Integrability in AdS/CFT, Part I: Classical Strings and Spin Chains

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Mathematical Structures in String Theory
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Introduction

AdS/CFT Conjecture
• $\mathcal{N} = 4$ gauge theory (exactly) dual to IIB superstrings on $AdS_5 \times S^5$.
• Spectrum should agree. Would like to test.
• Strong/weak duality: Cannot use perturbation theory on both sides.
• Hope: Both models appear integrable.

Outline
• Perturbative comparison of spectra
• Integrability of the classical superstring sigma model
• From a classical solution to an algebraic curve

Assume: Classical, non-interacting strings.
AdS/CFT as a Strong/Weak Duality

AdS/CFT predicts spectra of strings and gauge theory to match.

String theory expansion of energies $E$ at large $\lambda$

$$E(\lambda) = \lambda^{1/4} E_0 + \lambda^{-1/4} E_1 + \lambda^{-3/4} E_2 + \lambda^{-5/4} E_3 + \ldots.$$  

Gauge theory expansion of scaling dimensions $D$ at small $\lambda$

$$D(\lambda) = D^0 + \lambda D^1 + \lambda^2 D^2 + \lambda^3 D^3 + \ldots.$$  

How to confirm $E(\lambda) = D(\lambda)$?
Large Spin on $S^5$

Consider states with variable spin $J$ on $S^5$ (and lots of other parameters).

Effective spin $\mathcal{J}$, effective coupling $\tilde{\lambda}$:

$$\mathcal{J} = \frac{J}{\sqrt{\lambda}}, \quad \tilde{\lambda} = \frac{\lambda}{J^2} = \frac{1}{\mathcal{J}^2}.$$

String theory expansion of energies $E$ at large $\lambda$, fixed $\mathcal{J}$

$$E(\lambda, J) = \lambda^{1/2} E_0(\mathcal{J}) + \lambda^0 E_1(\mathcal{J}) + \lambda^{-1/2} E_2(\mathcal{J}) + \ldots.$$

Gauge theory expansion of scaling dimensions $D$ at small $\lambda$, fixed $J$

$$D(\lambda) = D^0(J) + \lambda D^1(J) + \lambda^2 D_2(J) + \lambda^3 D_3(J) + \ldots.$$
Outsmarting AdS/CFT

Expansion for large $J$ at large $\lambda$:

$$E(\lambda, J) = \lambda^{1/2} (J^0 E^0_0 + J^{-1} E^1_0 + J^{-3} E^2_0 + J^{-5} E^3_0 + \ldots)$$

$$+ \lambda^0 (J^{-2} E^1_1 + J^{-4} E^2_1 + J^{-6} E^3_1 + \ldots)$$

$$+ \lambda^{-1/2} (J^{-3} E^1_2 + J^{-5} E^2_2 + J^{-7} E^3_2 + \ldots)$$

$$+ \ldots$$

$$= J E^0_0$$

$$+ \frac{\lambda}{J} E^1_0$$

$$+ \frac{\lambda^2}{J^3} E^2_0$$

$$+ \frac{\lambda^3}{J^5} E^3_0$$

$$+ \frac{\lambda}{J^2} E^1_1$$

$$+ \frac{\lambda^2}{J^4} E^2_1$$

$$+ \frac{\lambda^3}{J^6} E^3_1$$

$$+ \frac{\lambda}{J^3} E^1_2$$

$$+ \frac{\lambda^2}{J^5} E^2_2$$

$$+ \frac{\lambda^3}{J^7} E^3_2$$

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Three-Loop Discrepancies

Expansion for large $J$ at small $\lambda$:

\[
D(\lambda, J) = J D_0^0
\]
\[
+ \lambda (J^{-1} D_1^1 + J^{-2} D_2^1 + J^{-3} D_3^1 + \ldots)
\]
\[
+ \lambda^2 (J^{-3} D_1^2 + J^{-4} D_2^2 + J^{-5} D_3^2 + \ldots)
\]
\[
+ \lambda^3 (J^{-5} D_1^3 + J^{-6} D_2^3 + J^{-7} D_3^3 + \ldots) + \ldots
\]

\[
= J D_0^0
\]
\[
+ \frac{\lambda}{J} D_0^1
\]
\[
+ \frac{\lambda}{J^2} D_1^1
\]
\[
+ \frac{\lambda}{J^3} D_2^1
\]
\[
+ \frac{\lambda^2}{J^3} D_0^2
\]
\[
+ \frac{\lambda^2}{J^4} D_1^2
\]
\[
+ \frac{\lambda^2}{J^5} D_2^2
\]
\[
+ \frac{\lambda^3}{J^5} D_0^3
\]
\[
+ \frac{\lambda^3}{J^6} D_1^3
\]
\[
+ \frac{\lambda^3}{J^7} D_2^3
\]

 NB, Minahan, Staudacher, Zarembo
 Serban, Staudacher (also Callan, Lee, McLoughlin, Schwarz, Swanson, Wu)
Outsmarted by AdS/CFT

Actual expansion for large $J$ at large $\lambda$:

$$E(\lambda, J) = \lambda^{0.5} \left( J E_0^0 + J^{-1} E_0^1 + J^{-3} E_0^2 + J^{-5} E_0^3 + \ldots \right)$$

$$+ \lambda^0 \left( J^{-2} E_1^1 + J^{-4} E_1^2 + J^{-5} E_1^{2.5} + J^{-6} E_1^3 + \ldots \right)$$

$$+ \lambda^{-0.5} \left( J^{-3} E_2^1 + J^{-5} E_2^2 + J^{-6} E_2^{2.5} + J^{-7} E_2^3 + \ldots \right)$$

$$+ \ldots$$

$$= J E_0^0$$

$$+ \frac{\lambda}{J} E_0^1$$

$$+ \frac{\lambda^2}{J^3} E_0^2$$

$$+ \frac{\lambda^3}{J^5} E_0^3$$

$$+ \frac{\lambda^2}{J^4} E_1^1$$

$$+ \frac{\lambda^2}{J^5} E_1^2$$

$$+ \frac{\lambda^{2.5}}{J^5} E_1^{2.5}$$

$$+ \frac{\lambda^3}{J^6} E_1^3$$

$$+ \frac{\lambda^3}{J^5} E_2^1$$

$$+ \frac{\lambda^2}{J^5} E_2^2$$

$$+ \frac{\lambda^{2.5}}{J^6} E_2^{2.5}$$

$$+ \frac{\lambda^3}{J^6} E_2^3$$
AdS and CFT

Attempt to avoid strong/weak duality at large spin $J$

- Coefficients of expansion different.
- Structure of expansion different.

What next?

- Compute AdS at large $\lambda$.
- Compute CFT at small $\lambda$.
- Notice similar structures.
- Some agreement up to $O(\lambda^2)$ or $O(1/J^4)$.
- Understand how to interpolate to finite $\lambda$.
- Three-loop mismatch related to new terms $E_0^3 - D_0^3 = -\frac{16}{3} E_{1.5}$. 
Overview Classical Strings

★ Cast of Characters
• Classical spinning string solutions
• Coset space sigma model
• Integrability, Lax connection, monodromy

★ Results
• Spectral curve
• Analytic properties
• String moduli (finite cut solutions)
• Integral equations
Spinning Strings

Many examples investigated:

- Gubser, Klebanov, Polyakov
- Frolov, Tseytlin: hep-th/0204226
- Minahan: hep-th/0209047
- Frolov, Tseytlin: hep-th/0304255

Folded circular pulsating higher modes plane waves

Ansatz, e.g. string on $\mathbb{R}_t \times S^2$: Energy $\mathcal{E} = E/\sqrt{\lambda}$, spin $\mathcal{J} = J/\sqrt{\lambda}$.

$$t(\tau, \sigma) = \mathcal{E} \tau, \quad \vec{X}(\tau, \sigma) = \begin{pmatrix} \sin \vartheta(\sigma) \cos \mathcal{J} \tau \\ \sin \vartheta(\sigma) \sin \mathcal{J} \tau \\ \cos \vartheta(\sigma) \end{pmatrix}.$$ 

Solve equations of motion and Virasoro constraint

$$\vartheta(\sigma) = \text{am}(\mathcal{E}(\sigma - \sigma_0), \eta), \quad \mathcal{J} = \eta \mathcal{E}.$$
Periodicity

Folded string: \( \vartheta(0) = 0 \) and \( \vartheta'(\pi/2n) = 0 \)

\[
J = \sqrt{\lambda} \mathcal{J} = \sqrt{\lambda} \frac{2n}{\pi} K(1/\eta), \quad E = \sqrt{\lambda} \mathcal{E} = \sqrt{\lambda} \frac{2n}{\eta \pi} K(1/\eta).
\]

Circular string: \( \vartheta(0) = 0 \) and \( \vartheta(2\pi/n) = 2\pi \)

\[
J = \sqrt{\lambda} \mathcal{J} = \sqrt{\lambda} \frac{2n\eta}{\pi} K(\eta), \quad E = \sqrt{\lambda} \mathcal{E} = \sqrt{\lambda} \frac{2n}{\pi} K(\eta).
\]

Global charges of generic solutions

\[
J_k = \sqrt{\lambda} \mathcal{J}_k(\eta_\alpha), \quad S_k = \sqrt{\lambda} \mathcal{S}_k(\eta_\alpha), \quad E = \sqrt{\lambda} \mathcal{E}(\eta_\alpha)
\]

with algebraic, elliptic, hyperelliptic, \ldots \ functions of moduli \( \{\eta_\alpha\} \).

- Why elliptic functions? What is the meaning of moduli?
Towards a General Solution

• Too difficult to solve the equations of motion in general.
  No direct way to quantization as in flat space or plane waves.
• Near plane waves: Very difficult to expand around plane waves.
  Only expansion, but good testing ground.

Now what?

• Give up on finding exact energy spectrum.
• Classify solutions to understand structure of spectrum.
• Try to quantise that.

How?!  

• Extract all conserved charges: Lax pair, monodromy.
• Investigate their analyticity properties.
• Reconstruct the corresponding algebraic curve.
• Discretise the curve.
Strings on $AdS_5 \times S^5$

IIB superstrings propagate on the curved superspace $AdS_5 \times S^5$

Coset space

$$AdS_5 \times S^5 \times \text{fermi} = \frac{PSU(2, 2|4)}{Sp(1, 1) \times Sp(2)}.$$ 

Decomposition of the algebra $u(2, 2|4)$ to $sp(1, 1) \times sp(2)$

$$j \in psu(2, 2|4), \quad j = h + q_1 + p + q_2, \quad h \in sp(1, 1) \times sp(2).$$

Algebra $j = [j_1, j_2]$ respects $\mathbb{Z}_4$-grading $h: 0$, $q_1: 1$, $p: 2$, $q_2: 3$.
Supersymmetric Sigma Model

Field $g(\sigma, \tau) \in \text{U}(2, 2|4)$ ($8 \times 8$ supermatrix) with flat connection $J$

$$J = -g^{-1}dg = H + Q_1 + P + Q_2, \quad dJ - J \wedge J = 0.$$ 

Coset $g \simeq gh$ with $h(\sigma, \tau) \in \text{Sp}(1, 1) \times \text{Sp}(2)$. Action

$$S_{\sigma} = \frac{\sqrt{\lambda}}{2\pi} \int \left( \frac{1}{2} \text{str} P \wedge *P - \frac{1}{2} \text{str} Q_1 \wedge Q_2 + \Lambda \wedge \text{str} P \right).$$

$\text{psu}(2, 2|4)$ Noether current $K$ and equation of motion

$$K = P + \frac{1}{2} *Q_1 - \frac{1}{2} *Q_2 - *\Lambda, \quad d*K - J \wedge *K - *K \wedge J = 0.$$ 

Virasoro constraints

$$\text{str} P_+^2 = \text{str} P_-^2 = 0.$$
Lax Connection

Integrability $\leadsto$ Lax pair: Family of connections

$$A(z) = H + \frac{1}{2}(z^{-2} + z^2)P + \frac{1}{2}(z^{-2} - z^2)(*P - \Lambda) + z^{-1}Q_1 + zQ_2.$$ 

Connection $A(z)$ flat for all values of the spectral parameter $z$

$$dA(z) - A(z) \wedge A(z) = 0.$$ 

Equivalent to flatness of $J$ and conservation of $K$.

- **Analytic** for all $z \in \mathbb{C}$.
- **Poles** at $z = 0, \infty$.
- **Point** $z = 1$ related to global symmetry: $A(1 + \epsilon) = J - 2\epsilon * K + \ldots$.

Alternative spectral parameter $x$ (double covering):

$$x = \frac{1 + z^2}{1 - z^2}, \quad z^2 = \frac{x - 1}{x + 1}.$$
Monodromy

Monodromy of Lax connection around closed string

\[ \Omega(z) = \left( \operatorname{P} \exp \oint_{-\gamma} J \right) \left( \operatorname{P} \exp \oint_{\gamma} A(z) \right). \]

Eigenvalues invariant under deformations of \( \gamma \)

\[ \Omega(z) \simeq \text{diag}(e^{i\hat{p}_1(z)}, \ldots, e^{i\hat{p}_4(z)}) \mid e^{i\tilde{p}_1(z)}, \ldots, e^{i\tilde{p}_4(z)}). \]

Transformation of solution \( g(\sigma, \tau) \) to set of quasi-momenta \( \{p_k(z)\} \).

- The \( p_k(z) \) are conserved, gauge-invariant quantities.
- No (conformal/kappa) gauge fixing required.
- Analytic functions of \( z \): Much physical information in \( \{p_k(z)\} \).
- \( \{p_k(z)\} \) contains all (?) action variables in Hamilton-Jacobi formalism.
- Diagonalising \( \Omega(z) \) introduces (solution-dependent) singular points.
Global Charges

Expansion of Lax connection at $z = 1$:

$$A(1 + \epsilon) = J - 2\epsilon * K + \mathcal{O}(\epsilon^2).$$

Global $\mathfrak{psu}(2,2|4)$ charges $S$ can be read off from monodromy at $z = 1$

$$\Omega(1 + \epsilon) = I - \epsilon \frac{4\pi S}{\sqrt{\lambda}} + \mathcal{O}(\epsilon^2).$$

Expansion of quasi-momenta (fix $\hat{p}_k(1) = \tilde{p}_k(1) = 0$)

$$\hat{p}_k(1 + \epsilon) \sim \epsilon \frac{4\pi (E, S_1, S_2)}{\sqrt{\lambda}} + \ldots, \quad \tilde{p}_k(1 + \epsilon) \sim \epsilon \frac{4\pi (J_1, J_2, J_3)}{\sqrt{\lambda}} + \ldots$$
Conjugation Symmetry

\[ \mathbb{Z}_4 \text{ property of supertranspose: } X^{ST,ST} = \eta X \eta, \ X^{ST,ST,ST,ST} = X. \]

Conjugation of connection \( J = H + Q_1 + P + Q_2 \)

\[ C \left( H, Q_1, P, Q_2 \right)^{ST} C^{-1} = (-H, -iQ_1, +P, +iQ_2). \]

Map \( z \mapsto iz \) conjugates Lax connection and monodromy

\[ A(iz) = -C A^{ST}(z) C^{-1}, \quad \Omega(iz) = C \Omega^{-ST}(z) C^{-1}. \]

Transformation of quasi-momenta with \( k' = (2, 1, 4, 3), \varepsilon_k = (+, +, -, -) \)

\[ \hat{p}_k(iz) = -\hat{p}_{k'}(z), \quad \tilde{p}_k(iz) = 2\pi m \varepsilon_k - \tilde{p}_{k'}(z). \]

\( z \mapsto -z \) is a trivial symmetry of quasi-momenta. Okay to use \( x \)

\[ x = \frac{1 + z^2}{1 - z^2} \quad \quad z^2 = \frac{x - 1}{x + 1}. \]
Analyticity

Monodromy $\Omega(z)$ is analytic in $z$ except at $z = 0, \infty$. Consider $z = 0$:
Diagonalise Lax connection perturbatively with regular $T(z)$

$$\partial_\sigma - \bar{A}_\sigma(z) = T(z)(\partial_\sigma - A_\sigma(z))T^{-1}(z).$$

Derivative $\partial_\sigma = \mathcal{O}(z^0)$ subleading w.r.t. $A_\sigma(z) = \mathcal{O}(1/z^2)$:

$$\bar{A}(z) = \frac{1}{2}T(P+ + \Lambda_\sigma)T^{-1}/z^2 + \mathcal{O}(1/z)$$
$$= \text{diag}(\alpha, \alpha, \beta, \beta||\alpha, \alpha, \beta, \beta)/z^2 + \mathcal{O}(1/z)$$

Degeneracies due to conjugation $CP^{ST}C^{-1} = P$, tracelessness $\text{str } P = 0$ and Virasoro $\text{str } P_+^2 = 0$.

$$\hat{p}_{1,2}(z) \sim \tilde{p}_{1,2}(z) \sim \alpha/z^2, \quad \hat{p}_{3,4}(z) \sim \tilde{p}_{3,4}(z) \sim \beta/z^2$$

at $z = 0$.

Diagonalization introduces new singularities $\{\hat{z}_a, \tilde{z}_a, z^*_a\}$ in $\hat{p}_k(z), \tilde{p}_k(z)$. 
Bosonic Branch Points

Eigenvalue crossing: Consider $2 \times 2$ bosonic submatrix $\Gamma$ of $\Omega(z)$

$$\Gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \gamma_{1,2} = \frac{1}{2} \left( a + d \pm \sqrt{(a - d)^2 + 4bc} \right).$$

Generic behaviour at degenerate eigenvalues $e^{ip_k(z_a)} = e^{ip_l(z_a)}$:

$$e^{ip_k(z_a)} \left( 1 \pm \alpha_a \sqrt{z - z_a} + O(z - z_a) \right).$$

Full turn around $z_a$ interchanges eigenvalues (labelling): Branch cut.
Fermionic Singularities

Mixed eigenvalue crossing: Consider $(1|1) \times (1|1)$ submatrix $\Gamma$ of $\Omega(z)$

$$\Gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \hat{\gamma} = \frac{bc}{d-a} + a, \quad \tilde{\gamma} = \frac{bc}{d-a} + d.$$  

Generic behaviour at degenerate eigenvalues $e^{i\hat{p}_k(z^*_a)} = e^{i\tilde{p}_l(z^*_a)}:

$$e^{i\hat{p}_k(z^*_a)} \left( \frac{\alpha^*_a}{z - z^*_a} + 1 + O(z - z^*_a) \right).$$

Residue of fermionic singularity $\alpha^*_a \sim bc$ is nilpotent.
Spectral Curve

- Singularities at $x = \pm 1$; asymptotics at $x = 0, \infty$; symmetry $x \mapsto 1/x$.
- Bosonic modes: Square-roots, branch cuts (Bose condensates).
- Fermionic excitations: Poles (Pauli principle).
- Stringy spectrum of physical excitations $4 + 4 \mid 8$. 

[Text references: NB, Kazakov, Sakai, Zarembo]
Spectral Transformation

From embedding of world-sheet $g(\sigma, \tau)$ to spectral curve $p'(x)$.

Spectral curve encodes **conserved charges** of a string solution.
Algebraic Curve

Can the Riemann surface $\mathbb{M}$ be embedded in $\mathbb{C}^2$ as an algebraic curve?

- Finite genus: Assume finitely many singularities $\{\hat{x}_a, \tilde{x}_a, x^*_a\}$. ✓
  Other solutions should be considered as limiting cases.
- Eigenvalues $e^{ip(x)}$ of $\Omega(x)$ are analytic almost everywhere. ✓
- Singularities $\{\hat{x}_a, \tilde{x}_a, x^*_a\}$ are square-root or pole singularities. ✓
- Monodromy $\Omega(x)$ has exponential singularities at $x = \pm 1$. ✗
- Quasi-momentum $p(x)$ is defined modulo $2\pi$. ✗
- $p'(x)$ is unique and has only square-root and pole singularities. ✓

$p'(x)$ is the algebraic curve associated to a classical string $g(\tau, \sigma)$

$$g(\tau, \sigma) \implies \frac{\hat{F}(x, p'(x))}{\tilde{F}(x, p'(x))} = 0, \infty \text{ with } \hat{F}, \tilde{F} \text{ polynomial (degree 4 in } p').$$

Simplest spinning strings have genus $0/1$: algebraic/elliptic functions.

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String Moduli

Single-valuedness of $e^{ip}$: All closed cycles must be integer

$$\oint dp \in 2\pi \mathbb{Z}.$$  

Cuts/singularities: “mode number” $n_a \in \mathbb{Z}$ and “amplitude” $K_a \in \mathbb{R}$

$$\int_{\mathcal{A}_a} dp = 0, \quad n_a = \frac{1}{2\pi} \int_{\mathcal{B}_a} dp, \quad K_a = -\frac{1}{2\pi i} \oint_{\mathcal{A}_a} \left(1 - \frac{1}{x^2}\right) p(x) \, dx.$$  

Solutions classified by: connection of sheets, mode numbers, amplitudes.
Integral Equations

Parametrise quasi-momenta $p(x)$ using 7 resolvents (cuts/poles)

$$G_j(x) = \int_{C_{j,a}} dy \, \rho_j(y) \frac{1}{1 - 1/y^2} \frac{1}{y - x} + \sum_a \frac{\alpha_{j,a}}{1 - 1/x_{j,a}^2} \frac{1}{x_{j,a} - x}.$$  

**Integral equations** with $H_j(x) = G_j(x) + G_j(1/x) - G_j(0)$

$$-2\pi n_{j,a} = \sum_{j'=1}^{7} M_{j,j'} H_{j'}(x) + F_j(x), \quad \text{for } x \in C_{j,a}, x_{j,a}.$$  

$M_{j,j'}$: Cartan matrix of $su(2,2|4)$.

$F_j(x)$: Potential terms made from $G_{j'}(0)$, $G_{j'}'(0)$ and $G_{j'}'(1/x)$. 
Conclusions

☆ AdS/CFT Spectral Comparison
- No exact perturbative comparison possible.

☆ IIB Strings in $AdS_5 \times S^5$
- Classical spectral curve derived & investigated.

☆ Outlook
- Integrability for gauge theory (tomorrow).
- Quantise string spectral curve.
- Find exact Bethe ansatz for AdS/CFT (if it exists).