longer than the diffusion time

Long Time Behavior of Magnetic Field in 2D (Dynamo Problem?)

1. What can be said about magnetic field asymptotics independent of dynamics?
2. How are 2D and 3D different?
3. Without dynamics – what happens to magnetic flux?

Kinematics:

$$\underline{b}_t + \underline{u}\cdot\nabla\underline{b} = \underline{b}\cdot\nabla\underline{u} + R_m^{-1}\nabla^2\underline{b}\ , \quad \underbrace{\nabla\cdot\underline{u}=0}_{\text{convenience}}$$

$\underline{u}$ is given:

"strong" kinematics – $\underline{u}$ is anything (almost)
upper bounds possible
(anti-dynamo thms)

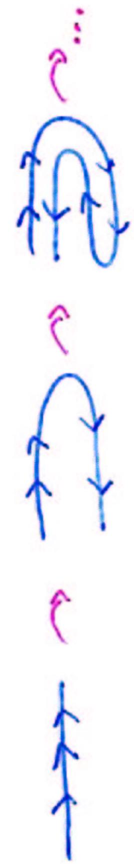
"weak" kinematics – $\underline{u}$ is specified
lower bounds?
(kinematic dynamos)

remarks: 1. The kinematic problem is linear!
(soln is available)

2. Longtime existence of MHD solns $\underline{u}, \underline{b}$ is unknown

Flux Production

2D:



a) small-scale flux

unless b) presence of a non-zero mean field provides a constant source

3D: (SFS)

large scale (length scale of helical motions) flux can be generated (without a mean field)

remark: flux growth rate bounded by line-stretching rate ($R_m \to \infty$)

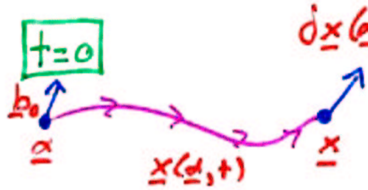
Solution of the Advection Equation

$$\frac{D}{Dt}\underline{b} = \underline{b}\cdot\nabla\underline{u} + \eta\nabla^2\underline{b}$$

$\eta = 0$: $\underline{b}(\underline{x}(\underline{\alpha},t),t) = \underline{\underline{J}}(\underline{x}(\underline{\alpha},t))\,\underline{b}(\underline{\alpha},0)$

where $J_{ij} = \dfrac{\partial x_i}{\partial \alpha_j}$

$d\underline{x}(\underline{\alpha},t) = \underline{u}(\underline{x},t)dt$, $\underline{x}(\underline{\alpha},0) = \underline{\alpha}$



$\underline{b}(\underline{x},t) = \underline{\underline{J}}\,\underline{b}_0$

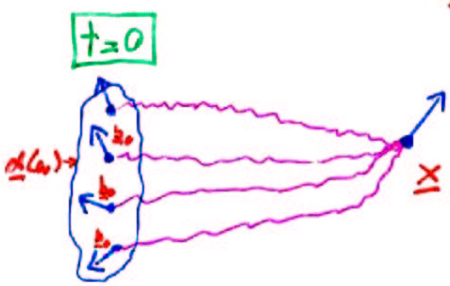
(Cauchy, or Frozen Field soln)

$\eta > 0$: $\underline{b}(\underline{x}(\underline{\alpha},t),t) = \langle \underline{\underline{J}}(\underline{x}(\underline{\alpha},t))\underline{b}(\underline{\alpha},0)\rangle$

where $J_{ij} = \dfrac{\partial x_i}{\partial \alpha_j}$

$d\underline{x} = \underline{u}\,dt + \sqrt{2\eta}\,d\underline{w}$

$\underline{w}$ is Brownian motion



$\underline{b}(\underline{x},t) = \int \underline{\underline{J}}\,\underline{b}_0(\underline{\alpha})\,P(\underline{\alpha})\,d\underline{\alpha}$

proof: operator splitting

remark: similar solns for passive scalars $\frac{D}{Dt}a = \eta\nabla^2 a$

Results:

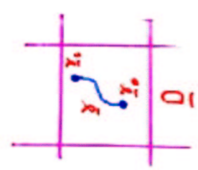
$\underline{D} = [0,1]^2$, periodic boundaries

$\underline{u}(\underline{x},t)$ Lipschitz, periodic, bounded on $D\times[0,\infty)$, $\bar{\underline{u}} = \underline{0}$

$\underline{b}(\underline{x},0)$ smooth, periodic

$\underline{\gamma}$ = flux section: smooth curve with endpoints $\underline{\gamma}_0, \underline{\gamma}_1$ s.t. $\underline{\gamma}_1 - \underline{\gamma}_0 \in D$

$\Phi_\gamma(t) = \int_\gamma \underline{b}\cdot\underline{n}\,ds$

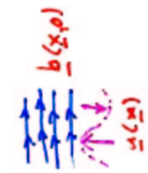


I. $\eta = 0$

(a) $\bar{\underline{b}} = \underline{0}$: $\Phi_\gamma(t)$ bounded $\forall t$, $\max_\gamma|\Phi_\gamma|(t)$ independent of t

(b) $\bar{\underline{b}} \neq \underline{0}$: $\Phi_\gamma(t)$ at most grows linearly

remark: linear growth is achievable



II. $\eta > 0$

(a) $\bar{\underline{b}} = \underline{0}$: (Zel'dovich) $|\Phi_\gamma(t)| \leq Ce^{-\eta t}$

$\max_\gamma|\Phi_\gamma|(t)$ decreases monotonically

(b) $\bar{\underline{b}} \neq \underline{0}$: $|\Phi_\gamma(t)|$ is bounded independently of $\underline{\gamma}$

(but not η, bound $\sim\eta^{-1}$ for fixed $\underline{u}$)

Flux: Magnetic Potential (1D)

Introduce a scalar potential $a(\underline{x},t)$ with

$$b_x = \partial_y a$$
$$b_y = -\partial_x a$$

Then

a) $\frac{D}{Dt} a = \eta \nabla^2 a$ $\qquad \eta = 0: \; a(\underline{x},t) = a(\underline{\alpha},0)$

$\qquad \eta > 0: \; a(\underline{x},t) = \langle a(\underline{\alpha},0) \rangle$

b) $\int_\gamma \underline{b} \cdot \underline{n} \, ds = a(\underline{\gamma}_1) - a(\underline{\gamma}_0) =$ flux through $\underline{\gamma}$

Problem: a is multivalued if $\bar{\underline{b}} \neq \underline{0}$

soln: $a(\underline{x},t) = \tilde{a}(\underline{x},t) + \underline{A} \cdot \underline{x}$

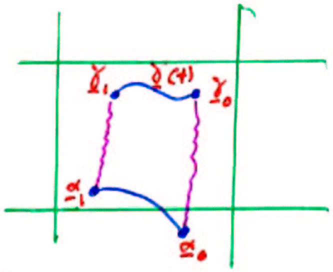
periodic $\qquad \underline{A} = (-\bar{b}_y, \bar{b}_x)$

remark: $\bar{\underline{b}} = \underline{0}$ then a is periodic + bounded

$\underline{b} \neq \underline{0}$ then a grows linearly

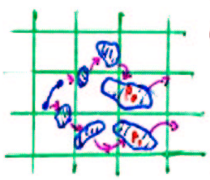
Some Details of Proof

$\boxed{\eta > 0, \; \bar{\underline{b}} = \underline{0}}$ (Zel'dovich)



$$\Phi_\gamma(t) = \int_\gamma \underline{b} \cdot \underline{n} \, ds = a(\underline{\gamma}_1,t) - a(\underline{\gamma}_0,t)$$
$$= \langle a(\underline{\alpha}_1,0) - a(\underline{\alpha}_0,0) \rangle$$
$$= \langle a(\underline{\alpha}_1,0) \rangle - \langle a(\underline{\alpha}_0,0) \rangle$$
$$= \int_D a(\underline{\alpha},0) P_1(\underline{\alpha}) \, d\underline{\alpha} - \int_D a(\underline{\alpha},0) P_0(\underline{\alpha}) \, d\underline{\alpha}$$

where $P_j(\alpha)$, $j=0,1$, is the prob. that a backwards noisy traj. $d\underline{x} = \underline{u}\,dt + \sqrt{2\eta}\, d\underline{w}$, $\underline{x}(t) = \underline{\gamma}_j$, arrives at $\underline{\alpha}$ at $t=0$. Note $P_0, P_1 \to 1$ exponentially at rate η (or faster) so $\Phi_\gamma \to 0$ exponentially at rate η (or faster);



$$P_{j,t} + \underline{u} \cdot \nabla P_j = -\eta \nabla^2 P_j, \quad P_j(\underline{x},t) = \delta(\underline{\gamma}_j - \underline{x})$$

so $\qquad \frac{1}{2}\frac{d}{dt} \int_D P_j^2 \, d\underline{x} = \eta \int_D (\nabla P_j)^2 \, d\underline{x}$

Set $\qquad P_j = 1 + \hat{P}_j$ and note

$$\int_D (\nabla \hat{P}_j)^2 d\underline{x} \geq \int_D \hat{P}_j^2 \, dx$$

i.e. $\qquad \frac{1}{2}\frac{d}{dt} \|\hat{P}_j\|_2^2 \geq \eta \|\hat{P}_j\|_2^2$

$\boxed{\eta > 0,\ \underline{\bar{b}} \neq \underline{0}}$

Remark: $a(\underline{\alpha}, 0) = \underbrace{\tilde{a}(\underline{\alpha}, 0)}_{\text{periodic}} + \underline{A} \cdot \underline{\alpha}$

By linearity, these initial conditions can be separated into two problems:

i) $a(\underline{\alpha}, 0) = \tilde{a}(\underline{\alpha}, 0)$ decays exponentially

ii) $a(\underline{\alpha}, 0) = \underline{A} \cdot \underline{\alpha}$ uniform initial field

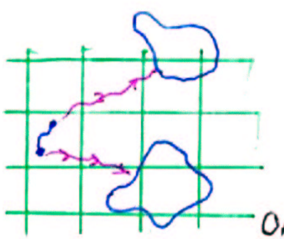
So we can assume $\tilde{a} = 0$, i.e., only source of long term flux is the mean field.

Then

$$\begin{aligned}
\Phi_\gamma(t) = \int_\gamma \underline{b} \cdot \underline{n}\, ds &= a(\gamma_1, t) - a(\gamma_0, t) \\
&= \langle a(\underline{\alpha}_1, 0) \rangle - \langle a(\underline{\alpha}_0, 0) \rangle \\
&= \underline{A} \cdot \int_D \underline{\alpha}\, (P_1(\underline{\alpha}) - P_0(\underline{\alpha}))\, d\underline{\alpha} \\
&= \underline{A} \cdot \int_{\mathbb{R}^2} \underline{\alpha}\, (\hat{P}_1(\underline{\alpha}) - \hat{P}_0(\underline{\alpha}))\, d\underline{\alpha}
\end{aligned}$$

$\hat{P}_i$ = prob. of landing at $\alpha \in \mathbb{R}^2$

$$= \underline{A} \cdot [(\text{Center of Mass } 1) - (\text{Center of Mass } 0)]$$



Once $\underbrace{P_0, P_1 \to 1}_{\text{time} \lesssim \eta^{-1}}$ then $\underbrace{(C.M._1 - C.M._0)}_{\text{length} \sim \eta^{-1} \max|\Delta u| = R_m} \to$ constant

$\Rightarrow |\Phi|$ is bounded by $C|A|R_m$

remark: can bound indep. of R_m if fluid is mixing

3D: A Special Case

Allow the forms

$$\underline{u}(\underline{x}, t) = (u(x,y,t), v(x,y,t), w(x,y,t))$$
$$\underline{b}(\underline{x}, 0) = (b_x(x,y), b_y(x,y), 0)\, e^{i\beta z}$$

and consider a flux section $\underline{\gamma} \times [-\frac{\Delta z}{2}, +\frac{\Delta z}{2}]$

Then $\boxed{\eta = 0}$

$$\begin{aligned}
\Phi(t) &= \int_{\text{section}(t)} \underline{b}(\underline{x}, t) \cdot \underline{n}\, d\underline{x} \\
&= \int_{\text{section}|_{t=0}} \underline{b}(\underline{x}, 0) \cdot \underline{n}\, d\underline{x} \\
&= \oint_{\partial\text{section}|_{t=0}} \underline{a}(\underline{x}, 0) \cdot d\underline{s} \qquad \underline{a} = (0,0,a)\, e^{i\beta z} \\
&= \frac{2\sin(\beta \frac{\Delta z}{2})}{\beta} \left(e^{i\beta z_1} a(\underline{\alpha}_1, 0) - e^{i\beta z_0} a(\underline{\alpha}_0, 0)\right) \qquad \text{old "2D" contr.} \\
&\quad + 2\sin(\beta \tfrac{\Delta z}{2}) \int a(\underline{x}, 0)\, e^{i\beta z}\, \underline{\gamma}_0 \cdot \hat{\underline{z}}\, ds \qquad \text{new "3D" contr. (SFS?)}
\end{aligned}$$