GENERALIZED COMPLEX STRUCTURES – AN INTRODUCTION

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BACKGROUND

BASIC SCENARIO

- manifold M^n
- replace T by $T \oplus T^*$
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- manifold M^n

- replace T by $T \oplus T^*$

- inner product of signature (n,n)

$$(X+\xi, X+\xi) = i_X \xi$$

- skew adjoint transformations:

$$\mathrm{End}\, T \oplus \Lambda^2 T^* \oplus \Lambda^2 T$$

- .... *in particular* $B \in \Lambda^2 T^*$

TRANSFORMATIONS

- exponentiate B:

  $X + \xi \mapsto X + \xi + i_X B$

- $B \in \Omega^2$ ... *the B-field*

- natural group $\text{Diff}(M) \ltimes \Omega^2(M)$

SPINORS

- Take $S = \Lambda^* T^*$

- $S = S^{ev} \oplus S^{od}$

- Define Clifford multiplication by

$$\begin{aligned} (X+\xi)\cdot\varphi &= i_X\varphi + \xi\wedge\varphi \\ (X+\xi)^2\cdot\varphi &= i_X\xi\varphi = (X+\xi, X+\xi)\varphi \end{aligned}$$

- $\exp B(\varphi) = (1 + B + \frac{1}{2}B\wedge B + \ldots)\wedge\varphi$

DERIVATIVES

- Lie bracket:

  $$2i_{[X,Y]}\alpha = d([i_X, i_Y]\alpha) + 2i_X d(i_Y\alpha) - 2i_Y d(i_X\alpha) + [i_X, i_Y]d\alpha$$

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- $u = X + \xi$, $v = Y + \eta$ use Clifford multiplication $u \cdot \alpha$ to define a bracket $[u, v]$:

  $$2[u, v] \cdot \alpha = d((u \cdot v - v \cdot u) \cdot \alpha) + 2u \cdot d(v \cdot \alpha)$$

  $$-2v \cdot d(u \cdot \alpha) + (u \cdot v - v \cdot u) \cdot d\alpha$$

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- *COURANT bracket*

  $$[X + \xi, Y + \eta] = [X, Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \frac{1}{2}d(i_X\eta - i_Y\xi)$$

Apply a 2-form B...

- $d \mapsto e^{-B} d e^{B} = d + dB$
- $[X + \xi, Y + \eta] \mapsto [X + \xi, Y + \eta] - 2 i_X i_Y dB$
- Diff$(M) \ltimes \Omega^2_{closed}(M)$ preserves inner product, exterior derivative and Courant bracket.

GENERALIZED GEOMETRIC STRUCTURES

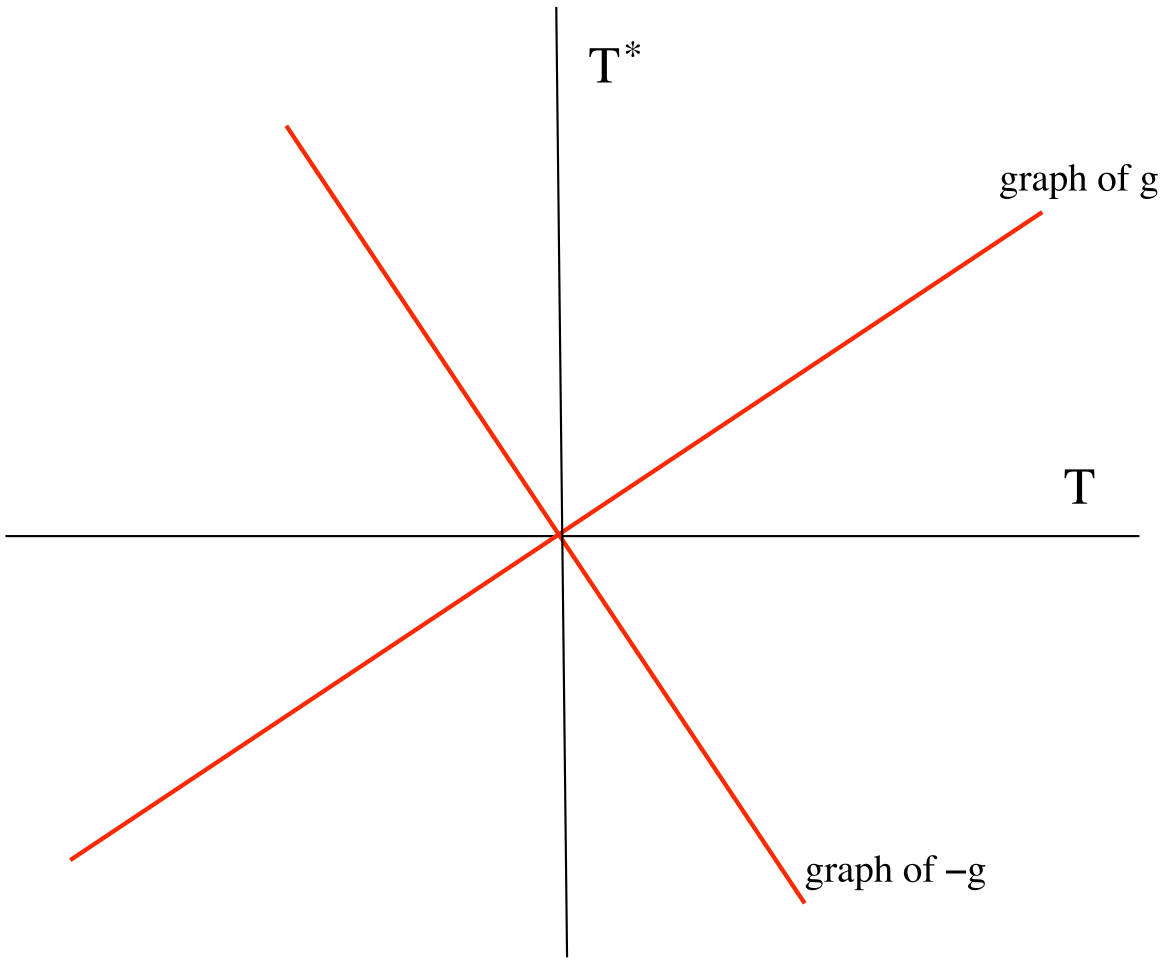
- $SO(n,n)$ compatibility

- integrability $\sim d$ or Courant bracket

- transform by $\mathrm{Diff}(M) \ltimes \Omega^2_{closed}(M)$

RIEMANNIAN METRIC

- Riemannian metric g_{ij}
- $X \mapsto g(X, -) : g : T \to T^*$
- *graph* of $g = V \subset T \oplus T^*$
- $T \oplus T^* = V \oplus V^\perp$


T*
graph of g
T
graph of −g

GENERALIZED RIEMANNIAN METRIC

- $V \subset T \oplus T^*$ positive definite rank n subbundle

- $=$ graph of $g + B : T \to T^*$

- $g + B \in T^* \otimes T^*$: g symmetric, B skew

GERBES

GERBES: ČECH 2-COCYCLES

- $g_{\alpha\beta\gamma} : U_\alpha \cap U_\beta \cap U_\gamma \to S^1$
- $(g_{\alpha\beta\gamma} = g^{-1}_{\beta\alpha\gamma} = \ldots)$
- $\delta g = g_{\beta\gamma\delta} g^{-1}_{\alpha\gamma\delta} g_{\alpha\beta\delta} g^{-1}_{\alpha\beta\gamma} = 1$ on $U_\alpha \cap U_\beta \cap U_\gamma \cap U_\delta$

This *defines* a gerbe.

CONNECTIONS ON GERBES

Connective structure:

$$A_{\alpha\beta} + A_{\beta\gamma} + A_{\gamma\alpha} = g^{-1}_{\alpha\beta\gamma} dg_{\alpha\beta\gamma}$$

Curving:

$$B_\beta - B_\alpha = dA_{\alpha\beta}$$

$\Rightarrow dB_\beta = dB_\alpha = H|_{U_\alpha}$ global three-form H

J.-L. Brylinski, *Characteristic classes and geometric quantization*, Progr. in Mathematics **107**, Birkhäuser, Boston (1993)

TWISTING $T \oplus T^*$

$$dA_{\alpha\beta} + dA_{\beta\gamma} + dA_{\gamma\alpha} = d[g^{-1}_{\alpha\beta\gamma} dg_{\alpha\beta\gamma}] = 0$$

- identify $T \oplus T^*$ on U_α with $T \oplus T^*$ on U_β by

  $X + \xi \mapsto X + \xi + i_X dA_{\alpha\beta}$

- defines a vector bundle E

  $0 \to T^* \to E \to T \to 0$

- with ... an inner product and a Courant bracket.

BASIC PROPERTIES OF COURANT BRACKET

$$[u, fv] = f[u, v] + (\pi(u)f)v - (u, v)df$$

$$\pi(u)(v, w) = ([u, v] + d(u, v), w) + (v, [u, w] + d(u, w))$$

where $\pi(X + \xi) = X$.

TWISTED COHOMOLOGY

- identify $\Lambda^* T^*$ on U_α with $\Lambda^* T^*$ on U_β by

  $\varphi \mapsto e^{dA_{\alpha\beta}}\varphi$

- defines a vector bundle $S =$ spinor bundle for E

- $d : C^\infty(S) \to C^\infty(S)$ well-defined

- $\ker d / \operatorname{im} d =$ twisted cohomology.

... WITH A CURVING

- $\varphi_\alpha = e^{dA_{\alpha\beta}} \varphi_\beta$
- $d\varphi_\alpha = 0$
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- $e^{B_\alpha} \varphi_\alpha = e^{B_\beta} \varphi_\beta = \psi$
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- Curving: $B_\beta - B_\alpha = dA_{\alpha\beta}$

- $e^{B_\alpha} \varphi_\alpha = e^{B_\beta} \varphi_\beta = \psi$

- $d\psi - H \wedge \psi = 0$

Definition: A **generalized metric** is a subbundle $V \subset E$ such that $\operatorname{rk} V = \dim M$ and the inner product is positive definite on V.

- $V \cap T^* = 0 \Rightarrow$ splitting of the sequence

  $0 \to T^* \to E \to T \to 0$

- $V^{\perp} \subset E$ another splitting

- difference $\in \operatorname{Hom}(T, T^*) = T^* \otimes T^* =$ Riemannian metric

SPLITTINGS IN LOCAL TERMS

- splitting: $C_\alpha \in C^\infty(U_\alpha, T^* \otimes T^*)$: $C_\beta - C_\alpha = dA_{\alpha\beta}$
- $Sym(C_\alpha) = Sym(C_\beta) =$ metric
- $Alt(C_\alpha) = B_\alpha =$ curving of the gerbe
- $H = dB_\alpha$ closed 3-form

- *two* splittings V and $V^\perp$ of $0 \to T^* \to E \to T \to 0$
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- X vector field, lift to $X^{+} \in C^{\infty}(V)$ and $X^{-} \in C^{\infty}(V^{\perp})$ in E

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- X vector field, lift to $X^+ \in C^\infty(V)$ and $X^- \in C^\infty(V^\perp)$ in E

- Courant bracket $[X^-, Y^+]$, Lie bracket $[X, Y]$

- $[X^-, Y^+] - [X, Y]^-$ is a one-form

- $[X^-, Y^+] - [X, Y]^- = 2g(\nabla_X Y)$

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- ∇ Riemannian connection with skew torsion $-H$

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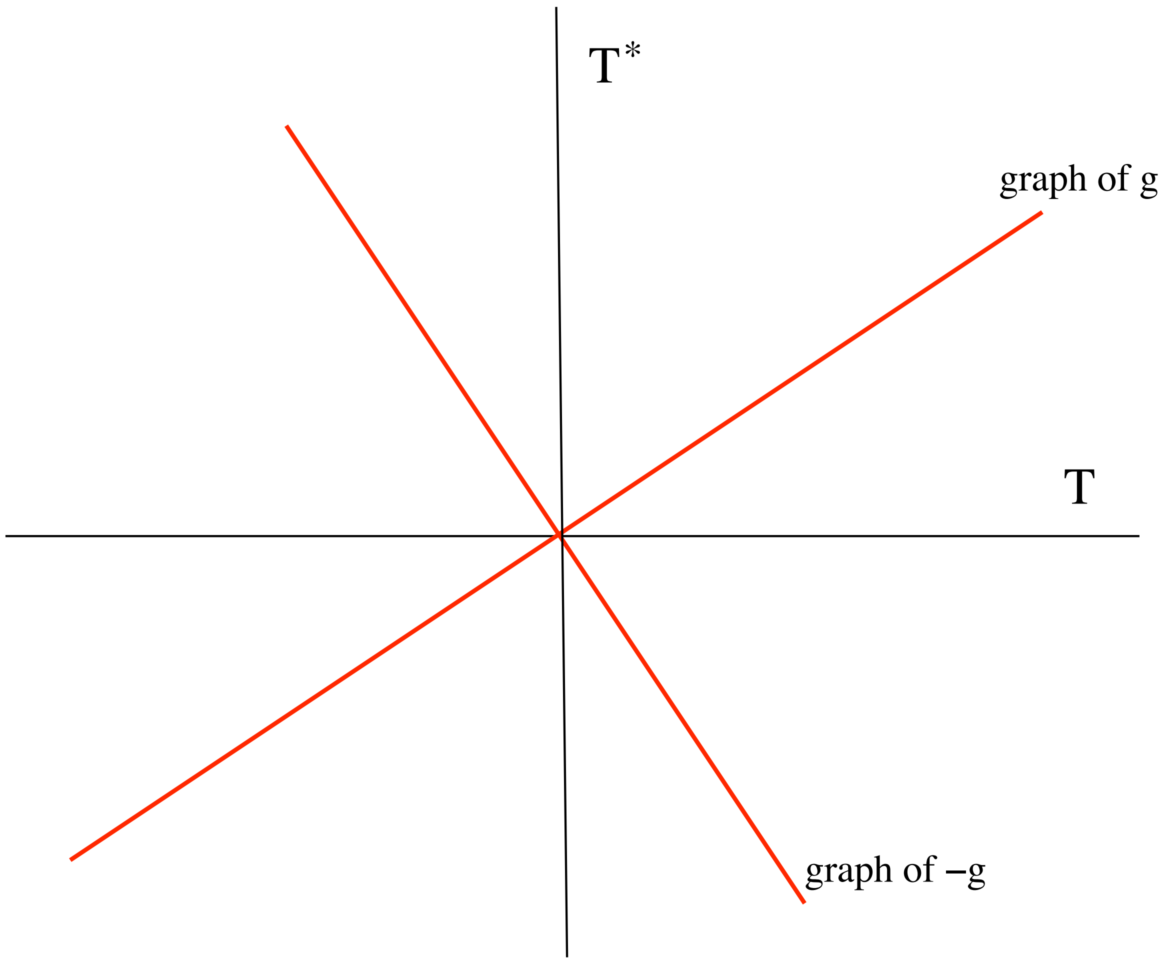
- $[X^+, Y^-] - [X, Y]^+$ has skew torsion H



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EXAMPLE: the Levi-Civita connection

$$\left[\frac{\partial}{\partial x_i} - g_{ik}dx_k, \frac{\partial}{\partial x_j} + g_{jk}dx_k\right] - \left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right]^{-} =$$

$$= \left(\frac{\partial g_{jk}}{\partial x_i} + \frac{\partial g_{ik}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_k}\right) dx_k = 2g_{\ell k}\Gamma_{ij}^{\ell} dx_k$$

EXAMPLE: connection with torsion db

$$\left[\frac{\partial}{\partial x_i} - g_{ik}dx_k + b_{ik}dx_k, \frac{\partial}{\partial x_j} + g_{jk}dx_k + b_{jk}dx_k\right] =$$

$$= \left(\frac{\partial g_{jk}}{\partial x_i} + \frac{\partial g_{ik}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_k}\right) dx_k + \left(\frac{\partial b_{jk}}{\partial x_i} - \frac{\partial b_{ik}}{\partial x_j}\right) dx_k$$

GENERALIZED COMPLEX STRUCTURES

A *generalized complex structure* is:

- $J : T \oplus T^* \to T \oplus T^*, J^2 = -1$ (or $J : E \to E$)

- $(Ju, v) + (u, Jv) = 0$

- if $Ju = iu, Jv = iv$ then $J[u, v] = i[u, v]$ (*Courant bracket*)

- $U(m, m) \subset SO(2m, 2m)$ structure on $T \oplus T^*$

M. Gualtieri: *Generalized complex geometry* math.DG/0401221

EXAMPLES

- complex manifold $J = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$

$$J = i : [\ldots \partial/\partial z_i \ldots, \ldots d\bar{z}_i \ldots]$$

- symplectic manifold $J = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}$

$$J = i : [\ldots, \partial/\partial x_j + i \sum \omega_{jk} dx_k, \ldots]$$

EXAMPLE: HOLOMORPHIC POISSON MANIFOLDS

$$\sigma = \sum \sigma^{ij} \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}$$

$$[\sigma, \sigma] = 0$$

$$J = i : \left[\ldots, \frac{\partial}{\partial z_j}, \ldots, d\bar{z}_k + \sum_{\ell} \bar{\sigma}^{k\ell} \frac{\partial}{\partial \bar{z}_\ell}, \ldots \right]$$

GENERALIZED KÄHLER MANIFOLDS

Kähler $\Rightarrow$ complex structure + symplectic structure

- complex structure $\Rightarrow J_1$ on $T \oplus T^*$
- symplectic structure $\Rightarrow J_2$ on $T \oplus T^*$
- compatibility ($\omega \in \Omega^{1,1}$): $J_1 J_2 = J_2 J_1$

A **generalized Kähler structure**: two commuting generalized complex structures J_1, J_2 such that $(J_1 J_2(X+\xi), X+\xi)$ is definite.

GUALTIERI'S THEOREM A generalized Kähler manifold is:

- two integrable complex structures I_+, I_- on M
- a metric g, hermitian with respect to I_+, I_-
- a 2-form B
- $U(m)$ connections ∇^+, ∇^- with skew torsion $\pm H = \pm dB$

V Apostolov, P Gauduchon, G Grantcharov, *Bihermitian structures on complex surfaces*, Proc. London Math. Soc. **79** (1999), 414–428

S. J. Gates, C. M. Hull and M. Roček, *Twisted multiplets and new supersymmetric nonlinear σ-models*. Nuclear Phys. B **248** (1984), 157–186.

- $(J_1 J_2)^2 = 1$

- eigenspaces $V, V^{\perp}$, metric on V positive definite

- connections $\Rightarrow \nabla^{+}, \nabla^{-}$ torsion $\pm dB$

- … also define on $0 \rightarrow T^* \rightarrow E \rightarrow T \rightarrow 0$

- $J_1 = -J_2$ on $V \Rightarrow I_+$

- $J_1 = J_2$ on $V^\perp \Rightarrow I_-$

- $\{J_1 = i\} \cap \{J_2 = -i\} = V^{1,0}$ is Courant integrable

INTEGRABILITY OF I_+

- $(I_+X)^+ = J_1 X^+$ so $I_+X = iX \Leftrightarrow J_1 X^+ = iX^+$

- ... but $J_1[X^+, Y^+] = i[X^+, Y^+]$

- so $I_+[X, Y] = i[X, Y]$

COMPATIBILITY WITH CONNECTION

- Require to show $\nabla^+ I_+ = 0$, or

  $g(\nabla^+_X I_+ Y, Z) = -g(\nabla^+_X Y, I_+ Z)$

- But

  $2g(\nabla^+_X Y, Z) = ([X^-, Y^+] - [X, Y]^-, Z^+) = ([X^-, Y^+], Z^+)$

- ... so require to show

  $([X^-, J_1 Y^+], Z^+) = -([X^-, Y^+], J_1 Z^+)$

- Nijenhuis condition

  $$[Ju, Jv] - J[Ju, v] - J[u, Jv] - [u, v] = 0$$

- For J_1:

  $$[J_1 X^-, J_1 Y^+] - J_1[J_1 X^-, Y^+] - J_1[X^-, J_1 Y^+] - [X^-, Y^+] = 0$$

- For J_2 (recall $J_1 = -J_2$ on V, $J_1 = J_2$ on $V^\perp$)

  $$-[J_1 X^-, J_1 Y^+] - J_2[J_1 X^-, Y^+] + J_2[X^-, J_1 Y^+] - [X^-, Y^+] = 0$$

- add...

  $$-(J_1 + J_2)[X^-, J_1 Y^+] - (J_1 - J_2)[X^-, J_1 Y^+] - 2[X^-, Y^+] = 0$$

- inner product with $J_1 Z^+$

  $((J_1 + J_2)[X^-, J_1 Y^+], J_1 Z^+) + ((J_1 - J_2)[X^-, J_1 Y^+], J_1 Z^+) + 2([X^-, Y^+], J_1 Z^+) = 0$

- but $J_1 = -J_2$ on V, so...

  $([X^-, J_1 Y^+], Z^+) + ([X^-, Y^+], J_1 Z^+) = 0$

- ... which is the required identity

COMPLEX POISSON STRUCTURES

- $[J_1, J_2] = 0$, $[I_+, I_-] \neq 0$ in general
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- $\Phi \in \Lambda^{2,0} + \Lambda^{0,2}$ for both complex structures
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- $\Rightarrow \sigma \in \Lambda^{0,2} \cong \Lambda^2 T$

- σ is holomorphic

- σ is a holomorphic Poisson structure

EXAMPLES

- T^4 and $K3$, $\sigma =$ holomorphic 2-form (D. Joyce)

- $\mathbf{C}P^2$, anticanonical divisor: $3L$ (NJH)

- $\mathbf{C}P^1 \times \mathbf{C}P^1$ anticanonical divisor: 2Δ (NJH)

- moduli space of ASD connections on above (NJH)

- $\mathbf{C}P^n$, blow-up of $\mathbf{C}P^2$ at n points, Hirzebruch surfaces (Y.Lin & S.Tolman, math.DG/0509069)