## New remarks on N=4

(G. Bossard + L.B, hep-th: 0507003)

N=4 SYM is beautiful: Finite, no anomaly for superconformal invariance, exact electromagnetic duality, etc...

Vafa + Witten twisted the theory in 94. In fact, this gives a dimensional reduction of the N=2, d=8 TQFT, relying on octonionic 8d self-duality Yang-Mills equation (L.B. + I.M. Singer)

$$F_{8i} = c_{ijk} F_{jk} \leftrightarrow F_{\mu\nu} = \Omega_{\mu\nu\rho\sigma} F_{\rho\sigma}$$

Here, I show (with G. Bossard), that the $\underline{16}$ supercharges:

$$\{Q^{\alpha}{}_i, \bar{Q}^{\dot\alpha i}\} \overset{SU(4)}{\longleftrightarrow} \{\underbrace{s, \bar{s}, s_\mu}_{\downarrow}, \bar{s}_\mu, \underset{\downarrow}{s_{\mu\nu^-}}, \bar{s}_{\mu\nu^-}\}$$

are such that:

- closed ("off-shell") algebra (6 generators)
- Fully determines the theory (and the rest of susy)
- $SL(2,R)$ symmetry is built-in, and sufficient to fix everything in $s$, $s_\mu$. $s_{\mu\nu^-}$.

Both scalar and vector TQFT symmetry have a geometrical interpretation, independent of supersymmetry. As we will see these properties follow from TQFT construction in $d=8 \to d=7 \to d=4$.

## Knowing the scalar BRST (YM) operator $Q=s$, where does come $s_\mu$??

↑05 04 224

Forget susy. Ask the question:

$$S_{top}(\text{gauge fields} + \text{fermionic partners}) = \int d(\;) + \{s, \underset{\uparrow ?}{\Lambda^{-1}}\}$$

- In fact $\boxed{T_{\mu\nu} = \{s, \Lambda^{-1}_{\mu\nu}\}}$ (Witten 1988)
  (1) $s$ is geometrical scalar operator, well defined in curved space
  (2) The equivariant $s_c$, is a combinaison of susy generators.
- Is $\Lambda^{-1}_{\mu\nu}$ conserved, as $T_{\mu\nu}$? If yes, $\Lambda^{-1}_{\mu\nu}$ is the Noether current of a new vector symmetry, $s_\mu$. Then $\boxed{\{s, s_\mu\} = \partial_\mu}$ $\underline{s_\mu \text{ fixes } \Lambda^{-1}_{\mu\nu} \text{ or } \Lambda^{-1}}$.
- Can we build $s_\mu$ geometrically, as we did for $s$? $\boxed{\text{Yes.}}$
- Therefore, one can propose the following path: build $s$ and $s_\mu$; ask for a $s$- and $s_\mu$- invariant action.

→ This determines, in twisted form, all YM susy actions!

<u>Note</u> $(s, s_\mu)$ + inv. action $\to (s, s_\mu, \underset{\uparrow}{s_{\mu\nu^-}}) \underset{\text{twist}}{\approx} Q^\alpha$

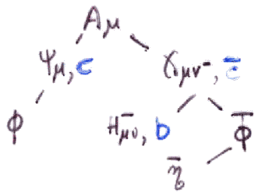
$s_{\mu\nu^-}$ concentrates all non "off-shell" closure of susy.

2

Initiating $N=4, d=4$ susy : $N=2$ $d=8$ : $\boxed{\text{vector symmetry}}$

~~2004~~ 0504224

$(A_\mu, \lambda^\alpha, \bar\lambda_{\dot\alpha}, \varphi) \Leftrightarrow (A_\mu, \Psi_\mu, \chi_{\mu\nu^-}, \bar\eta, \Phi, \bar\Phi)$




note : $\chi_{\mu\nu^-} = \frac{1}{2} \Omega_{\mu\nu\rho\sigma} \chi_{\mu\nu^-}$ $\quad (28 = \underset{\text{spin}(7)}{21} \oplus 7)$

$\rightarrow (Q^\alpha, Q_{\dot\alpha}) \leftrightarrow (s, s_\mu, s_{\mu\nu^-})$

Call $\boxed{\delta \equiv K^\mu s_\mu}$ $\quad \boxed{K^\mu = \text{covariantly constant vector.}}$

$$\boxed{\begin{aligned} \tilde{\mathcal{F}} \equiv (d + s + \delta - i_K)(A + c + |K|\bar c) + (A + c + |K|\bar c)^2 = \\ F + \Psi + g(K)\bar\eta + i_K \chi + \Phi + |K|^2 \bar\Phi \end{aligned}}$$

+ Bianchi identity

$\boxed{s^2 = 0}$ $\quad \boxed{\{s_\mu, s_\nu\} = 0}$ $\quad \boxed{\{s, s_\mu\} = \partial_\mu}$

- $\left\{ \begin{array}{l} s\int\mathcal{L} = \delta\int\mathcal{L} = 0 \\ \text{no } K \text{ dependence} \end{array} \right.$ $\quad \Rightarrow \int \mathcal{L} = \int s(\underset{\text{Fixed}}{\cdots}) = \int s\delta(\cdots)$

$\underset{\text{by twist}}{=} \mathcal{L}_{N=2, d=8}$

- $\delta$ fixes the Morse Function.

$\rightsquigarrow$ off shell closed algebra. $s, \ldots$ come for free.

$g(K) = g_{\mu\nu} dx^\mu K^\nu$ ; $i_K dx^\mu = K^\mu$



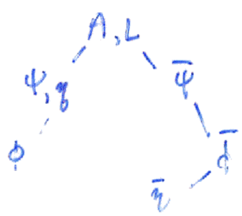
Going down to $d=7$. First step $\rightarrow$ $SL(2,R)$ symmetry

$M_8 = N_7 \times S_1$. take $K^\mu$ tg. to $S_1$.

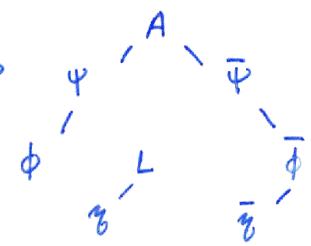
$$\left\{ \begin{array}{lcl} \overset{d=8}{(s, s_\mu)} & \rightarrow & \overset{d=7}{(s, \bar s, s_i)} \quad \text{(still 9 generators)} \\ A_\mu & \rightarrow & (L, A_i) \\ s_{\mu\nu^-} & \rightarrow & \bar s_i \\ \chi_{\mu\nu^-} & \rightarrow & \bar\Psi_i \end{array} \right.$$

Last equation : $\boxed{\begin{aligned} \tilde{\mathcal{F}}' \equiv (d + s + \bar s)(A + c + \bar c) + (A + c + \bar c)^2 = \\ F + \Psi + \bar\Psi + \Phi + L + \bar\Phi \end{aligned}}$

+ Bianchi identity




$\rightarrow$




An $SL(2,R)$ symmetry has been created $\varphi^{g, G-g}_{p-G}$

| | | | | |
|---|---|---|---|---|
| $(\Psi, \bar\Psi)$ | $\rightarrow$ | $SL(2,R)$ | doublet | $\Psi^\alpha$ |
| $(\Phi, L, \bar\Phi)$ | $\rightarrow$ | " " | triplet | $\Phi^{ij}$ |
| $s, \bar s$ | $\rightarrow$ | " " | doublet | $s^\alpha$ |
| $\eta, \bar\eta$ | $\rightarrow$ | " " | " " | $\eta^\alpha$ |
| $s_i, \bar s_i$ | $\rightarrow$ | " " | " " | $s_i^\alpha$ |

The fields $(\Psi,\bar\Psi)$ and $(\eta,\bar\eta)$ can be identified as $SL(2,\mathbb{R})$ doublets, $\Psi^\alpha$ and $\eta^\alpha$, $\alpha=1,2$, and the three scalar fields $(\Phi,L,\bar\Phi)$ as a $SL(2,\mathbb{R})$ triplet, $\Phi^i$, $i=1,2,3$. The index $\alpha$ and $i$ are respectively raised and lowered by the volume form $\varepsilon_{\alpha\beta}$ of $SL(2,\mathbb{R})$ and the Minkowski metric $\eta_{ij}$ of signature $(2,1)$. Both BRST and antiBRST operators can be assembled into a $SL(2,\mathbb{R})$ doublet $s^\alpha=(s,\bar s)$.

The horizontality condition () can be solved, with the introduction of three 0-form Lagrange multipliers, $\eta,\bar\eta,b$ and a 1-form $T$.

$$
\begin{cases}
sA=\Psi-d_Ac & \bar sA=\bar\Psi-d_A\bar c\\
s\Psi=-d_A\Phi-[c,\Psi] & \bar s\Psi=-T-d_AL-[\bar c,\Psi]\\
s\Phi=-[c,\Phi] & \bar s\Phi=-\bar\eta-[\bar c,\Phi]\\
s\bar\Phi=\eta-[c,\bar\Phi] & \bar s\Phi=-[\bar c,\bar\Phi]\\
s\eta=[\Phi,\bar\Phi]-[c,\eta] & \bar s\eta=-[\bar\Phi,L]-[\bar c,\eta]\\
sL=\bar\eta-[c,L] & \bar sL=-\eta-[\bar c,L]\\
s\bar\eta=[\Phi,L]-[c,\bar\eta] & \bar s\bar\eta=[\Phi,\bar\Phi]-[\bar c,\bar\eta]\\
s\bar\Psi=T-[c,\bar\Psi] & \bar s\bar\Psi=-d_A\bar\Phi-[\bar c,\bar\Psi]\\
sT=[\Phi,\bar\Psi]-[c,T] & \bar sT=-d_A\eta+[L,\bar\Psi]-[\bar\Phi,\Psi]-[\bar c,T]\\
 & \\
sc=\Phi-c^2 & \bar sc=L-b\\
s\bar c=b-[c,\bar c] & \bar s\bar c=\bar\Phi-\bar c^2\\
sb=[\Phi,\bar c]-[c,\bar c] & \bar sb=\eta+[\bar c,L]
\end{cases}
$$

These equations are not $SL(2,\mathbb{R})$ covariant because of our simplest choice of the transformation of antighosts transformations, like $s\bar c=b-[c,\bar c]$. By suitable redefinitions of auxiliary fields, one can get, however, a manifestly $SL(2,\mathbb{R})$ covariant formulation of the symmetry, as follows:

$$
\begin{cases}
s^\alpha A=\Psi^\alpha-d_Ac^\alpha & s^\alpha\eta_\beta=-2\sigma^{ij}{}_\beta{}^\alpha[\Phi_i,\Phi_j]-[c^\alpha,\eta_\beta]\\
s^\alpha\Psi_\beta=\delta^\alpha_\beta T-\sigma^i{}_\beta{}^\alpha d_A\Phi_i-[c^\alpha,\Psi_\beta] & s^\alpha T=\frac12 d_A\eta^\alpha+\sigma^{i\,\alpha\beta}[\Phi_i,\Psi_\beta]-[c^\alpha,T]\\
s^\alpha\Phi_i=\frac12\sigma_i{}^{\alpha\beta}\eta_\beta-[c^\alpha,\Phi_i] & \\
s^\alpha c_\beta=-\delta^\alpha_\beta b+\sigma^i{}_\beta{}^\alpha\Phi_i-\frac12[c^\alpha,c_\beta] & \\
s^\alpha b=-\frac12\eta^\alpha+\frac12\sigma^{i\,\alpha\beta}[\Phi_i,c_\beta]+\frac1{12}[c^\beta,[c_\beta,c^\alpha]]-\frac12[c^\alpha,b] &
\end{cases}
$$

See the idea of $\mathcal{N}_T=2$ ~~balanced~~ balanced TQFT, by Dijgkraaft and Moore.



The Cartan algebra (that we will denote by the subindex $(c)$) is obtained by adding gauge transformations with parameters $c$ and $\bar c$, from $s$ and $\bar s$, respectively. It reads :

$$
\begin{array}{ll}
s^\alpha_{(c)}A=\Psi^\alpha & s^\alpha_{(c)}\eta_\beta=-2\sigma^{ij}{}_\beta{}^\alpha[\Phi_i,\Phi_j]\\
s^\alpha_{(c)}\Psi_\beta=\delta^\alpha_\beta T-\sigma^i{}_\beta{}^\alpha d_A\Phi_i & s^\alpha_{(c)}T=\frac12 d_A\eta^\alpha+\sigma^{i\,\alpha\beta}[\Phi_i,\Psi_\beta]\\
s^\alpha_{(c)}\Phi_i=\frac12\sigma_i{}^{\alpha\beta}\eta_\beta &
\end{array}
$$

The equivariant (Cartan) algebra is the one that will match by twist with the relevant part of the twisted supersymmetry algebra. Its closure is only modulo gauge transformations, with parameters that are ghosts of ghosts.

TQFT gauge function is $F_{ij} - c_{ijk} D_i L$

$$S = \int C_3 \wedge \mathrm{Tr}(F\wedge F) + s\big(\bar\Psi(CF + DL + T) + \bar\phi D\Psi + \eta(\Phi,\bar\Phi) + \bar\eta(\bar\phi L)\big)$$

$\Rightarrow$ SUSY 7-d action.

The $SL(2,R)$ doublet $s^\alpha$ of vector symmetries in $d=7$

What happened to the vector symmetry in $d=8$? In fact there are many ways of choosing the circle $S^1$ of dimensional reduction from 8 to 7 dimensions. This generates an automorphism $\Gamma$ of the 7 dimensional symmetry in 7d. Basically, we will have the (commuting) 7 dimensional derivative $\Gamma$, with

$$\delta^\alpha_{(c)} = [s^\alpha_{(c)}, \Gamma]$$ so we expect the

16 Fermionic generators: $(s, \bar{s}, s_i, \bar{s}_i)$

with: $(i_\kappa * C^* \Psi \equiv c_{ijk} \kappa^i dx^j \Psi^k)$ $\delta = \kappa^i s_i$ $\bar{\delta} = \kappa^i \bar{s}_i$

$$\boxed{\begin{aligned}(d + s + \bar{s} + \delta + \bar{\delta})(A + c + \bar{c} + |\kappa|(\gamma + \bar{\gamma})) + (A + c \ldots)^2 \\ = F + \Psi + \bar{\Psi} + g(\kappa)(\eta + \bar{\eta}) + i_\kappa * C^*(\Psi + \bar{\Psi}) \\ + (1 + |\kappa|^2)(\Phi + L + \bar{\Phi})\end{aligned}}$$

This equation is $SL(2,R)$ invariant. Notice that

$$\{s,\delta\} + \{\bar{s},\bar{\delta}\} = 0 \qquad \leadsto \boxed{\begin{cases}[s,\delta] = \mathcal{L}_\kappa \\ [\bar{s},\bar{\delta}] = -\mathcal{L}_\kappa\end{cases}}$$

$\Rightarrow$ need new eqs., in a $SL(2,R)$ invariant way.

9

**The complete Faddeev–Popov ghost dependent vector and scalar topological symmetries in seven dimensions**

We now directly construct the scalar and vector BRST topological operators $\boxed{s^\alpha \text{ and } \delta^\alpha}$ the equivariant analogs of which are $s^\alpha_{(c)}$ and $\delta^\alpha_{(c)}$. One has scalar Faddeev–Popov ghosts, $c, \bar{c}, \gamma, \bar{\gamma}$, which are associated to the equivariant BRST operators $s_{(c)}, \bar{s}_{(c)}, \delta_{(c)}, \bar{\delta}_{(c)}$, respectively.

The relations (14) suggests the following horizontality condition, with $(d+s+\bar{s}+\delta+\bar{\delta})^2 = 0$:

$$\begin{aligned}&(d + s + \bar{s} + \delta + \bar{\delta})(A + c + \bar{c} + |\kappa|\gamma + |\kappa|\bar{\gamma}) + (A + c + \bar{c} + |\kappa|\gamma + |\kappa|\bar{\gamma})^2 \\ &\qquad = F + \Psi + \bar{\Psi} + g(\kappa)(\eta + \bar{\eta}) + i_\kappa \star C^*(\Psi + \bar{\Psi}) + (1 + |\kappa|^2)(\Phi + L + \bar{\Phi})\end{aligned} \tag{18}$$

It is $SL(2,R)$ invariant. By construction, this equation has the following indetermination:

$$\{s,\delta\} + \{\bar{s},\bar{\delta}\} = 0 \tag{19}$$

This degeneracy is raised, owing to the introduction of the constant vector $\kappa$, with,

$$\boxed{\{s,\delta\} = \mathcal{L}_\kappa \qquad \{\bar{s},\bar{\delta}\} = -\mathcal{L}_\kappa} \tag{20}$$

This relation is fulfilled by completing Eq. (18) by the following ones:

$$\begin{cases}(d + s + \delta - i_\kappa)(A + c + |\kappa|\gamma) + (A + c + |\kappa|\gamma)^2 \\ \qquad = F + \Psi + g(\kappa)\eta + i_\kappa \star C^*(\bar{\Psi}) + (\Phi + |\kappa|^2\bar{\Phi}) & (21) \\ (d + \bar{s} + \bar{\delta} + i_\kappa)(A + \bar{c} + |\kappa|\bar{\gamma}) + (A + \bar{c} + |\kappa|\bar{\gamma})^2 \\ \qquad = F + \bar{\Psi} + g(\kappa)\bar{\eta} + i_\kappa \star C^*(\Psi) + (\bar{\Phi} + |\kappa|^2\Phi) & (22)\end{cases}$$

8

Thus, in $d=7$, by stepping from $d=8$:

- $SL(2,R)$ symmetry has appeared
- 16 geometrical BRST operators exist:
  $(s, s_i, \bar{s}, \bar{s}_i)$
- Asking invariance under $s$ and $s_i$ (8 generat.)
  $$S = \int d(.) + s^\alpha(\Psi_\alpha)$$

In fact $d(..) = \int_{N_4} C_3 \wedge T_2(F \wedge F)$ to match susy action and $(\sigma^i)^{\alpha\beta} \delta_\alpha \Psi_\beta = 0$

- $(s, \bar{s}, s_i)$ form the maximal off-shell closed algebra of $d=7$ susy.
  (remember $s_{\mu\nu^-} \sim \bar{s}_i$)

→

9

## From $d=7$ to $d=4$

The twisted $N=4$ susy of Vaffa-Witten is:

$$S_{N=4} = \int H_{\mu\nu^-}\left(\tfrac{1}{2} H_{\mu\nu^-} + F_{\mu\nu^-}\right) + H_\mu\left(\tfrac{1}{2} H_\mu + D_\mu L + D_\nu h_{\mu\nu^-}\right) + D_\mu \Phi D_\mu \bar{\Phi} + \text{Fermi terms} + \text{higher order} )$$

- So fields are:
  $$\big(\underset{4}{A_\mu}, \underset{3}{h_{\mu\nu^-}}, \underset{1}{L}, \underset{1}{\phi}, \underset{1}{\bar{\phi}}, (\Psi_\mu, \bar{\chi}_{\mu\nu^-}, \bar{\eta}), (\bar{\Psi}_\mu, \chi_{\mu\nu^-}, \eta)\big)$$
- It is clearly a TQFT with $3+4$ topological gauge functions, for $A_\mu, L, h_{\mu\nu^-}$ (see LB, Kanno, Singer 1998))
- In hyperkahler space, one can untwist, and one has 3 constant hyperKähler forms $J^I_{\mu\nu^-}$, and
  $$\Phi^I \overset{\in SU(2)}{=} (J^I_{\mu\nu^-})\, h_{\mu\nu^-}$$
- The full eventual $SO(5,1)$ ($SO(6)$ after Wick rotation) is for free.

10

$s, \bar{s}, s_\mu, \bar{s}_\mu$ are defined, elegantly, by:

$$\begin{cases} (d+s+\bar{s}+\delta+\bar{\delta})(A+c+\bar{c}+|\kappa|\gamma+|\kappa|\bar{\gamma})+(A+c+\bar{c}+|\kappa|\gamma+|\kappa|\bar{\gamma})^2 \\ \qquad = F+\Psi+\bar{\Psi}+g(\kappa)(\eta+\bar{\eta})+g(J_I\kappa)(\chi^I+\bar{\chi}^I)+(1+|\kappa|^2)(\Phi+L+\bar{\Phi}) \\ (d_A+s_{(c)}+\bar{s}_{\bar{c}}+\delta_\gamma+\bar{\delta}_{\bar{\gamma}})h^I = d_A h^I+\bar{\chi}^I-\chi^I+i_{J^I\kappa}(\bar{\Psi}-\Psi) \end{cases} \tag{36}$$

$$\begin{cases} (d+s+\delta-i_\kappa)(A+c+|\kappa|\gamma)+(A+c+|\kappa|\gamma)^2 = F+\Psi+g(\kappa)\eta+g(J_I\kappa)\chi^I+\Phi+|\kappa|^2\bar{\Phi} \\ (d_A+s_{(c)}+\delta_\gamma-i_\kappa)h^I = d_A h^I+\bar{\chi}^I+i_{J^I\kappa}\bar{\Psi} \end{cases} \tag{37}$$

$\delta = \kappa^\mu s_\mu$ and $\bar{\delta} = \kappa^\mu \bar{s}_\mu$

- The 16 generators $(s, s_i, \bar{s}, \bar{s}_i)$ in $d=4$ have become $(s, s_\mu, \bar{s}, \bar{s}_\mu, s_{\mu\nu^-}, \bar{s}_{\mu\nu^-})$ in $d=8$
- The max. off shell closed algebra with 9 generators is $(s, s_\mu, s_{\mu\nu^-}, \bar{s})$
- But the action, and the full sym. is determined by 6 gene $\boxed{(s, \bar{s}, s_\mu)}$

11

The Cartan BRST scalar algebra is:

$$\begin{aligned} s^\alpha_{(c)} A &= \Psi^\alpha \\ s^\alpha_{(c)} \Psi_\beta &= \delta^\alpha_\beta T - \sigma^i{}_\beta{}^\alpha d_A \Phi_i \\ s^\alpha_{(c)} \Phi_i &= \frac{1}{2}\sigma_i{}^{\alpha\beta}\eta_\beta \\ s^\alpha_{(c)} \eta_\beta &= -2\sigma^{ij}{}_\beta{}^\alpha[\Phi_i,\Phi_j] \\ s^\alpha_{(c)} T &= \frac{1}{2} d_A \eta^\alpha + \sigma^{i\,\alpha\beta}[\Phi_i,\Psi_\beta] \end{aligned} \qquad \begin{aligned} s^\alpha_{(c)} h^I &= \chi^{\alpha\, I} \\ s^\alpha_{(c)} \chi^I_\beta &= \delta^\alpha_\beta H^I + \sigma^i{}_\beta{}^\alpha[\Phi_i, h^I] \\ s^\alpha_{(c)} H^I &= \frac{1}{2}[\eta^\alpha, h^I] + \sigma^{i\,\alpha\beta}[\Phi_i,\chi^I_\beta] \end{aligned}$$

One has the closure relation $s^{\{\alpha}_{(c)} s^{\beta\}}_{(c)} = \sigma^{i\,\alpha\beta}\delta_{\mathrm{gauge}}(\Phi_i)$. The Cartan vector algebra is:

$$\begin{aligned} \delta^\alpha_{(c)} A &= g(\kappa)\eta^\alpha + g(J_I\kappa)\chi^{\alpha\, I} \\ \delta^\alpha_{(c)} \Psi_\beta &= \delta^\alpha_\beta(i_\kappa F - g(J_I\kappa)H^I) + \sigma^i{}_\beta{}^\alpha g(J_I\kappa)[\Phi_i,h^I] - 2\sigma^{ij}{}_\beta{}^\alpha g(\kappa)[\Phi_i,\Phi_j] \\ \delta^\alpha_{(c)} \Phi_i &= -\frac{1}{2}\sigma_i{}^{\alpha\beta} i_\kappa \Psi_\beta \\ \delta^\alpha_{(c)} \eta_\beta &= -\delta^\alpha_\beta i_\kappa T + \sigma^i{}_\beta{}^\alpha \mathscr{L}_\kappa \Phi_i \\ \delta^\alpha_{(c)} T &= \frac{1}{2} d_A i_\kappa \Psi^\alpha - g(J_I\kappa)([\eta^\alpha,h^I] + \sigma^{i\,\alpha\beta}[\Phi_i,\chi^I_\beta]) + g(\kappa)\sigma^{i\,\alpha\beta}[\Phi_i,\eta_\beta] - \mathscr{L}_\kappa \Psi^\alpha \\ \delta^\alpha_{(c)} h^I &= -i_{J^I\kappa}\Psi^\alpha \\ \delta^\alpha_{(c)} \chi^I_\beta &= \delta^\alpha_\beta(\mathscr{L}_\kappa h^I + i_{J^I\kappa}T) + \sigma^i{}_\beta{}^\alpha \mathscr{L}_{J^I\kappa}\Phi_i \\ \delta^\alpha_{(c)} H^I &= \frac{1}{2}[i_\kappa\Psi^\alpha, h^I] + \mathscr{L}_{J^I\kappa}\eta^\alpha + \sigma^{i\,\alpha\beta}[\Phi_i, i_{J^I\kappa}\Psi_\beta] - \mathscr{L}_\kappa \chi^{\alpha\, I} \end{aligned}$$

12

**Invariant action**

There are two gauge functions which fit in a fundamental multiplet of $SL(2,\mathbb{R})$, and satisfy:

$$\sigma^{i\,\alpha\beta}\delta_{(c)\,\alpha}\Psi_\beta = 0 \qquad (28)$$

The action is defined as:

$$\boxed{S = -\frac{1}{2}\int_M \mathrm{Tr}\ F_\wedge F + s^\alpha_{(c)}\Psi_\alpha} \qquad (29)$$

Eq. (28) completely constrains the gauge function (up to a global scale factor), as follows:

$$\Psi_\alpha = \int_M \mathrm{Tr}\ \Big(\star\chi^I_\alpha H_I + \chi^I_\alpha J_I \star F - \Psi_\alpha \star T + J^I \star \Psi_{\alpha\wedge} d_A h_I - \sigma^i{}_\alpha{}^\beta \Psi_\beta \star d_A\Phi_i$$
$$- 2\star\sigma^{ij}{}_\alpha{}^\beta \eta_\beta[\Phi_i,\Phi_j] - \star\sigma^i{}_\alpha{}^\beta \chi^I_\beta[\Phi_i,h_I] + \frac{1}{2}\star\varepsilon_{IJK}\chi^I_\alpha[h^J,h^K]\Big) \qquad (30)$$

The action (29) is $\delta^\alpha$ and $s^\alpha$ invariant. Indeed, one can check that it verifies:

$$\boxed{S = -\frac{1}{2}\int_M \mathrm{Tr}\ F_\wedge F + s^\alpha_{(c)} s_{(c)\,\alpha}\mathscr{F} = -\frac{1}{2}\int_M \mathrm{Tr}\ F_\wedge F + s^\alpha_{(c)}\delta_{(c)\,\alpha}\mathscr{G}} \qquad (31)$$

with

$$\mathscr{F} = \int_M \mathrm{Tr}\ \Big(\star h_I H^I + h_I J^I \star F + \frac{1}{3}\varepsilon_{IJK}h^I h^J h^K - \frac{1}{2}\Psi^\alpha \star \Psi_\alpha + \frac{1}{2}\star\eta^\alpha\eta_\alpha\Big) \qquad (32)$$

$$\mathscr{G} = \int_M \mathrm{Tr}\ \Big( -\frac{1}{2}g(\kappa)_\wedge((A-\mathring{A})_\wedge(F+\mathring{F}) - \tfrac{1}{3}(A-\mathring{A})^3)$$
$$- \frac{1}{2}\star\varepsilon_{IJK}h^I \mathscr{L}_{J^J{}_\kappa}h^K + \star s^\alpha_{(c)}\delta_{(c)\,\alpha}\left(\tfrac{1}{2}h_I h^I - \tfrac{2}{3}\Phi^i\Phi_i\right)\Big)$$

These facts remind us that we are in the context of a $\mathcal{N}_T = 2$ theory. The critical points of the Morse function $\mathscr{F}$ in the field space are given by the equations

see balanced TQFT by Moore and Dijkgraaf.

$$J^I \star F + \frac{1}{2}\star\varepsilon^I{}_{JK}[h^J,h^K] = 0$$
$$d_A \star h_I J^I = 0$$

One has also the interesting expression : (see Sorella)

$$\boxed{S = \int \epsilon_{\mu\nu\rho\sigma}\ s_\mu s_\nu s_\rho s_\sigma\ \mathrm{Tr}\ \bar{\Phi}^2}$$

It is remarkable that $s$, $\bar{s}$ and $s_\mu$ invariances determine the action (29).
These 6 generators control the whole $N=4$ susy theory.



Expanding the action $S$, and integrating out $T$ and $H^I$, reproduces the $\mathcal{N}=4$ action in its twisted form

$$S \approx \int_M \mathrm{Tr}\ \Big(-\frac{1}{2}F\star F + \frac{1}{4}d_A h_I \star d_A h^I + 2d_A\Phi^i \star d_A\Phi_i - 2\chi^\alpha_I J^I \star d_A\Psi_\alpha + 2\Psi^\alpha \star d_A\eta_\alpha$$
$$+ 2\star\eta^\alpha[h_I,\chi^I_\alpha] + J_I \star \Psi^\alpha[h^I,\Psi_\alpha] + \star\varepsilon_{IJK}\chi^{\alpha I}[h^J,\chi^K_\alpha]$$
$$- 2\star\sigma^{i\alpha\beta}\chi_{\alpha I}[\Phi_i,\chi^I_\beta] - 2\star\sigma^{i\alpha\beta}\eta_\alpha[\Phi_i,\eta_\beta] - 2\sigma^{i\alpha\beta}\Psi_\alpha\star[\Phi_i,\Psi_\beta]$$
$$- \frac{1}{8}\star[h_I,h_J][h^I,h^J] - 2\star[\Phi^i,h_I][\Phi_i,h^I] - 4\star[\Phi^i,\Phi^j][\Phi_i,\Phi_j]\Big)$$

In fact, it is not necessary to ask $SL(2,\mathbb{R})$-invariance from the beginning. Rather, looking for a $\delta$, $s$ and $\bar{s}$ invariant action, with ghost number zero, determines a unique action, Eq. (29). This action has the additional $SL(2,\mathbb{R})$ and $\bar{\delta}$ invariances. Thus the $\mathcal{N}=4$ supersymmetric action is determined by the invariance under the action of only 6 generators $s, \bar{s}, \delta$, with a much smaller internal symmetry than the $SL(2,\mathbb{R})$ R-symmetry, namely the ghost number symmetry.

## Conclusion

- The $N=4$ $d=4$ SYM theory is much "simpler" and geometrical in terms of the 10 Fermi symmet

$$(s, \bar{s}, s_\mu, \bar{s}_\mu)$$

- These 10 operators can be constructed from the geometry of the YM Field, with an internal $SL(2,R)$ symmetry. Coupling to $k_{\mu\nu}$, $L$.
- Only the knowledge of the 6 operators $(s, \bar{s}, s_\mu)$ is needed to build the theory.
- The existence of the operators $s_{\mu\nu}$, $\bar{s}_{\mu\nu}$ appears as an accidental symmetry of the inv. action.
- By untwisting, on a HyperKähler space, the $N=4$ susy theory is revealed, with all its additional symmetries (like $SU(4)$).
- The maximal off-shell closed sector is made of:

$$(s, \bar{s}, s_\mu, s_{\mu\nu}) \qquad (9 \text{ generators})$$

(The unclosure concentrates in the sector with 7 generat

- All of this is quite clear by descending from $d=8$.
- A direct application is a forthcoming "simple" proof of finiteness of $N=4$, $d=4$.

15