The S-Matrix Reloaded: Twistors, Unitarity, Gauge Theories and Gravity

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with I. Bena, N.E.J. Bjerrum-Bohr, V. Del Duca, L. Dixon, D. Dunbar,
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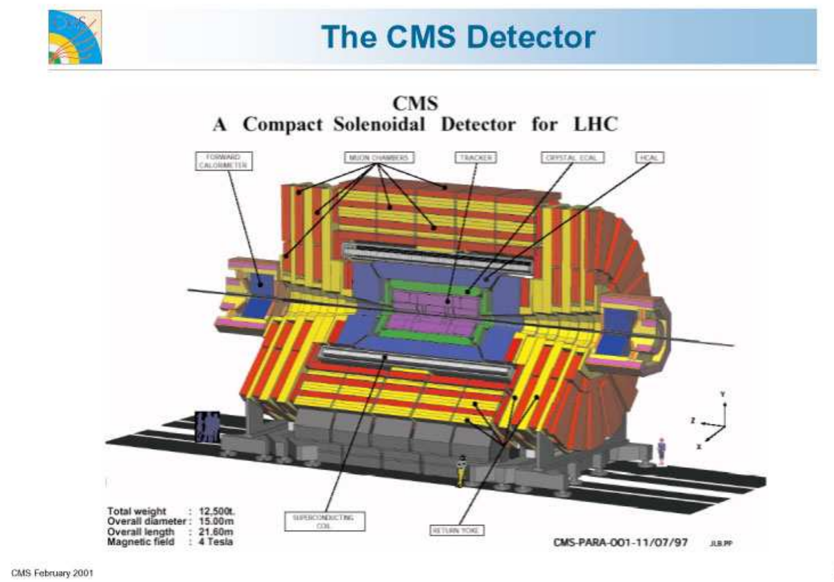
Outline

- Motivation
  (a) QCD and applications to colliders, especially the LHC
  (b) Try to solve $N = 4$ maximally supersymmetric Yang-Mills theory
  (c) Reexamine question of supergravity divergences.
- Twistors.
- $N = 4$ super-Yang-Mills loop amplitudes
  (a) Unitarity method
  (c) Twistor space structure
  (c) Higher loops resummation
- Supergravity.
- Summary and Outlook.

CERN LHC

The issues of perturbation theory in quantum field theory are central to particle physics. Entire month of the 2004 KITP collider physics workshop was devoted to the issues of pushing QCD perturbative calculations to higher order.



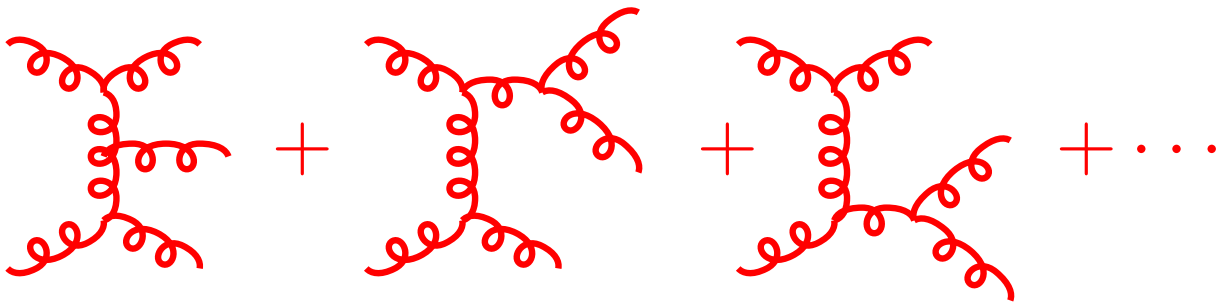
Enormous resources devoted to these experiments

Very rapid recent progress in perturbation theory: unitarity method, twistors, on-shell recursion.

Helicity

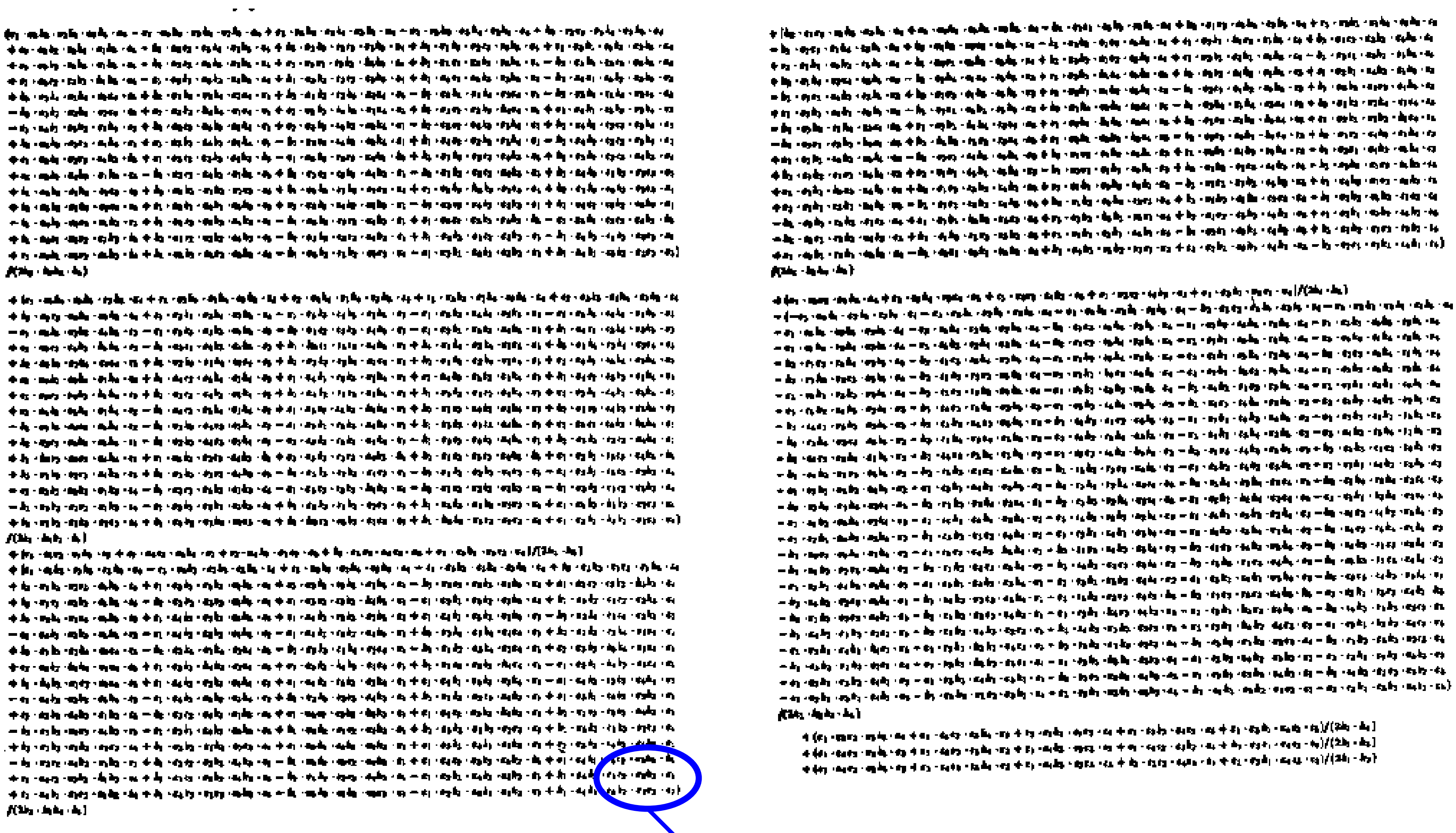
Consider the five-gluon tree-level amplitude of QCD. Enters in calculation of multi-jet production at hadron colliders.

Described by following Feynman diagrams:



If you follow the textbooks you discover a disgusting mess.

Result of a brute force calculation:



$$k_1 \cdot k_4 \, \varepsilon_2 \cdot k_1 \, \varepsilon_1 \cdot \varepsilon_3 \, \varepsilon_4 \cdot \varepsilon_5$$

Helicity

Xu, Zhang and Chang
F.A.Berends, R.Kleiss, P.De Causmaecker
R. Gastmans and T. T. Wu
J.F. Gunion and Z. Kunszt
& many others

Vector polarizations

$$\varepsilon_\mu^+(k;q) = \frac{\langle q^- | \gamma_\mu | k^- \rangle}{\sqrt{2}\,\langle q\, k\rangle}, \qquad \varepsilon_\mu^-(k,q) = \frac{\langle q^+ | \gamma_\mu | k^+ \rangle}{\sqrt{2}\,[k\, q]}$$

More sophisticated version of circular polarization: $\varepsilon_\mu^\pm = (0, 1, \pm i, 0)$

All required properties of polarization vectors satisfied:

$$\varepsilon_i^2 = 0\,, \qquad k \cdot \varepsilon(k,q) = 0\,, \qquad \varepsilon^+ \cdot \varepsilon^- = -1$$

Notation $$\epsilon^{ab}\lambda_{ja}\lambda_{lb} \longleftrightarrow \langle j\, l\rangle = \langle k_{j\,-} | k_{l\,+}\rangle = \sqrt{2k_j \cdot k_l}\, e^{i\phi}$$

$$\epsilon_{\dot a \dot b}\tilde\lambda_j^{\dot a}\tilde\lambda_l^{\dot b} \longleftrightarrow [j\, l] = \langle k_{j\,+} | k_{l\,-}\rangle = -\sqrt{2k_j \cdot k_l}\, e^{-i\phi}$$

Changes in reference momentum q are equivalent to gauge transformations.

Graviton polarization tensors are the squares of these!

$$\varepsilon_{\mu\nu}^{++} = \varepsilon_\mu^+ \varepsilon_\nu^+\,, \qquad 2 = 1 + 1$$

Five Gluon Results with Helicity

Following contains the physical content of the messy formula:

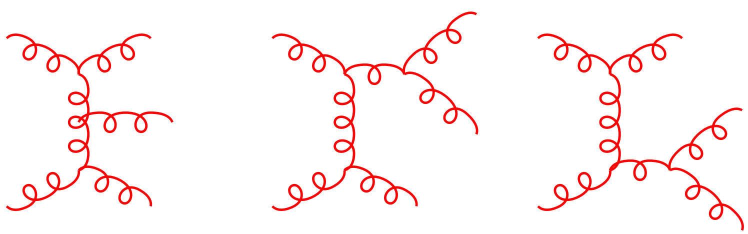
$$A_5(1^\pm, 2^+, 3^+, 4^+, 5^+) = 0$$

$$A_5(1^-, 2^-, 3^+, 4^+, 5^+) = i\frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5\,1\rangle}$$

These are color stripped amplitudes.

$$\mathcal{A}_5(1,2,3,4,5) = \sum_{\text{perms}} \mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4}T^{a_5})\, A_5(1^-, 2^-, 3^+, 4^+, 5^+)$$

Motivated by the Chan-Paton factors of open string theory. Mangano and Parke



Feynman diagrams scramble together kinematics and color.

Twistor Space and Topological String Theory

In a beautiful paper Ed Witten demonstrated that "twistor space" can reveal hidden structures of scattering amplitudes. Precursor from Nair

Link to string theory is for $N = 4$ super-Yang-Mills theory, but at tree level it might as well be QCD.

Twistor space given by Fourier transform with respect to plus helicity spinors.

$$\widetilde{A}(\lambda_i, \mu_i) = \int \prod_i \frac{d^2\widetilde{\lambda}_i}{(2\pi)^2} \exp\left(\sum_j \mu_j^{\dot{a}} \widetilde{\lambda}_{j\dot{a}}\right) A(\lambda_i, \widetilde{\lambda}_i)$$

Tree-level QCD scattering amplitudes $\leftrightarrow$ 'Twistor-space' $\leftrightarrow$ Topological String Theory

E. Witten; Roiban, Spradlin, and Volovich

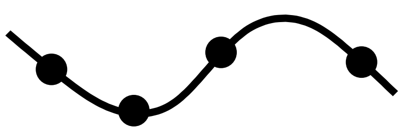
Witten observed that in twistor space external points lie on certain curves. Very constraining. Non-trivial Duality

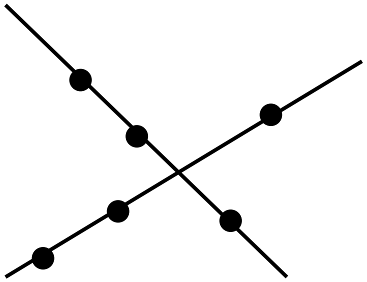
$N = 4$ non-MHV Amplitudes

Ed Witten conjectured that amplitudes should be supported on curves in twistor space of degree

$$d = q - 1 + L, \quad q = \# \text{ negative helicities}, \quad L = \# \text{ loops},$$

In twistor space external points of amplitudes have support on curves:





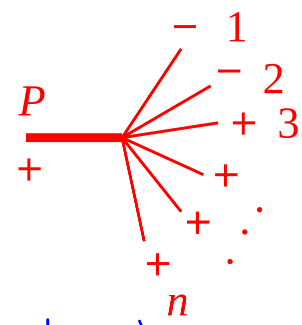
Connected and disconnected pictures.

Witten
Roiban, Spradlin and Volovich
Georgiou, Glover and Khoze
Cachazo, Svrček and Witten
Gukov, Motl and Neitzke
Bena, Bern and Kosower
Bedford, Brandhuber, Spence and Travaglini
Britto, Cachazo and Feng
Bjerrum-Bohr, Dixon and Dunbar
Bena, Bern, Kosower, Roiban
and many others

MHV Vertices

Motivated by twistor space structure Cachazo, Svrček and Witten define an off-shell "MHV vertex" based on Parke-Taylor amplitudes

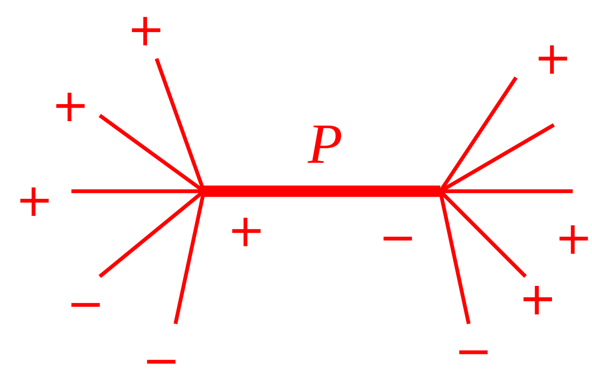
$$V(1^-, 2^-, 3^+, \ldots, n^+, P^+) = \frac{\langle 1\, 2\rangle^4}{\langle 1\, 2\rangle \cdots \langle n-1,\, n\rangle \langle n\, P\rangle \langle P\, 1\rangle}$$

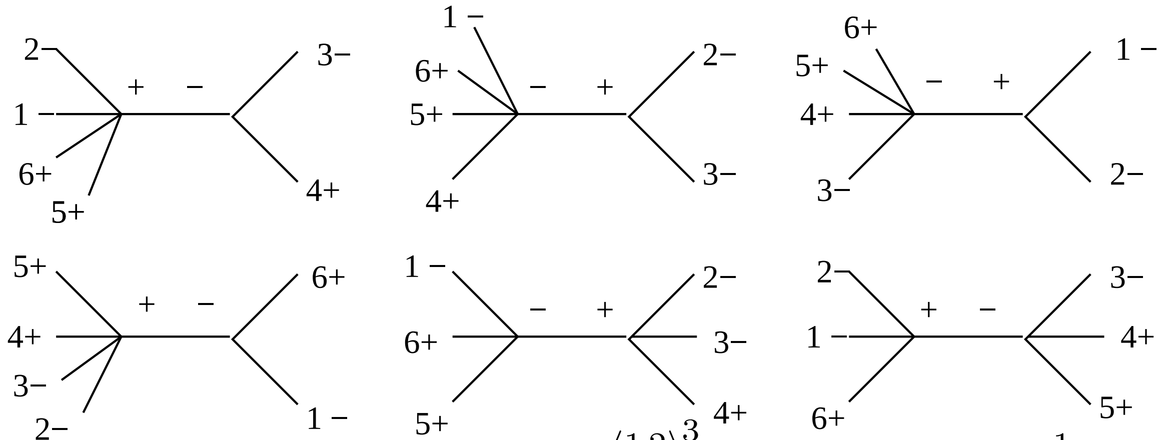


Continue spinor off-shell $(P^2 \neq 0)$: $\langle i\, P\rangle = \eta \sum_{j=1}^{n} \langle i^- | \not{k}_j | q^- \rangle$
where $P = k_1 + k_2 + \cdots k_n$ and q auxiliary, satisfying $q^2 = 0$.

Non-MHV amplitudes obtained by sewing together MHV vertices.

Holds generally for any massless gauge theory, including QCD. Georgiou and Khoze; Wu and Zhu





$$A_6(1^-,2^-,3^-,4^+,5^+,6^+) = \frac{\langle 1\,2\rangle^3}{\langle 5\,6\rangle\,\langle 6\,1\rangle\,\langle 2|\,5+6+1\,|q\rangle\,\langle 5|\,6+1+2\,|q\rangle} \times \frac{1}{s_{34}} \times \frac{\langle 3|\,4\,|q\rangle^3}{\langle 3\,4\rangle\,\langle 4|\,3\,|q\rangle}$$

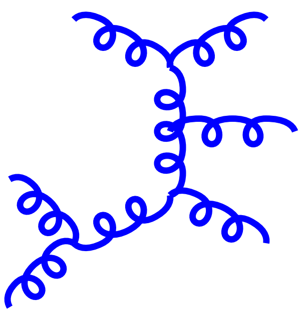
$$+ \frac{\langle 1|\,4+5+6\,|q\rangle^3}{\langle 4\,5\rangle\,\langle 5\,6\rangle\,\langle 6\,1\rangle\,\langle 4|\,5+6+1\,|q\rangle} \times \frac{1}{s_{23}} \times \frac{\langle 2\,3\rangle^3}{\langle 3|\,2\,|q\rangle\,\langle 2|\,3\,|q\rangle}$$

$$+ \frac{\langle 3|\,4+5+6\,|q\rangle^3}{\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,\langle 6|\,3+4+5\,|q\rangle} \times \frac{1}{s_{12}} \times \frac{\langle 1\,2\rangle^3}{\langle 2|\,1\,|q\rangle\,\langle 1|\,2\,|q\rangle}$$

$$+ \frac{\langle 2\,3\rangle^3}{\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5|\,2+3+4\,|q\rangle\,\langle 2|\,3+4+5\,|q\rangle} \times \frac{1}{s_{61}} \times \frac{\langle 1|\,6\,|q\rangle^3}{\langle 6\,1\rangle\,\langle 6|\,1\,|q\rangle}$$

$$+ \frac{\langle 1|\,5+6\,|q\rangle^3}{\langle 5\,6\rangle\,\langle 6\,1\rangle\,\langle 5|\,6+1\,|q\rangle} \times \frac{1}{s_{561}} \times \frac{\langle 2\,3\rangle^3}{\langle 3\,4\rangle\,\langle 4|\,2+3\,|q\rangle\,\langle 2|\,3+4\,|q\rangle}$$

$$+ \frac{\langle 1\,2\rangle^3}{\langle 6\,1\rangle\,\langle 2|\,6+1\,|q\rangle\,\langle 6|\,1+2\,|q\rangle} \times \frac{1}{s_{612}} \times \frac{\langle 3|\,4+5\,|q\rangle^3}{\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5|\,3+4\,|q\rangle}$$



$\langle 1|\,2+3\,|4\rangle \equiv \left\langle 1^- \middle| \not{k}_2 + \not{k}_3 \middle| 4^- \right\rangle$

q arbitrary but null

Key message from twistors: For general helicities tree-level scattering amplitudes are much much simpler than anyone anticipated.

$N = 4$ Super-Yang-Mills

In 1974 't Hooft suggested that we could solve QCD in the planar limit.

This is too hard. We should look instead at a simpler theory.

$N = 4$ super-Yang-Mills is by far the simplest $D = 4$ gauge theory.

$N = 4$ theory is a cousin of QCD, but with specially arranged matter.

1 gluon, 4 real fermions and 6 scalars.

- $N = 4$ super-Yang-Mills is a conformal field theory (CFT). UV finite.

- It is the CFT appearing in Maldacena's AdS/CFT correspondence.

- Maldacena conjecture suggests a magical simplicity, especially in the planar limit with strong coupling – dual to weakly coupled gravity.

Can we solve $N = 4$ super-Yang-Mills theory?

This is an important question not just in string theory community.

An AdS/CFT puzzle

For large 't Hooft coupling get weakly coupled gravity on AdS side.

Weakly coupled gravity on AdS side is relatively simple.

Quantities protected by susy are generally simple on the CFT side.

What about unprotected quantities?

Heuristically, to match the simplicity of the AdS side, the perturbation series should be resummable. Expect an iterative structure to allow for a resummation.

How can we identify the iterative structure?

Our approach is to look at scattering amplitudes. Well defined (in dim. reg.), gauge invariant, and independent of field variable choices.

Loop Amplitudes

Bern, Dixon, Dunbar, Kosower
hep-ph/9403226,9409265
Bern and Morgan, hep-ph/9511336

Summary of results from our early papers on the subject:

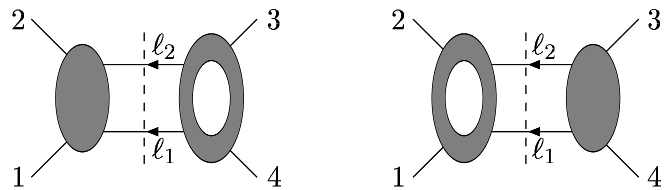
- Key Theorem: *Any* amplitude in any massless theory is fully determined from D-dimensional tree amplitudes to *all* loop orders. Off-shell formulations unnecessary. Unitarity is all that is necessary.
- Four-dimensional cut constructibility: At one-loop, any amplitude in a massless susy gauge theory is fully constructible from *four-dimensional* tree amplitudes (even in the presence of IR and UV singularities).
- Simplicity: One-loop $N = 4$ amplitudes are much much simpler than they ought to be. Twistor space and topological string theory finally points to the origin of this simplicity.

Textbook field theory ideas *not* needed: Green functions, Feynman rules, counterterms, Faddeev-Popov ghosts, BRST, superspace, etc.

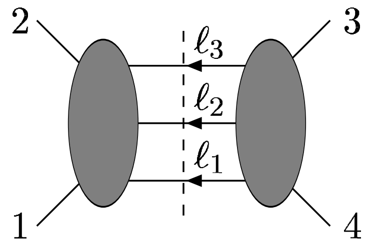
Generalized Cuts

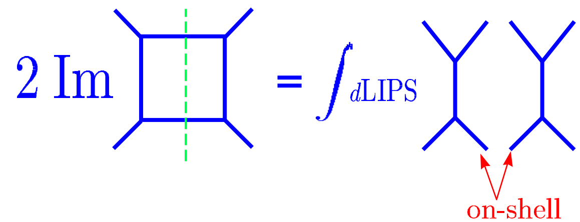
Bern, Dixon and Kosower, hep-ph/9708239
Bern, Dixon and Kosower, hep-ph/0404293
Britto, Cachazo and Feng, hep-th/0412103

Two-particle cuts:

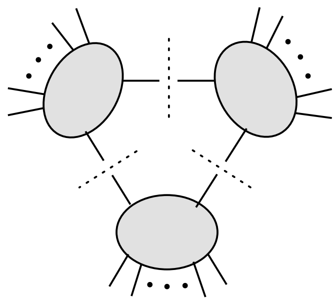


Three-particle cuts:





Generalized triple cut:

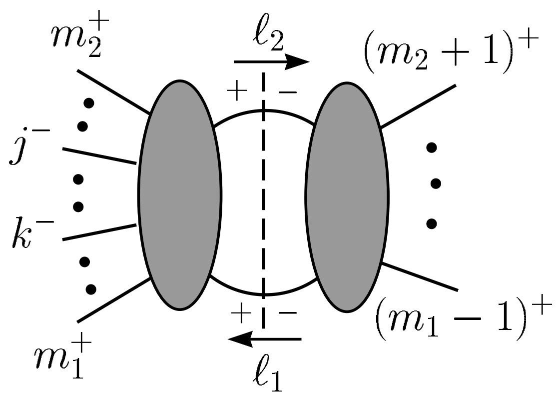


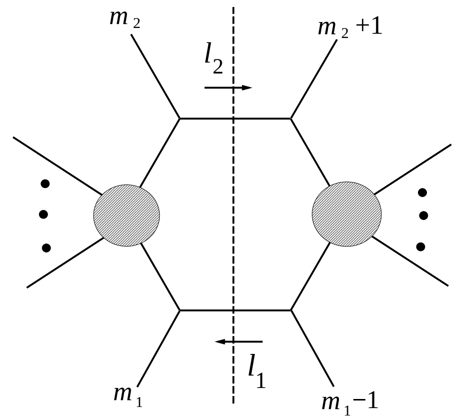
It should be interpreted as demanding that cut propagators do not cancel.

The unitarity method is a potent tool for state-of-the-art calculations. It very effectively combines with twistor methods.

Arbitrary Number of Legs at One Loop

Consider cuts of maximally helicity violating one-loop amplitudes.





Bern, Dixon
Dunbar and Kosower

The tree-level Parke-Taylor amplitudes for n gluons have a remarkable property:

$$A^{\rm tree}(\ell_1^+, m_1^+, \cdots, k^-, \cdots, j^-, \cdots, m_2^+, \ell_2^+) = \frac{\langle k\, j\rangle^4}{\langle \ell_1\, m_1\rangle\, \langle m_1,\, m_1+1\rangle \cdots \langle m_2-1,\, m_2\rangle\, \langle m_2\, \ell_2\rangle\, \langle \ell_2\, \ell_1\rangle}$$

Only 2 denominators in each tree have non-trivial dependence on loop momentum.

Together with 2 cut propagators the 4 denominators from the trees give at worst a hexagon integral (which simplifies in susy cases).

Bern, Dixon, Dunbar and Kosower (1994)

At one loop in our earlier papers we obtained:

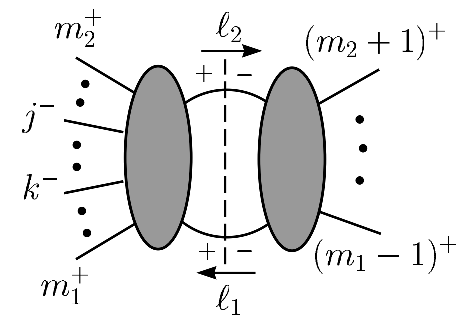
- All MHV amplitudes in maximal $N = 4$ super-Yang-Mills theory.

- All MHV amplitudes in $N = 1$ super-Yang-Mills

- All helicities for $N = 4$ super-Yang-Mills six-points amplitudes.

$$A_5^{\text{1-loop}} = A_5^{\text{tree}} \left[-\frac{1}{\epsilon^2} \sum_{i=1}^{5} \left(\frac{\mu^2}{-s_{i,i+1}} \right)^{\epsilon} + \sum_{i=1}^{5} \ln\left(\frac{-s_{i,i+1}}{s_{i-2,i-1}} \right) \ln\left(\frac{-s_{i+2,i+3}}{s_{i-2,i-1}} \right) + \frac{5\pi^2}{6} \right]$$

These amplitudes are the one-loop analogs of the Parke-Taylor tree-level amplitudes.



The amplitudes are much much simpler than they ought to be.

$N = 4$ next-to-MHV Amplitudes

To uncover the twistor space structure of loop amplitudes we computed NMHV amplitudes using the unitarity method.

- 7 points, e.g. $A_7(1^-, 2^-, 3^+, 4^-, 5^+, 6^+, 7^+)$ — equivalent to 227,585 Feynman diagrams.

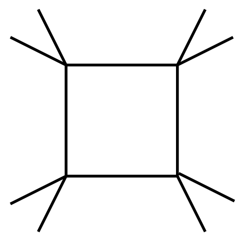
  Britto, Cachazo and Feng, hep-th/0410179

  Bern, Del Duca, Dixon and Kosower, hep-th/0410224

- n-points – needed to fully expose the twistor structure

  Bern, Dixon and Kosower hep-th/0412210

$$A_n^{\text{1-loop}} = \sum_i c_i B_i$$



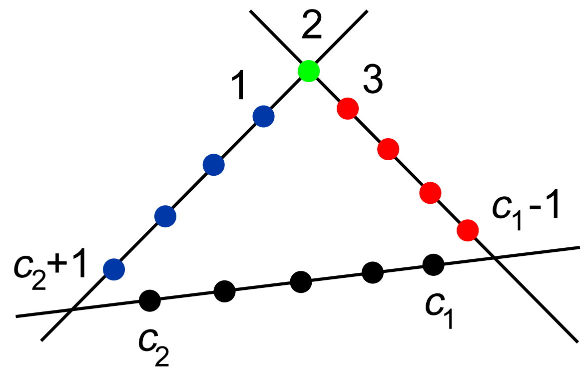
The B_i are known scalar box functions given in terms of polylogs.
Coefficients for all NMHV n-point amplitudes are listed in our paper hep-th/0412210. Example:

$(1+2) \equiv \not{k}_1 + \not{k}_2$

$$c_{136} = \frac{\left(\left\langle 7^+\right| (2+4) \left|3^+\right\rangle \langle 5\,4\rangle + \left\langle 7^+\right| 6 \left|5^+\right\rangle \langle 3\,4\rangle\right)^4}{\langle 2\,3\rangle \langle 3\,4\rangle \langle 4\,5\rangle \langle 5\,6\rangle [7\,1] \langle 1^+| (2+3) |4^+\rangle \langle 7^+| (5+6) |4^+\rangle \langle 4^-| (5+6)(7+1) |2^+\rangle \langle 4^-| (2+3)(7+1) |6^+\rangle}$$

A key result: Beautiful twistor-space picture for terms in integral function coefficients:

Bern, Dixon and Kosower



General coplanarity of NMHV integral coefficients proven.

Bern, Del Duca, Dixon and Kosower; Britto, Cachazo and Feng

Complete determination of *all* one-loop next-to-MHV amplitudes.

Bern, Dixon and Kosower

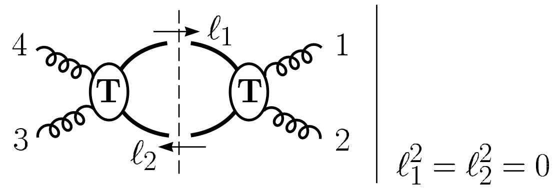
Points to further twistor space marvels awaiting discovery and exploitation.

A full understanding of the twistor space structure of loop amplitudes should lead to new insights. Twistor string interpretation?

N=4 Multi-Loop Amplitudes

ZB, Rozowsky, Yan

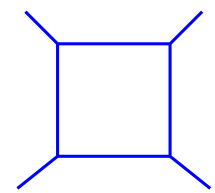
Consider $N = 4$ super-Yang-Mills.



The basic D-dimensional two-particle sewing equation:

$$\sum_{N=4 \text{ states}} A_4^{\text{tree}}(-\ell_1, 1, 2, \ell_2) \times A_4^{\text{tree}}(-\ell_2, 3, 4, \ell_1) = -\frac{st A_4^{\text{tree}}(1,2,3,4)}{(\ell_1 - k_1)^2 (\ell_2 - k_3)^2}$$

Applying this equation at one-loop we have



$$\mathcal{A}_4^{\text{1-loop}}(1,2,3,4) = -st A_4^{\text{tree}} \mathcal{I}_4^{\text{1-loop}}(s,t)$$

This amplitude has the correct s and t channel cuts in all dimensions. It agrees with the results of Green, Schwarz and Brink.

Since we get back A_4^{tree} we can recycle the two-particle cut algebra to all loop orders!

Exact Two-loop Expressions

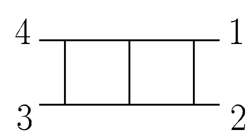
ZB, Rozowsky, Yan

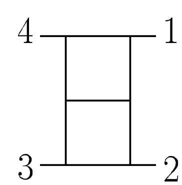
The two-loop two-particle cut sewing algebra is identical to the one-loop case.

We have also verified that the three particle cuts contain no other functions than those found with two-particle cuts.

Combining all cuts into a single function gives

$$A_4^{\rm planar}(1^-,2^-,3^+,4^+) = -st\, A_4^{\rm tree}(1^-,2^-,3^+,4^+)\left(s\,\mathcal{I}_4^{\text{2-loop}}(s,t) + t\,\mathcal{I}_4^{\text{2-loop}}(t,s)\right)$$

$$- st\, A_4^{\rm tree} \Bigg\{ s$$ 

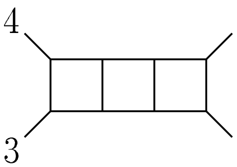
$$+ \; t$$ 

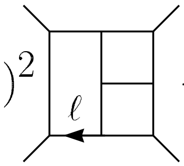
$$\Bigg\}$$

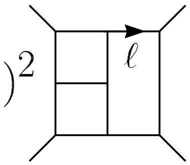
This is the exact expression for planar contributions in terms of what are now known scalar integrals. Non-planar is similar.

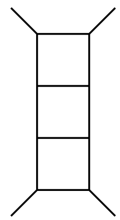
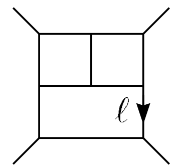
The Structure of the L-loop Amplitude

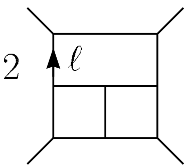
Apply same cut construction to three loops:

$-ist\,A_4^{\text{tree}} \Big\{ s^2$ 

$+\, s(\ell+k_2)^2$ 

$+\, s(\ell+k_4)^2$ 

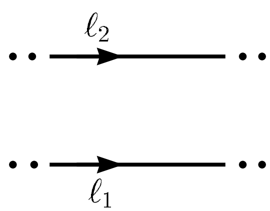
$+\, t^2$ 
$+\, t(\ell+k_1)^2$ 

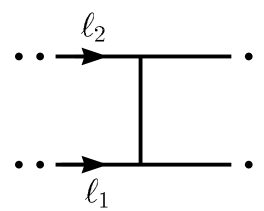
$+\, t(\ell+k_3)^2$ 

$\Big\}$

Have verified 2 and 3 particle cuts.

For higher loops pattern appears to be to add extra line with given factor. No triangle or bubble sub-diagrams allowed.

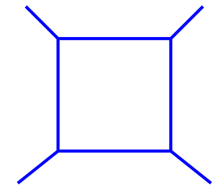


$\longrightarrow\; i\,(\ell_1+\ell_2)^2 \times$ 

Note: So far this is prior to carrying out loop integration.

Loop Iteration of the Amplitude

The four-point one-loop $D = 4, N = 4$ amplitude:

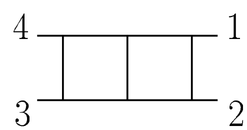


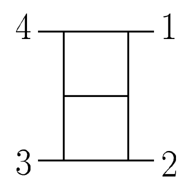
$$A_4^{\text{1-loop}}(s,t) = -st A_4^{\text{tree}} \mathcal{I}_{\text{1-loop}}(s,t)$$

$$I^{\text{1-loop}}(s,t) \sim \frac{1}{st}\left[\frac{2}{\epsilon^2}\left((-s)^{-\epsilon} + (-t)^{-\epsilon}\right) - \ln^2\left(\frac{t}{s}\right) - \pi^2\right] + \mathcal{O}(\epsilon)$$

To check for iteration we need to evaluate the loop integrals Smirnov

$$A_4^{\text{2-loop}}(1^-,2^-,3^+,4^+) = -st\, A_4^{\text{tree}}(1^-,2^-,3^+,4^+)\,\left(s\,\mathcal{I}_4^{\text{2-loop}}(s,t) + t\,\mathcal{I}_4^{\text{2-loop}}(t,s)\right)$$

$-st\, A_4^{\text{tree}} \Big\{ s$ 

$+ \; t$ 

$\Big\}$

Near $D = 4$ the double box integral is a rather intricate object involving up to 4th order polylogarithms.

Nevertheless, the *planar* two-loop amplitude undergoes an amazing simplification: Anastasiou, Bern, Dixon, Kosower

$$M_4^{\text{2-loop}}(s,t) = \frac{1}{2}\Big(M_4^{\text{1-loop}}(s,t)\Big)^2 + f(\epsilon) M_4^{\text{1-loop}}(s,t)\Big|_{\epsilon\to 2\epsilon} - \frac{1}{2}\zeta_2^2$$

where

$$M_4^{\text{loop}} = A_4^{\text{loop}}/A_4^{\text{tree}}, \qquad f(\epsilon) = -\zeta_2 - \zeta_3\,\epsilon - \zeta_4\,\epsilon^2$$

$f(\epsilon)$ is a universal IR function given in terms of anomalous dimensions of leading twist operators.

Thus, we have succeeded to express the two-loop amplitude as an iteration of the one loop amplitude together with a universal IR function.

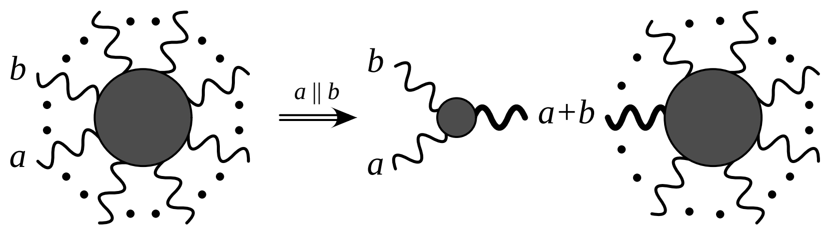
Non-trivial polylogarithm and Nielsen function identities needed to demonstrate the above.

Generalization to n-Points

Anastasiou, Bern, Dixon, Kosower

Not yet feasible to explicitly evaluate $n > 4$ point two-loop integrals

But we have tools for obtaining results: Collinear behavior



Have calculated the two-loop splitting amplitudes which determine the behavior of amplitudes as momenta become collinear.

Following ansatz satisfies all collinear constraints:

$$M_n^{2\text{-loop}}(\epsilon) = \frac{1}{2}\left(M_n^{1\text{-loop}}(\epsilon)\right)^2 + f(\epsilon) M_n^{1\text{-loop}}(2\epsilon) - \frac{1}{2}\zeta_2^2$$

where
$$M_n^{\text{loop}} = A_n^{\text{loop}} / A_n^{\text{tree}}\,, \qquad f(\epsilon) = -\zeta_2 - \zeta_3\,\epsilon - \zeta_4\,\epsilon^2$$

Interesting quantity is finite remainder after subtracting IR divergences.

The conjecture is almost certainly true for MHV amplitudes.

Multi-loop Generalization

ZB, Dixon and Smirnov
hep-th/0505205

Does the above iteration hold to higher loop orders?

To check this we explicitly integrated the known three loop integrand.

used Smirnov's techniques

Answer in terms of several pages of harmonic polylogarithms.

Remiddi and Vermaseren

After applying several hundred harmonic polylogarithm identities:

$$M_4^{\text{3-loop}}(\epsilon) = -\frac{1}{3}\Big[M_4^{\text{1-loop}}(\epsilon)\Big]^3 + M_4^{\text{1-loop}}(\epsilon)\, M_4^{\text{2-loop}}(\epsilon) + f^{\text{3-loop}}(\epsilon)\, M_4^{\text{1-loop}}(3\,\epsilon) + C^{(3)} + \mathcal{O}(\epsilon)$$

where

$$f^{\text{3-loop}}(\epsilon) = \frac{11}{2}\,\zeta_4 + \epsilon(6\zeta_5 + 5\zeta_2\zeta_3) + \epsilon^2(c_1\zeta_6 + c_2\zeta_3^2)\,,$$

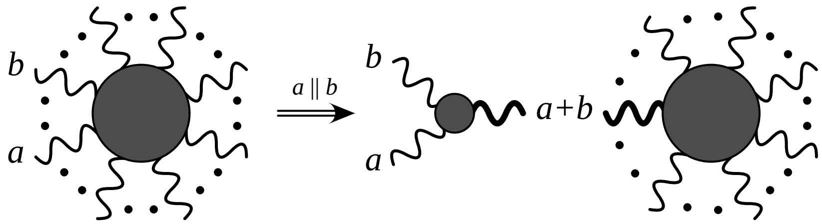
and

$$C^{(3)} = \left(\frac{341}{216} + \frac{2}{9}c_1\right)\zeta_6 + \left(-\frac{17}{9} + \frac{2}{9}c_2\right)\zeta_3^2\,.$$

Rational numbers c_1 and c_2 are undetermined since they actually cancel from the expression. (A five-point calculation would determine these constants.)

All-Leg Bootstrap

Repeat two-loop discussion, but at three loops.



Although we don't have a three-loop calculation of the splitting amplitude, it is clear by now it too should iterate.

Following exactly the same logic as at two loops gives us immediately an n-point generalization for MHV amplitudes:

$$M_n^{\text{3-loop}}(\epsilon) = -\frac{1}{3}\left[M_n^{\text{1-loop}}(\epsilon)\right]^3 + M_n^{\text{1-loop}}(\epsilon)\, M_n^{\text{2-loop}}(\epsilon) + f^{\text{3-loop}}(\epsilon)\, M_n^{\text{1-loop}}(3\,\epsilon) + C^{(3)} + \mathcal{O}(\epsilon)$$

With this ansatz, three-loop MHV amplitudes have proper factorization limits.

All Loop Bootstrap

Key observation: through 3 loops the iteration is exactly the same as the known iteration of IR singularites.

In any unbroken gauge theory the IR structure is understood to *all* loop orders. Sterman and Magnea; Catani; Sterman and Tejeda-Yeomans

Cleaning up Sterman and Magnea IR formula for planar $N = 4$ super-Yang-Mills theory gives a beautiful formula for all loop orders:

$$\mathcal{M}_n = \exp\left[\sum_{l=1}^{\infty} a^l \left(f^{(l)}(\epsilon) M_n^{(1)}(l\epsilon) + h_n^{(l)}(\epsilon) \right)\right]$$

where $M_n^{(1)}$ is the one-loop amplitude and h_n is an undetermined finite function.

$$a = \frac{N_c \alpha_s}{2\pi}(4\pi e^{-\gamma})^{\epsilon} \qquad f^{(l)}(\epsilon) = f_0^{(l)} + \epsilon f_1^{(l)} + \epsilon^2 f_2^{(l)}$$

$$f_0^{(l)} = \frac{1}{4}\hat{\gamma}_K^{(l)}, \qquad f_1^{(l)} = \frac{l}{2}\hat{\mathcal{G}}_0^{(l)},$$

γ_K has various names: cusp anomalous dimension, soft anomalous dimension, high spin limit of the leading twist operators, high moment limit of Altarelli-Parisi kernel.

$$\gamma_K = 4\,a - 4\zeta_2\,a^2 + 22\zeta_4\,a^3 + \cdots\,,$$

$$\gamma(j) = \frac{1}{2}\gamma_K(\ln(j) + \gamma_e) - B(\alpha_s) + \mathcal{O}(\ln(j)/j)\,,$$

$\gamma(j)$ is the anomalous dimension of leading twist operator at spin j.

- Our determination of the cusp anomalous dimension agrees with that of Kotikov, Lipatov, Onishchenko and Velizhanin (KLOV) as extracted from the QCD computation of Moch, Vermaseren and Vogt (MVV).
- Also agrees with the results of Bethe ansatz integrability results of Staudacher. New ansatz for all orders γ_K!
- By assuming iteration of splitting amplitudes, it seems possible to evaluate γ_K to all loop orders. Problem for the future.
- Some recent progress on constructing a proof for higher numbers of legs from Cachazo.

Key Formula for Finite Remainder

We can determine the finite remainder function through 3 loops by comparison to our explicit computations.

Left over finite parts are constants in planar $N = 4$ theory! We will assume this to be true to all loop orders.

Subtracting the known IR divergence (which cancels from any physical quantity) gives (taking $D = 4$ or $\epsilon = 0$ to recover conformal limit)

$$\mathcal{F}_n = \exp\left[\frac{1}{4}\gamma_K F_n^{(1)} + C\right] .$$

where $F_n^{(1)}$ are the known one-loop finite parts of scattering amplitudes.

$$\gamma_K = 4\,a - 4\zeta_2\,a^2 + 22\zeta_4\,a^3 + \cdots ,$$

$$C = -\frac{1}{2}\zeta_2^2\,a^2 + \left[\left(\frac{341}{216} + \frac{2}{9}c_1\right)\zeta_6 + \left(-\frac{17}{9} + \frac{2}{9}c_2\right)\zeta_3^2\right]a^3 + \cdots .$$

All loops are expressed in terms of 1-loop finite remainder!

This is almost certainly connected to integrability. Minahan and Zarembo; Beisert, Kristjansen and Staudacher; Bena, Polchinski and Roiban; Lipatov; Faddeev and Korchemsky; Tseytlin; and many others

Connection of Gravity and Gauge Theory Amplitudes

At tree-level, Kawai, Lewellen and Tye have given a complete description of the relationship between closed string and open string amplitudes.

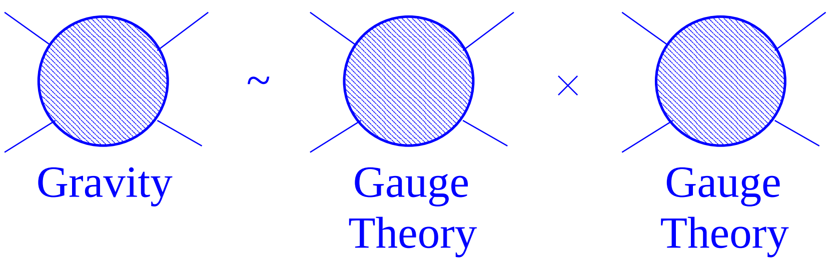
In the field theory limit ($\alpha' \to 0$) $\qquad s_{ij} = (k_i + k_j)^2$

$$M_4^{\rm tree}(1,2,3,4) = s_{12} A_4^{\rm tree}(1,2,3,4)\, A_4^{\rm tree}(1,2,4,3)\,,$$

$$M_5^{\rm tree}(1,2,3,4,5) = s_{12}s_{34} A_5^{\rm tree}(1,2,3,4,5) A_5^{\rm tree}(2,1,4,3,5) + s_{13}s_{24} A_5^{\rm tree}(1,3,2,4,5)\, A_5^{\rm tree}(3,1,4,2,5)$$

where we have stripped all coupling constants. M_n is gravity amplitude and A_n is color stripped gauge theory amplitude.

$\mathcal{A}_4^{\rm tree} = g^2 \sum_{\rm non-cyclic} {\rm Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4}) A_4^{\rm tree}(1,2,3,4)$



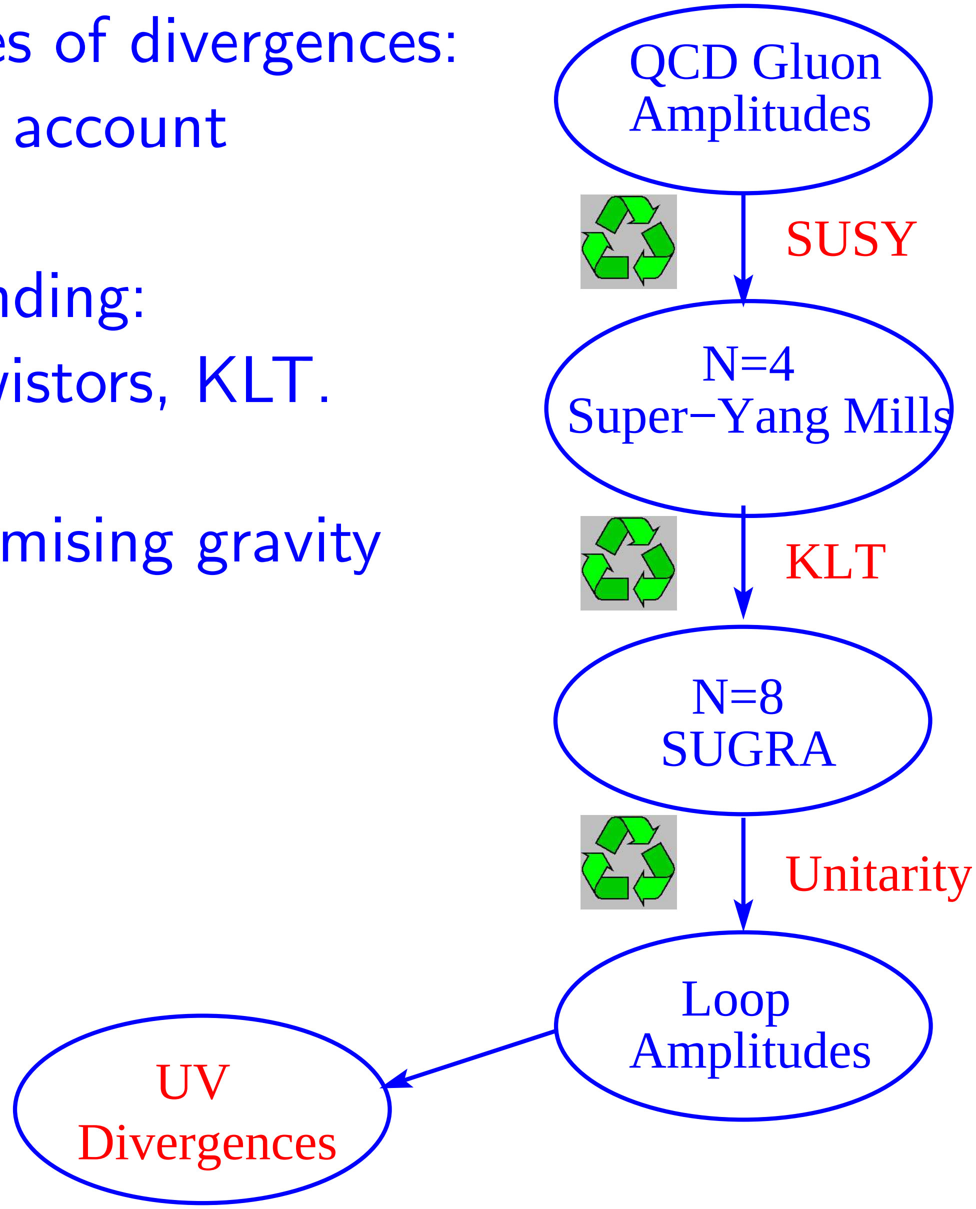
Holds for any external states.
See review: gr-qc/0206071

Supergravity Loops

- Serious flaw with all previous studies of divergences: Rely on powercounting, taking into account only supersymmetry.
  Now have a much deeper understanding: hidden symmetries and dualities, twistors, KLT.

- $N = 8$ supergravity is the most promising gravity theory to investigate for finiteness.

- More susy $\longrightarrow$ simpler calculations (with the right formalism).



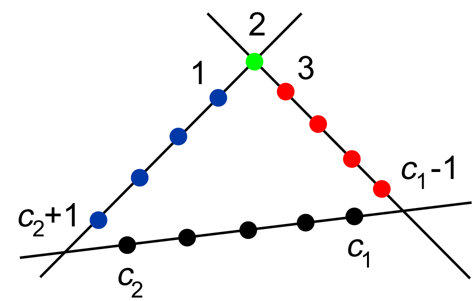
Comments on Gravity Amplitudes

- $N = 8$ supergravity definitely is less divergent than previously thought with the divergence delayed until *at least* 5 (instead of 3) loops.

Bern, Dixon, Dunbar, Perelstein, Rozowsky; Howe and Stelle

- Infinite sequences of one-loop MHV gravity amplitudes have been obtained by exploiting relationship to gauge theory. Gravity amplitudes inherit properties from gauge theory ones. Bern, Dixon, Rozowsky, Yan

- Twistor space structure of tree and one-loop amplitudes in gravity *inherited* from gauge theory, except derivative of delta-function support.



Witten; Bern, Bjerrum-Bohr, Dunbar; Bjerrum-Bohr, Dunbar, Ita; Bjerrum-Bohr, Dunbar, Ita, Perkins,Risager

- In one-loop n-graviton amplitudes a remarkable set of cancellations: amplitudes have same UV behavior as $N = 4$ super-Yang-Mills theory.

Bern, Dixon, Perelstein, Rozowsky; Bern, Bjerrum-Bohr, Dunbar, Ita

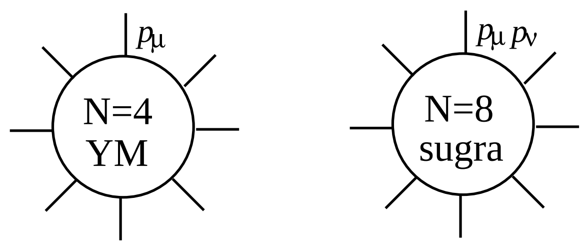
Bern, Dixon, Perelstein, Rozowsky
Bern, Bjerrum-Bohr, Dunbar, Ita
hep-th/9811140; hep-th/0501137

$N = 8$ Cancellations

Well known that all one loop supergravity amplitudes are finite. No supersymmetric counterterm exists.

Closer examination of the scattering amplitudes reveals striking set of cancellations, beyond what is needed for one-loop finiteness.

Compare $N = 4$ Yang-Mills with $N = 8$ supergravity:



Relative degree of divergence seems to gets worse.

However, all complete calculations to date find $N = 8$ sugra has *exactly* the same degree of divergence as $N = 4$ Yang-Mills.

Unitarity method directly feeds lower loop amplitudes into higher loops.

Serious re-examination of the UV properties of multi-loop $N = 8$ supergravity using modern tools is needed.

Summary

1. Motivation for studying amplitudes.
   (a) LHC demands QCD loop calculations
   (b) Can we solve $N = 4$ super-Yang-Mills theory?
   (c) Is $N = 8$ supergravity finite, contrary to accepted wisdom?
2. Generalized unitarity method: Loop amplitudes from tree amplitudes.
3. Important new twistor space idea: Amplitudes are surprisingly simple, even for general helicities.
4. Presented non-trivial evidence that planar $N = 4$ super-Yang-Mills scattering amplitudes can be solved to all loop orders. Precise ansatz for MHV amplitudes to all loop orders.
5. Standard arguments that supergravity diverges has a serious flaw. For $N = 8$ some evidence to the contrary.
6. There are a variety of exciting avenues for further exploration in QCD, super-Yang-Mills and supergravity.