \\ K(X) \arrow[r, "\sim"] & K^{-1}(\tilde{X}) \end{tikzcd}$$

for unoriented strings:

$$\begin{tikzcd} ? \arrow[r, "\sim"] \arrow[d] & ? \arrow[d] \\ KR^{-2m}(X) \arrow[r, "T"] & KR^{-3}(\tilde{X}) \end{tikzcd}$$

What are "?"'s ?

Before orientifold (IIB)

$E = E_0 \oplus E_1$ has a finer $\mathbb{Z}$-grading

$= \bigoplus_{i\in\mathbb{Z}} \mathcal{E}_i \qquad E_0 = \bigoplus_{\text{even}} \mathcal{E}_i \quad , \; E_1 = \bigoplus_{\text{odd}} \mathcal{E}_i$

$T = Q + Q^\dagger \qquad Q$ grade $+1$

holomorphic

$Q^2 = 0$

$\cdots \to \mathcal{E}_{-1} \xrightarrow{Q_{-1}} \mathcal{E}_0 \xrightarrow{Q_0} \mathcal{E}_1 \xrightarrow{Q_1} \cdots$ complex of sheaves

$\rightsquigarrow D^b(\mathrm{Coh}\,X)$

With orientifold

$\gamma : E^* \to \tau^* E \quad , \; \gamma^* T = \gamma T^T \gamma^{-1}$

$\mathrm{grading} = -\gamma(\mathrm{grading})^T \gamma^{-1} + \#$

$$\begin{tikzcd}
\to \tau^*\mathcal{E}_{-1} \arrow[r, "\tau^* Q_{-1}"] & \tau^*\mathcal{E}_0 \arrow[r, "\tau^* Q_0"] & \tau^*\mathcal{E}_1 \to \\
\to \mathcal{E}^*_{q+1} \arrow[u, "\gamma"] \arrow[r, "Q_q^T"] & \mathcal{E}^*_q \arrow[u, "\gamma"] \arrow[r, "Q^T_{q-1}"] & \mathcal{E}^*_{q-1} \arrow[u, "\gamma"] \to
\end{tikzcd}$$

but $\mathcal{E}^* = ? \qquad \mathcal{E}^{**} \overset{?}{=} \mathcal{E}$

B-branes in LG model

M. Kontsevich

$W =$ Polynomial in $X_1, \dots, X_N \longrightarrow$ LG superpotential

Data of D-brane: matrix factorization of W
(Object)

$Q(x) = \begin{pmatrix} 0 & f(x) \\ g(x) & 0 \end{pmatrix}$ $2r\times 2r$ entries $\in \mathbb{C}[X] = \mathbb{C}[X_1, \dots, X_N]$

$Q(x)^2 = W(x)\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \quad (\text{i.e. } fg = gf = W\mathbb{1}_r)$

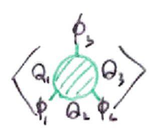
equivalently $M = \mathbb{C}[X]^{\oplus r} \oplus \mathbb{C}[X]^{\oplus r}$ $\mathbb{Z}_2$ graded free $\mathbb{C}[X]$ module

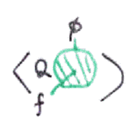
$Q \in \mathrm{End}^{\mathrm{od}}_{\mathbb{C}[X]}(M)\ , \; Q^2 = W\,\mathrm{id}_M$

Open string states: $\mathcal{H}_{Q_1,Q_2} = \mathrm{Hom}^{\mathrm{ev}}_{\mathbb{C}[X]}(M_1, M_2) \oplus \mathrm{Hom}^{\mathrm{od}}_{\mathbb{C}[X]}(M_1, M_2)$
(morphisms)

$\mathcal{D}\phi = Q_2\phi - (-1)^{|\phi|}\phi Q_1$

ground states : $H^{\mathrm{ev/od}}(Q_1, Q_2) = \mathcal{D}$-cohomology

 $= \mathrm{res}_{\mathrm{Crit}(W)}\left(\dfrac{\mathrm{Str}(\phi_3\phi_2\phi_1(dQ_1)^N)}{\partial_1 W \cdots \partial_N W}\right)$

 $= \mathrm{res}_{\mathrm{Crit}(W)}\left(\dfrac{f\,\mathrm{Str}(\phi(dQ)^N)}{\partial_1 W \cdots \partial_N W}\right)$ Kapustin-Li

in LG orbifold

H
H - Walcher
Walcher
(also Dell Aquilla et al)

$W(x_1, \ldots, x_N)$ homogeneous degree d

$$LG \xrightarrow{IR} SCFT \qquad \hat{c} = N(1 - \tfrac{2}{d})$$

R-symmetry : $X_i \to e^{2i\alpha/d} X_i$

$(W \to e^{2i\alpha} W)$

$Q(x_1, \ldots, x_N)$ also "homogeneous" (for BSCFT)

$$(*) \quad e^{i\alpha R} Q(e^{2i\alpha/d} X) e^{-i\alpha R} = e^{i\alpha} Q(X)$$

IR R-symmetry : $\phi(x) \to e^{i\alpha R_2} \phi(e^{2i\alpha/d} X) e^{-i\alpha R_1}$ on $\mathcal{H}_{1,2}$

$\Rightarrow H^{ev,od}_{(om)}$ refined to $H^p_{(om)} \quad p \in \mathbb{Q}$

$$\xrightarrow{Q} H^p_{om} \xrightarrow{Q} H^{p+1}_{om} \xrightarrow{Q}$$

$\mathbb{Z}_d$ -orbifold by $X_i \to e^{2\pi i/d} X_i$

$$(*) \Rightarrow \quad e^{i\pi R} Q(e^{2\pi i/d} X) e^{-i\pi R} = -Q(X)$$

$$(*) \Rightarrow \quad e^{i\pi R} Q(e^{2\pi i/d} X) e^{-i\pi R} = -Q(X)$$

$\sigma = (-1)^F$: $\quad \sigma e^{i\pi R} Q(e^{2\pi i/d} X)(\sigma e^{i\pi R})^{-1} = Q(X)$

— invariant.

Choose $e^{i\pi\varphi}$ so that

$\rho_\varphi(e^{2\pi i/d}) = e^{i\pi\varphi} \sigma e^{i\pi R}$ has order d.

($\varphi \equiv \varphi + 2$ labels $\mathbb{Z}_d$ - representation on M)

$\mathbb{Z}_d$ action on $\mathrm{Hom}(M_1, M_2)$: $\phi(x) = \rho_{\varphi_2}(\omega) \phi(\omega x) \rho_{\varphi_1}(\omega)^{-1}$

$$\mathcal{H}^{orb}_{(Q_1,\varphi_1),(Q_2,\varphi_2)} = \mathrm{Hom}(M_1, M_2)^{\mathbb{Z}_d}$$

R-sym : $\phi(x) \to e^{i\alpha R_2} \phi(e^{2i\alpha/d} X) e^{-i\alpha R_1}$

R-charge can be made $\subset \mathbb{Z}$ by dressing $e^{i\alpha(\varphi_2 - \varphi_1)}$

$\rightsquigarrow \mathbb{Z}$ - grading

$\varphi \not\equiv \varphi + 2$ for "$\mathbb{Z}$ - graded B-branes" matrix factorizations of W $\overset{\text{Eisenbud}}{\longleftrightarrow}$ Maximal Cohen Macaulay modules over $R = \mathbb{C}[X]/W$

1980's Auslander, Reiten
Knorrer, Buchweitz, ...

Buchweitz (1980's)

$$\underline{MF}(W) \cong \underline{MCM}(R) \cong \underline{D}^b(R)$$

$\underline{MCM}$: projectives $= 0$; $\underline{D}^b$: modulo perfect cplxs

Orlov (2005) $W = G(\phi_1, \cdots, \phi_N)$ degree d

$M = \{W = 0\} \subset \mathbb{CP}^{N-1}$ hypersurface

$d = N$: $\mathrm{gr}\underline{MF}(W) \xrightarrow[\sim]{\Psi_t} D^b(\mathrm{Coh}\, M)$

$\mathrm{gr}\underline{MF}$: $\mathbb{Z}$ graded B-branes in LG orbifold $W = G/\mathbb{Z}_d$

$d > N$: $\mathrm{gr}\underline{MF}(W) \xrightarrow{\sim} \langle \underbrace{\bullet, \cdots, \bullet}_{(d-N)\text{ objects}}, D^b(\mathrm{Coh}M) \rangle$ semi-orthogonal decompo

$d < N$: $D^b(\mathrm{Coh}\,M) \xrightarrow{\sim} \langle \underbrace{\mathcal{O}_M(1-(N-d)), \cdots, \mathcal{O}_M}_{(N-d)\text{ objects}}, \mathrm{gr}\underline{MF}(W) \rangle$

B-parity in LG model

$$\mathcal{L} \supset \int d\theta^+ d\theta^- \, W(X)$$

B-parity $\theta^+ \leftrightarrow \theta^- \Rightarrow \tau$ must obey $W(\tau X) = -W(X)$

action on Matrix factorizations

$\cdots$ Functor $MF(W) \xrightarrow{P} MF(W)^{op}$, over τ, Commuting with SUSY $\mathcal{D}$

$$(M, Q, \sigma) \to (P(M), P(Q), P(\sigma))$$

$Q_1 \to Q_2$ $\qquad P_{Q_1,Q_2} : \mathcal{H}_{Q_1,Q_2} \longrightarrow \mathcal{H}_{P(Q_2),P(Q_1)}$ $\qquad P(Q_1) \leftarrow P(Q_2)$

$\circlearrowleft \mathcal{D} \qquad \circlearrowleft \mathcal{D}$

s.t. $P^2 \cong \mathrm{id}$

Example $\quad P(M) = M^*$ dual module

$P(\sigma) = \sigma^*$ $\quad$ (1 on M^{ev*}, -1 on M^{od*})

$P(Q)(x) = -Q(\tau x)^T$

$P_{Q_1,Q_2}(\phi)(x) = \phi(\tau x)^T$

$A \to A^T$ is graded transpose $\langle A^T f, e\rangle = (-1)^{|A||f|}\langle f, Ae\rangle$

$$(AB)^T = -B^T A^T (-1)^{|B||A|} \ast$$

$$(P(Q)(x))^2 = Q(\tau x)^T Q(\tau x)^T \overset{\ast}{=} -(Q(\tau x) Q(\tau x))^T$$

$$= -(W(\tau x)\, id_M)^T = W(x)\, id_{P(M)} \quad ✓$$

$\phi \in \mathcal{H}_{1,2}$

$$\mathcal{D}P(\phi)(x) = P(Q_1)P(\phi)(x) - (-1)^{|\phi|} P(\phi)(x) P(Q_2)(x)$$

$$= -Q_1(\tau x)^T \phi(\tau x)^T + (-1)^{|\phi|}\phi(\tau x)^T Q_2(\tau x)^T$$

$$= -(-1)^{|\phi|}(\phi(\tau x) Q_1(\tau x))^T + (Q_2(\tau x)\phi(\tau x))^T$$

$$= P(\mathcal{D}\phi)(x) \quad ✓$$

$$P^2(M) = M^{**}$$

$\exists$ isomorphism $\iota : M \to M^{**}$ $\quad \langle \iota(e), f\rangle = (-1)^{|e||f|}\langle f, e\rangle$

$$\iota : (M, Q, \sigma) \xrightarrow{\cong} (P^2(M), P^2(Q), P^2(\sigma)) \quad \therefore ✓$$

Another example : $\tilde{P} = [1]\circ P$

$$(M, Q, \sigma) \to (M^*, -\tau^* Q^T, -\sigma^*)$$

Invariant brane configurations

$\{B_i\}_{i\in I}$ $\quad B_i = (M_i, Q_i, \sigma_i)$

$\exists\, p : I \to I$ involution s.t.

$P(B_i) \underset{\cong}{\xrightarrow{U_i}} B_{p(i)}$ $\quad$ i.e. $U_i : M_i^* \longrightarrow M_{p(i)}$

s.t. $\quad Q_{p(i)}(x) = -U_i Q_i(\tau x)^T U_i^{-1}$

$\sigma_{p(i)} = U_i \sigma_i^* U_i^{-1}$

Parity action

$$B_i \to B_{p(i)}$$

$$\mathcal{H}_{i,j} \to \mathcal{H}_{p(j),p(i)}$$

$$\phi(x) \mapsto U_i \phi(\tau x)^T U_j^{-1}$$

Require $\underline{P^2 = id}$ equal! (not just isomorphic)

$$\phi(x) \overset{P}{\mapsto} U_i\phi(\tau x)^T U_j^{-1} \overset{P}{\longmapsto} U_{p(j)} U_j^{-T} \underbrace{\phi(\tau^2 x)^{TT}}_{\iota_j \phi(x) \iota_i^{-1}} U_i^T U_{p(i)}^{-1}$$

$\overset{\text{This} = \phi(x)}{\Longrightarrow}$ $\quad U_{p(j)} U_j^{-T} \iota_j = \epsilon \quad$ (j-independent sign)

Orbifold case

Γ finite abelian group, W : Γ-invariant
$W(gx) = W(x)$

$\to$ LG orbifold W/Γ.

$MF^{\Gamma}(W) \ni (M, Q, \rho, \sigma)$ $\quad \rho : \Gamma \to \mathrm{Aut}(M)$

Assume $\tau^2 \in \Gamma$

$$P : (M, Q, \rho, \sigma) \to (M^*, -\tau^* Q^T, P(\rho), \sigma^*)$$

$$P(\rho)(g) = \chi(g)\rho(g)^{-T}$$

($\chi : \Gamma \to \mathbb{C}^*$ character) does the job.

Inv. brane config.

$\{B_i\}_{i \in I}$ $\quad B_i = (M_i, Q_i, \rho_i, \sigma_i)$

$\exists p : I \to I$ involution s.t. $P(B_i) \xrightarrow[\cong]{U_i} B_{p(i)}$

i.e. $U_i : M_i^* \to M_{p(i)}$ s.t. $Q_{p(i)}(x) = -U_i Q_i(\tau x)^T U_i^{-1}$

$$\sigma_{p(i)} = U_i \sigma_i^* U_i^{-1}$$

$$\rho_{p(i)}(g) = U_i \rho_i(g)^{-T} U_i^{-1} \cdot \chi(g)$$

Parity action

$B_i \to B_{p(i)}$

$\mathcal{H}_{i,j} \to \mathcal{H}_{p(j),p(i)}$

$\phi(x) \mapsto U_i \phi(\tau x)^T U_j^{-1}$

Require $P^2 = id$ (equal)

$$\phi(x) \to U_i \phi(\tau x)^T U_j^{-1} \Rightarrow U_{p(j)} U_j^{-T} \underbrace{\phi(\tau^2 x)^{TT}} U_i^T U_{p(i)}^{-1}$$

$$\phi(\tau^2 x)^{TT} = \sigma_j \underbrace{\phi(\tau^2 x)} \sigma_i^{-1}, \qquad \phi(\tau^2 x) = \rho_j(\tau^2)^{-1}\phi(x)\rho_i(\tau^2)$$

this $= \phi(x)$

$\Longrightarrow$

$$U_{p(j)} U_j^{-T} \sigma_j \rho_j(\tau^2)^{-1} = c(p)$$

(j-indep phase)

$$U_j = U_{p^2(j)} = c(p)\rho_{p(j)}(\tau^2)\sigma_{p(j)}^{-1} U_{p(j)}^T$$

$$= c(p)^2 \underbrace{\rho_{p(j)}(\tau^2) U_j \rho_j(\tau^2)^T}_{= \chi(\tau^2) U_j}$$

$$\therefore \quad c(p)^2 = \chi(\tau^2)^{-1}$$

Alternatively we may consider "representation" of $0 \to \Gamma \to \hat{\Gamma} \to \mathbb{Z}_2 \to 0$ on $\{B_i\}_{i \in I} \subset MF(W)$

Invariant irreducible brane

$$B = (M, Q, \rho, \sigma)$$

$P(B) \overset{U}{\underset{\cong}{\to}} B$ invariant

$H^0(Q,Q) = \mathbb{C}\, \mathrm{id}_M$ irreducible

$$\Rightarrow \quad U U^{-T} \iota = \epsilon_B\, C(P)\, \rho(\tau^2) \qquad \epsilon_B = \pm 1$$

$B^{\oplus N}$ is compatible with $\{B_i\}_{i\in I}$

if $P(B^{\oplus N}) \overset{U\otimes\gamma}{\underset{\cong}{\longrightarrow}} B^{\oplus N}$, $\gamma\gamma^{-T} = \epsilon_B$

$$\begin{cases} \epsilon_B = 1 & : \ SO(N) \text{ gauge group} \\ \epsilon_B = -1 & : \ USp(N) \text{ gauge group} \end{cases}$$

Example

$$W = X^{k+2} \xrightarrow{IR} \mathcal{N}=2 \text{ minimal model } \ c = \frac{3k}{k+2}$$

Cardy brane : $Q_L = \begin{pmatrix} 0 & X^{L+1} \\ X^{k+1-L} & 0 \end{pmatrix} \quad \sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$R_L = \begin{pmatrix} \frac{1}{2} - \frac{L+1}{k+2} & 0 \\ 0 & -\frac{1}{2} + \frac{L+1}{k+2} \end{pmatrix}$$

$\mathbb{Z}_{k+2}$-orbifold by $X \to \omega X \quad \omega^{k+2} = 1$

$$\rho_M(\omega) = \begin{pmatrix} \omega^{-\frac{L+1+M}{2}} & 0 \\ 0 & \omega^{\frac{L+1-M}{2}} \end{pmatrix} \qquad \begin{matrix} M \in \mathbb{Z}_{2(k+2)} \\ L+1+M : \text{even} \end{matrix}$$

$$(Q_L, \rho_M, \sigma) \overset{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}{\cong} (Q_{k-L}, \rho_{M+k+2}, -\sigma)$$

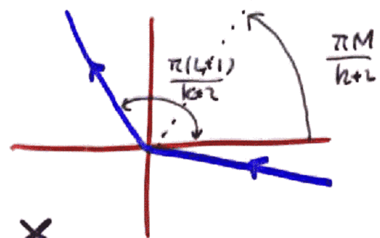
$\tau^{k+2} = -1 \Rightarrow$ Parity $W(\tau X) = -W(X)$

$$P(Q_L, \rho_M, \sigma) \underset{\begin{pmatrix} 0 & \tau^{-\frac{L+1-M}{2}} \\ -\tau^{\frac{L+1+M}{2}} & 0 \end{pmatrix}}{\cong} (Q_L, \rho_{-M}, -\sigma)$$

$$W = X^{k+2} / \mathbb{Z}_{k+2} \quad \overset{\text{mirror}}{\longleftrightarrow} \quad W = X^{k+2}$$

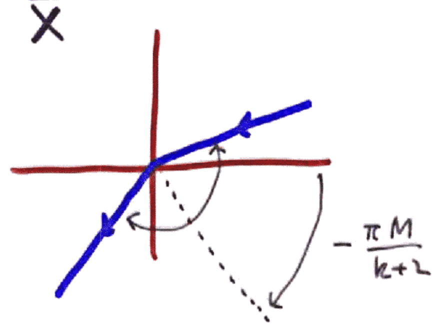
B(A)-brane orientifold $\longleftrightarrow$ A(B)-brane orientifold

(Q_L, P_M, σ) $\longleftrightarrow$



$\big\downarrow P$ $\quad$ $X \downarrow \bar{X}$ $\quad$ $\downarrow P$

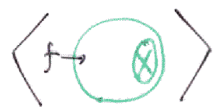
$(Q_L, P_{-M}, -\sigma)$ $\longleftrightarrow$



Crosscap state in topological LG

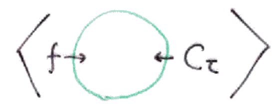
involution $\tau : X_i \to \tau_{ij} X_j$ $\quad$ $W(\tau X) = -W(X)$

Compute topologically twisted $\mathbb{R}P^2$-diagram



$f \in \mathbb{C}[X] / (\partial_1 W, \cdots, \partial_N W)$

Answer $=$



$= \mathrm{Res}_{\mathrm{Crit}(W)} \left(\frac{f\, C_\tau}{\partial_1 W \cdots \partial_N W} \right)$

- $C_\tau = 0$ if X^τ is not middle dimensional
- If $\dim X^\tau = \frac{1}{2} \dim X$ (requires $\dim_{\mathbb{C}} X$ even)

C_τ = boundary state for brane wrapped on X^τ

$$x = (\underset{-1}{x_\perp}, \underset{+1}{x_\parallel}) \qquad W = \sum_{i=1}^{N/2} x_\perp^i R_i(x) \qquad \{\eta_i, \bar{\eta}_j\} = \delta_{ij}$$

$$Q_\tau = \sum_{i=1}^{N/2} (x_\perp^i \eta_i + R_i(x) \bar{\eta}_i)$$

$$C_\tau = \frac{(-1)^{\frac{N}{2}(\frac{N}{2}+1)/2}}{N!} \mathrm{Str}((dQ_\tau)^N)$$

Index formula

$$\mathrm{Tr}_{i\,P(i)}(-1)^F P \;=\; i$$  $$P \;=\; ?$$

$W(X_1,\cdots,X_N)$ degree d $/\mathbb{Z}_d : X_i \to \omega X_i \quad \omega^d=1$

$$\tau : X_i \to \tau_i X_i \qquad \tau_i^d=-1 \qquad \tau_i^2=\bar{\omega}\in\mathbb{Z}_d$$

$B=(M,Q,\rho,\sigma)$

$$\mathrm{Tr}^{orb}_{B,P(B)}(-1)^F P = \frac{1}{d}\sum_{\omega\in\mathbb{Z}_d}\mathrm{Tr}_{\underbrace{\mathcal{H}_{B,P(B)}}_{\shortparallel}}\left((-1)^F P_\tau \omega\right)$$

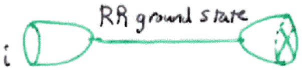
$(\text{Chan-Paton})\otimes(\text{polynomial})$

$$\mathrm{Tr}_{CP}\left((-1)^F P_\tau \omega\right) = \mathrm{Tr}_M\left(\sigma\rho(\omega^2\tau^2)^{-1}\right) C(P_{\omega\tau})^{-1}$$

$$\mathrm{Tr}_{\text{polynomial}}(P_\tau\omega) = \sum_{\{n_i\}}\tau_i^{n_i}\omega^{n_i} = \frac{1}{\prod_{i=1}^N(1-\omega\tau_i)}$$

$$\underline{\mathrm{Tr}^{orb}_{B,P(B)}(-1)^F P = \frac{1}{d}\sum_{\omega\in\mathbb{Z}_d}\mathrm{Tr}_M\left(\sigma\rho(\omega^2\tau^2)^{-1}\right)\frac{C(P_{\omega\tau})^{-1}}{\prod_{i=1}^N(1-\omega\tau_i)}}$$

RR charge

i  $P \;=\; i$  P

 ω (M,Q,ρ,σ) $$= \frac{1}{\sqrt{d}}\frac{1}{(1-\omega)^{N/2}}\mathrm{Tr}_M\left(\sigma\rho(\omega)^{-1}\right)$$ Walcher

ω  P $$= \underline{\frac{(1-\omega)^{N/2}}{\sqrt{d}}\sum_{\substack{\tilde{\omega}\in\mathbb{Z}_d\\ \tilde{\omega}^2\tau^2=\omega}}\frac{C(P_{\tilde{\omega}\tau})^{-1}}{\prod_{i=1}^N(1-\tilde{\omega}\tau_i)}}$$

$$\overset{d:\text{odd}}{=} \frac{C(P_{\tilde{\omega}\tau})^{-1}}{\sqrt{d}}\left(\frac{1+\tilde{\omega}\tau}{1-\tilde{\omega}\tau}\right)^{N/2} \qquad \exists!\,\tilde{\omega}\in\mathbb{Z}_d \quad \tilde{\omega}^2\tau^2=\omega$$

$$\overset{d:\text{even}}{=} \begin{cases} \dfrac{1}{\sqrt{d}}\left[C(P_{\tilde{\omega}\tau})^{-1}\prod_{i=1}^N\left(\dfrac{1+\tilde{\omega}_i\tau_i}{1-\tilde{\omega}\tau_i}\right)^{\frac{1}{2}}+C(P_{-\tilde{\omega}\tau})^{-1}\prod_{i=1}^N\left(\dfrac{1-\tilde{\omega}\tau_i}{1+\tilde{\omega}\tau_i}\right)^{\frac{1}{2}}\right] & \text{if} \quad \omega=\tilde{\omega}^2\tau_i^2 \\ 0 & \text{otherwise} \end{cases}$$

Other cases : W = quasi-homogeneous degree $d_1, \cdots, d_N$

$$\text{mod } \mathbb{Z}_d : X_i \to \omega_i X_i \qquad \omega_i = e^{\frac{2\pi i}{d_i} m}$$

$$\tau : X_i \to \tau_i X_{\sigma(i)}, \quad \tau_i^d = -1 \quad d_{\sigma(i)} = d_i \quad \tau_i \tau_{\sigma(i)} = \bar{\omega}_i \in \mathbb{Z}_d$$

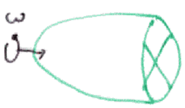
$$\mathrm{Tr}_{poly} = \frac{1}{\prod_{\sigma(i)=i}(1-\omega_i\tau_i)} \; \frac{1}{\prod_{\{i,\sigma(i)\}}(1-\omega_i^2\tau_i\tau_{\sigma(i)})}$$ This may vanish

$\Rightarrow$ use TFT correlator  in such sector

$\omega \in \mathbb{Z}_d$

(ω) : restriction to X_i with $\omega_i = 1$ $\quad N^{(\omega)} = \#$ such fields

$\{\phi_\beta^{(\omega)}\} \subset \mathbb{C}[X^{(\omega)}]/\partial W^{(\omega)}$ chiral ring basis

$\phi_\beta^{(\omega)}$ ω  P

Crosscap state in (ω)-theory

$$= \sum_{\substack{\tilde{\omega} \in \mathbb{Z}_H \\ \omega = \tilde{\omega}^2\tau^2}} \frac{c(P_{\tilde{\omega}\tau})^{-1}}{\sqrt{d}\, N^{(\omega)}!} \prod_{\substack{\sigma(i)=i \\ \omega_i \neq 1}} \left(\frac{1+\tilde{\omega}_i q_i}{1-\tilde{\omega}_i q_i}\right)^{\frac{1}{2}} \mathrm{Res}^{(\omega)}\left(\frac{\phi_\beta^{(\omega)}\, C^{(\omega)}_{P_{\tilde{\omega}\tau}}}{\partial_1 W^{(\omega)} \cdots \partial_{N^{(\omega)}} W^{(\omega)}}\right)$$

Orientifold

data : (X, τ) manifold with involution (2d QFT)

gauge the QFT by $\mathbb{Z}_2 = \{1, \tau\cdot\Omega\}$

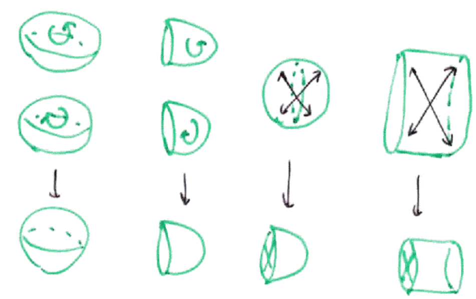
$$\Omega : (t, \sigma) \to (t, -\sigma)$$

for σ-model : sum over maps

oriented double cover of Σ

$$\begin{array}{ccc} \tilde{\Sigma} & \longrightarrow & X \\ \Omega \downarrow & \circlearrowleft & \downarrow \tau \\ \tilde{\Sigma} & \longrightarrow & X \end{array}$$

$\tilde{\Sigma}$



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