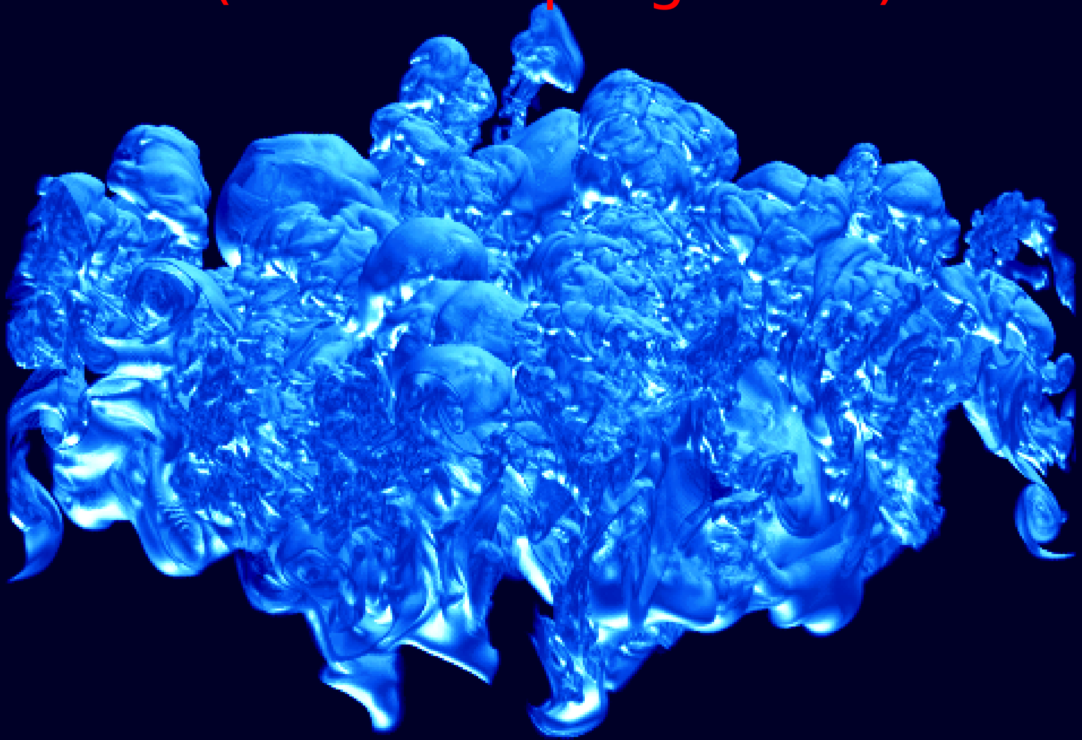
New Algorithms for Simulating Low Mach Number Astrophysical Flows

(a work in progress...)



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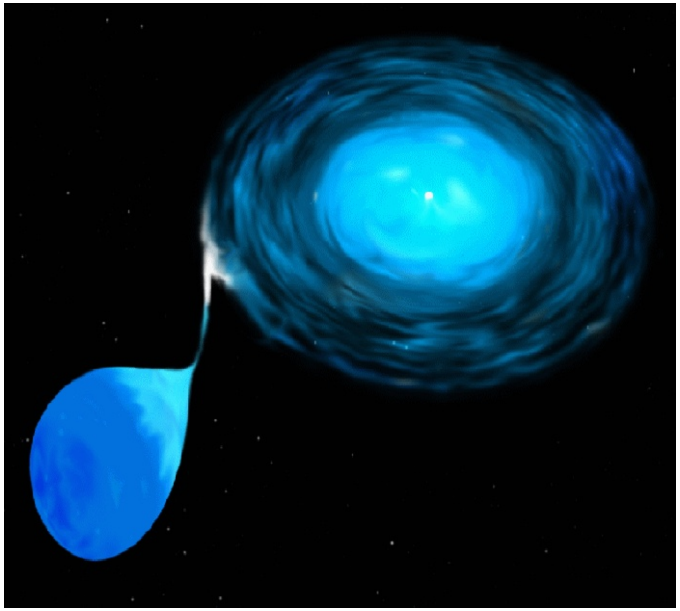
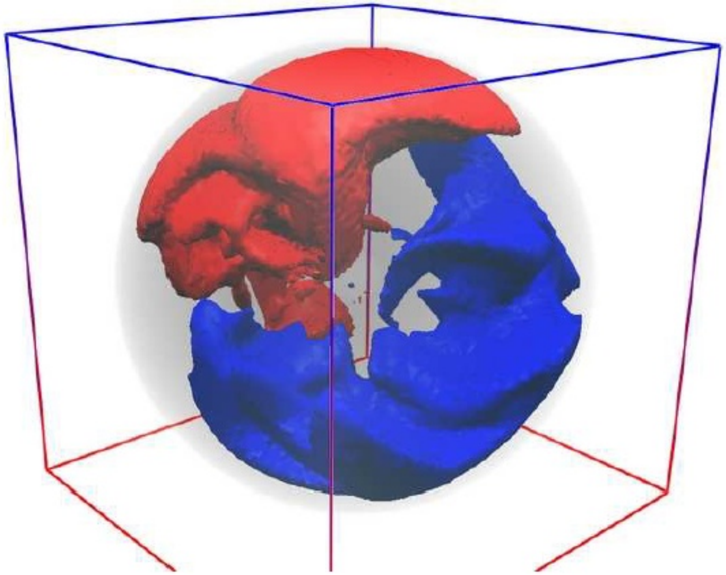
Type Ia Supernovae Theory



The best model for SNe Ia is the thermonuclear explosion of a Chandrasekhar mass carbon/ oxygen white dwarf. (see Hillebrandt and Niemeyer 2000 for a nice review)



(Kuhlen et al. 2006)

As material piles on the star, the central temperature increases, and highly screened ^{12}C + ^{12}C reactions begin driving convection. (see Woosley, Wunsch, & Kuhlen 2003)

Eventually, the cooling cannot keep up with the energy release from reactions, and a burning front is born.

Equations of Hydrodynamics

Traditionally, the fully compressible reactive Euler equations are used to model flames and explosions in SNe Ia

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{U}) = 0$$

$$\frac{\partial (\rho \vec{U})}{\partial t} + \vec{\nabla} \cdot (\rho \vec{U} \vec{U}) + \vec{\nabla} p = \rho \vec{g}$$

$$\frac{\partial (\rho E)}{\partial t} + \vec{\nabla} \cdot (\,(\rho E + p) \vec{U}) = \rho \vec{U} \cdot \vec{g} - \sum_k \rho q_k \dot{\omega}_k$$

$$\frac{\partial (\rho X_k)}{\partial t} + \vec{\nabla} \cdot (\rho \vec{U} X_k) = \rho \dot{\omega}_k$$

with

$$E = e + \frac{1}{2} |\vec{U}|^2$$

$$p = p(\rho, e, X_k)$$

Simulating Low Mach Phenomena

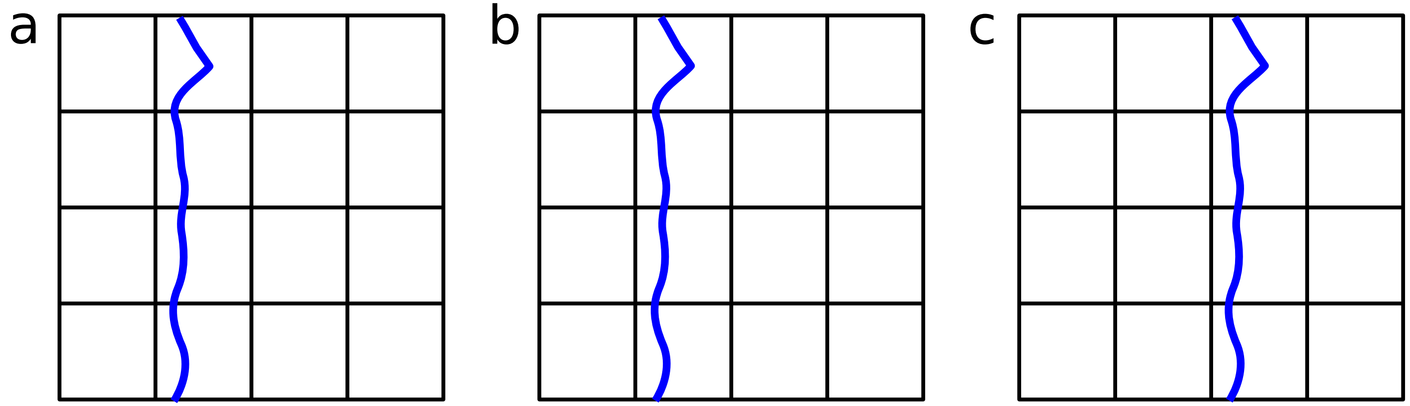
In an explicit algorithm, information cannot propagate more than one zone per step

$$\Delta t = \min\left\{\frac{\Delta x}{|u| + c}\right\}$$

For $M \ll 1$ this is

$$\Delta t = \min\left\{\frac{\Delta x}{c}\frac{1}{1+|M|}\right\} \approx \min\left\{\frac{\Delta x}{c}\left(1 - |M|\right)\right\} \approx \frac{\Delta x}{c}$$

For very low Mach number flows, it takes $\sim 1/M$ timesteps for a fluid element to move more than one zone in the simulation.



◀ A Mach 0.01 front moving to the right (a) initially, (b) after 1 step, (c) after 100 steps.

Can't we do better?

Popular Low Speed Approximations

Incompressible

Formally the $M \to 0$ limit of the Navier-Stokes equations—no compressibility effects are modeled.

Velocity satisfies $\nabla \cdot U = 0 \implies D\rho/Dt = 0$

Boussinesq

Density is allowed to vary only in the buoyancy term (usually through a linear relation with temperature)

Velocity satisfies $\nabla \cdot U = 0$

Constant density background—cannot follow convection over a significant fraction of a scale height.

Anelastic

Small thermodynamic perturbations from a *static* hydrostatic background

Perturbational density is ignored in continuity equation giving

$$\nabla \cdot (\rho_0 U) = 0$$

and as a result, approximations are made to the buoyancy term.

Low Mach Number Combustion

For the small scale problems, we can neglect the stratification of the star and use low Mach number methods developed for terrestrial combustion.

- Pressure is decomposed into dynamic and thermodynamic parts:

$$p(x,t) = p_0(t) + M^2 \pi(x,t)$$

- The total pressure is replaced by p_0 everywhere except in the momentum equation
  - This decouples the pressure and density variations and filters the sound waves from the system.
- The energy equation is replaced by an enthalpy equation:

$$\rho \frac{Dh}{Dt} = \cancelto{0}{\frac{Dp_0}{Dt}} + \nabla \cdot (\kappa \nabla T) - \sum_k \rho q_k \dot{\omega}_k$$

Low Mach Number Combustion

The equation of state is recast as a constraint on the velocity.

In an open domain, the pressure is constant, $p = p_0$, so

$$\frac{Dp_0}{Dt} = 0 = \frac{\partial p}{\partial \rho}\frac{D\rho}{Dt} + \frac{\partial p}{\partial T}\frac{DT}{Dt} + \sum_k \frac{\partial p}{\partial X_k}\frac{DX_k}{Dt}$$

Now, continuity of mass and species tell us that

$$\frac{D\rho}{Dt} = -\rho\vec{\nabla}\cdot\vec{U} \qquad\qquad \frac{DX_k}{Dt} = \dot{\omega}_k$$

So

$$\vec{\nabla}\cdot\vec{U} = \frac{1}{\rho\partial p/\partial\rho}\left\{\frac{\partial p}{\partial T}\frac{DT}{Dt} + \sum_k \frac{\partial p}{\partial X_k}\dot{\omega}_k\right\}$$

The temperature evolution is found from the enthalpy giving:

$$\vec{\nabla}\cdot\vec{U} = \frac{1}{\rho\partial p/\partial\rho}\left\{\frac{1}{\rho c_p}\frac{\partial p}{\partial T}\left[\vec{\nabla}\cdot\kappa\vec{\nabla}T - \sum_k \rho\left(q_k + \frac{\partial h}{\partial X_k}\right)\dot{\omega}_k\right] + \sum_k \frac{\partial p}{\partial X_k}\dot{\omega}_k\right\}$$

Simulation Method

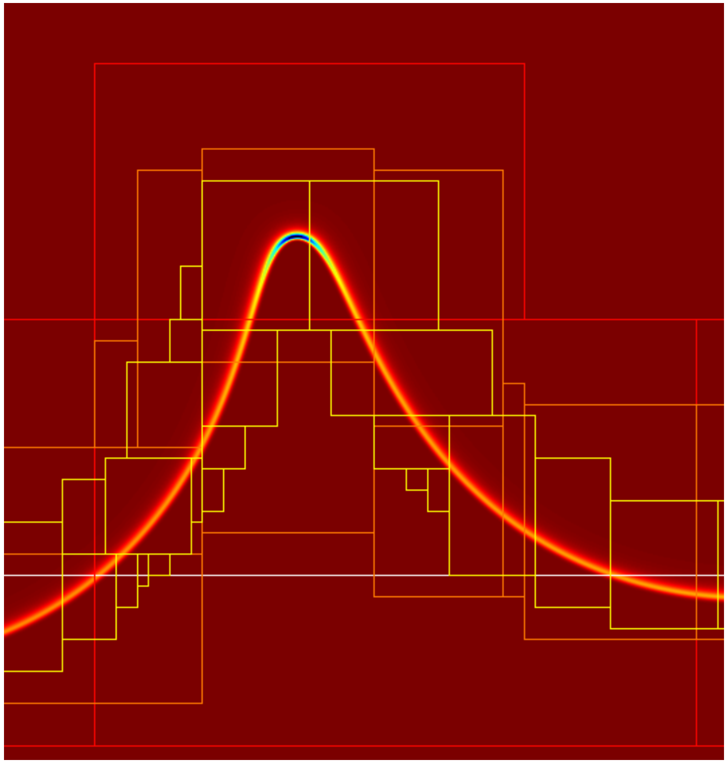
(Bell et al. 2004 JCP 195, 677)

The resulting system of equations makes no assumption about the amplitude of density and temperature variations

It is only required that the pressure remain close to the background pressure

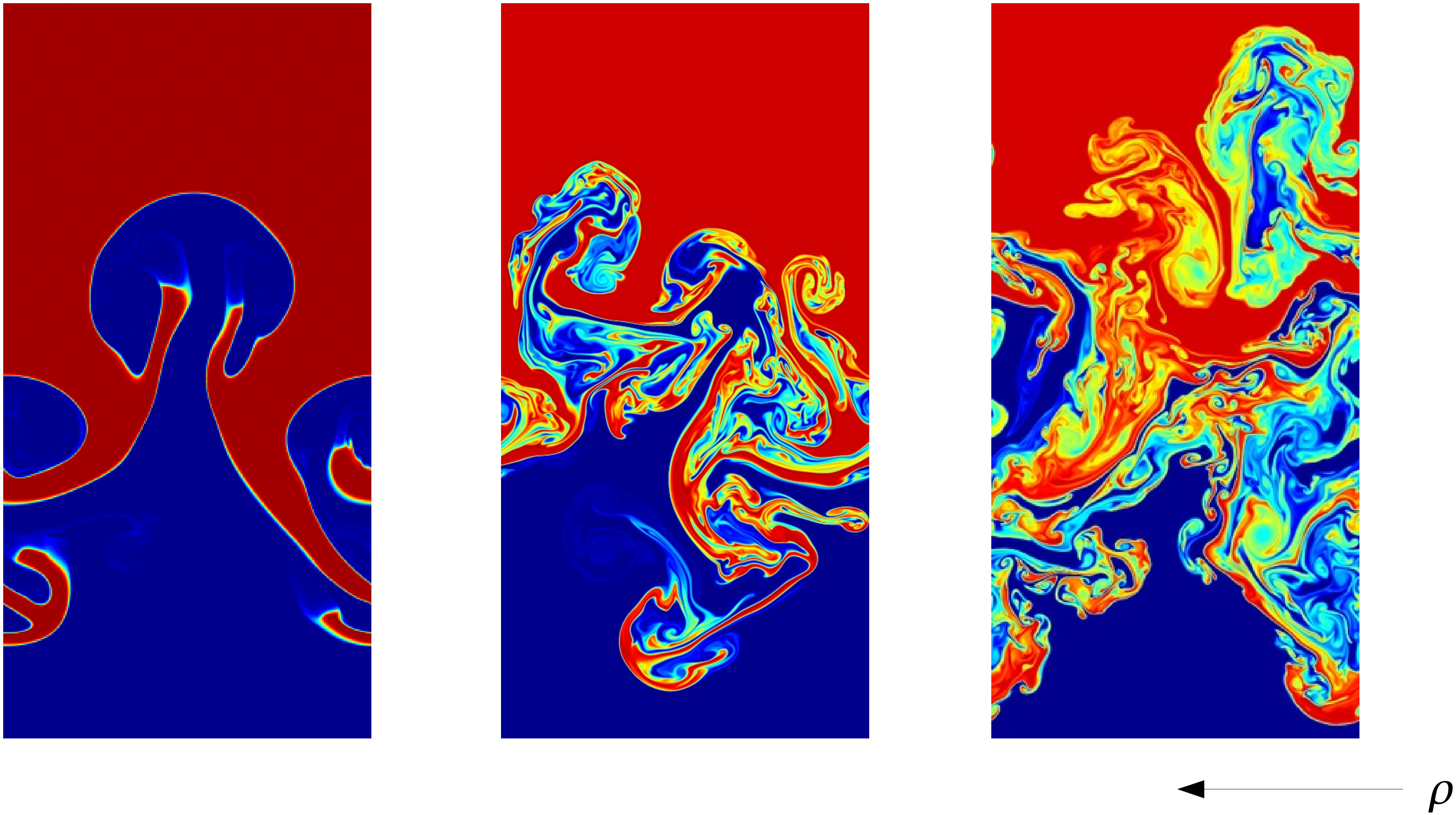
This algorithm has been implemented for astrophysical flows

- Advection/projection/reaction
- Block structured adaptive mesh
- Timestep restricted by $|v|$ not $|v| + c$
- Degenerate/Relativistic EOS used.



Transition to Distributed Burning

(Bell et al. 2004, ApJ, 608, 883)



This algorithm has been applied to small scale studies of Rayleigh-Taylor unstable flames and flame bubbles.

Maestro: Stratified Low Mach Number Code

(Almgren et al. 2006 ApJ, 637, 922; Almgren et al. 2006, ApJ, 649, 927; Almgren et al. 2007 submitted)

Extending the low Mach number method to the full star

- Reformulation of the pseudo-incompressible method by Durran (1989) to general equations of state
- Compressibility effects from both the background stratification and localized heating are incorporated
  - Finite amplitude density/temperature perturbations allowed
- The pressure must remain close to the background pressure
  - Expansion of background state is incorporated

The constraint equation becomes

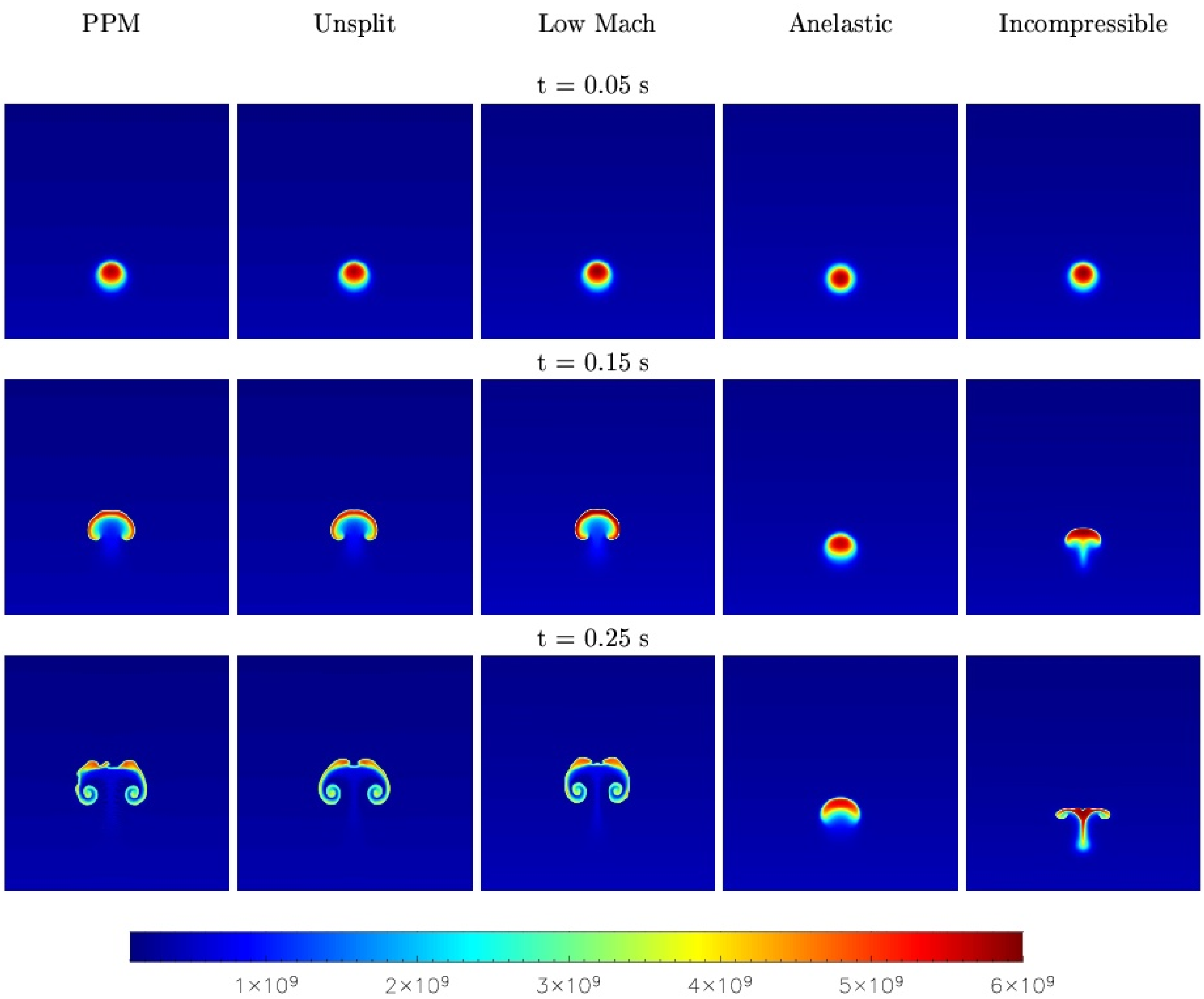
$$\vec{\nabla} \cdot (\beta_0 \vec{U}) = \beta_0 \left(S - \frac{1}{\Gamma_1 p_0} \frac{\partial p_0}{\partial t} \right)$$

with

$$\beta_0(r,t) = \beta(0,t) \exp\left(\int_0^r \frac{1}{(\Gamma_1 p)_0} \frac{\partial p_0}{\partial r'} dr' \right)$$

Comparison to Other Algorithms

(Almgren et al. 2006 ApJ, 637, 922; Almgren et al. 2006, ApJ, 649, 927; Almgren et al. 2007, submitted)



To test the code, we turn to comparisons with other algorithms.

- Comparing to fully compressible codes requires operating at a Mach ~ 0.1 – 0.2.

Performance

(Almgren et al. 2006 ApJ, 637, 922)

Some performance numbers for the rising bubble test (note, this is the algorithm as described in paper I)

6×10^9 K bubble to 0.25 s ($M < 0.2$)

	FLASH/PPM	low Mach
# of steps	2148	246
wallclock time	14200 s	1480 s

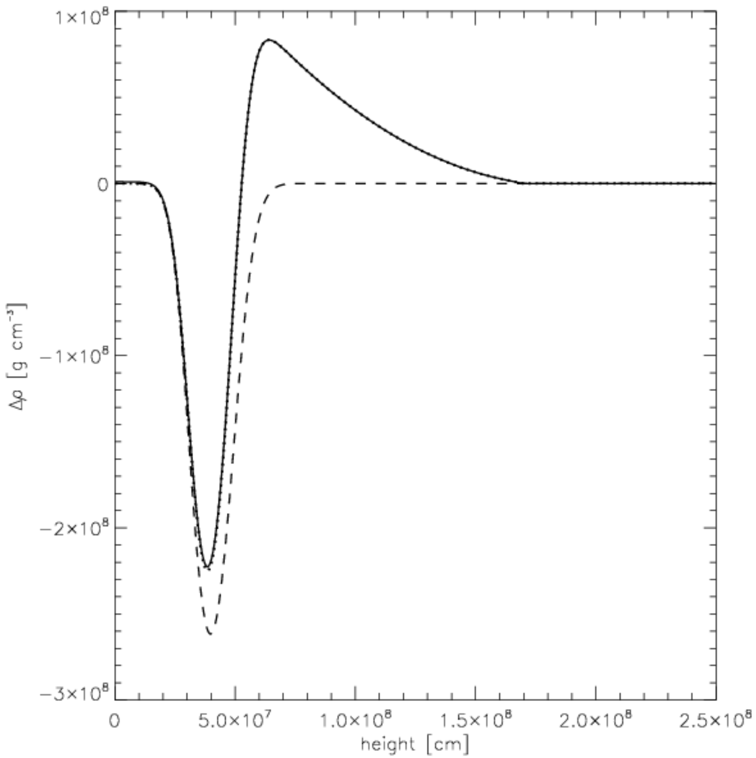
10^9 K bubble to 1.0 s ($M < 0.05$)

	FLASH/PPM	low Mach
# of steps	7482	252
wallclock time	52100 s	1560 s

All timings are single processor using Intel compilers with similar optimization options.

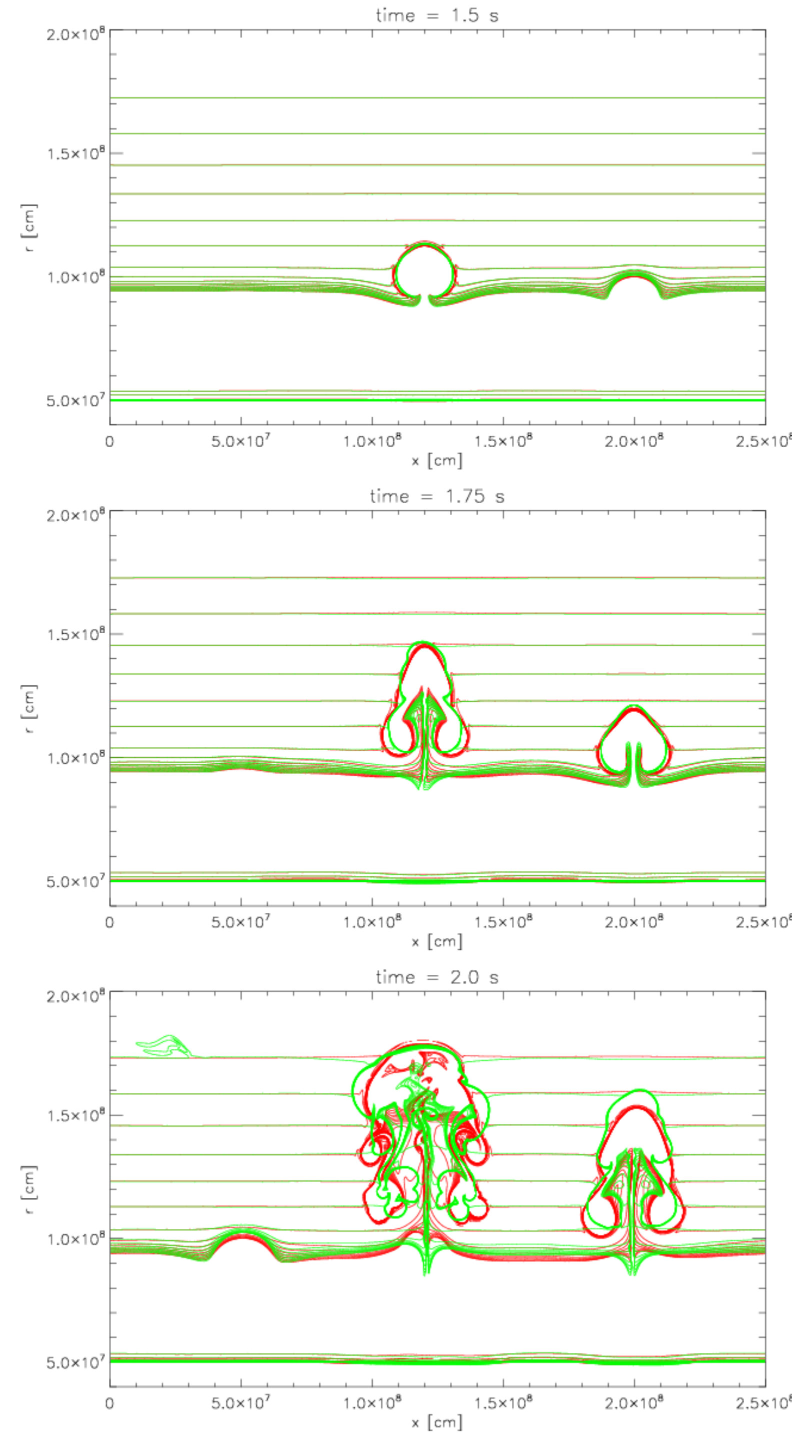
Base State Evolution

(Almgren et al. 2006 ApJ, 637, 922; Almgren et al. 2006, ApJ, 649, 927)



◀ 1-d comparison of compressible (solid), low Mach with base state adjustment (dotted), and low Mach with fixed base state (dashed)

▶ 2-d comparison of fully compressible (red) and low Mach (green) with uniform and localized heating sources

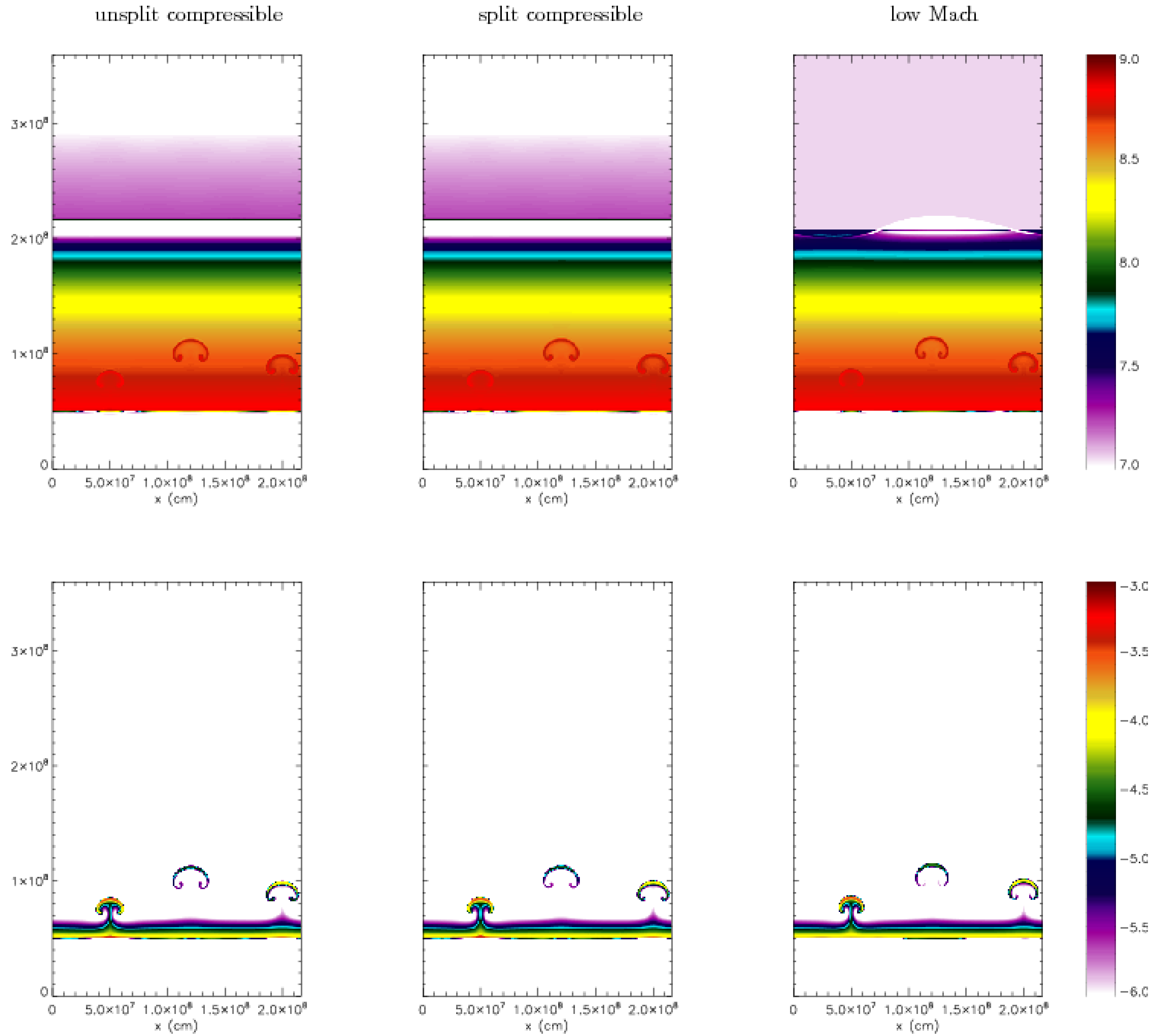


The low Mach number method accurately captures the expansion of a plane-parallel background state in response to heating.

This is important—in the low Mach number approximation, the pressure must always stay close to the background state.

Reactions

(Almgren et al. 2007 submitted)



The latest version of the algorithm evolves the species equations and allows for arbitrary reaction networks.

Again we see strong agreement with compressible codes.

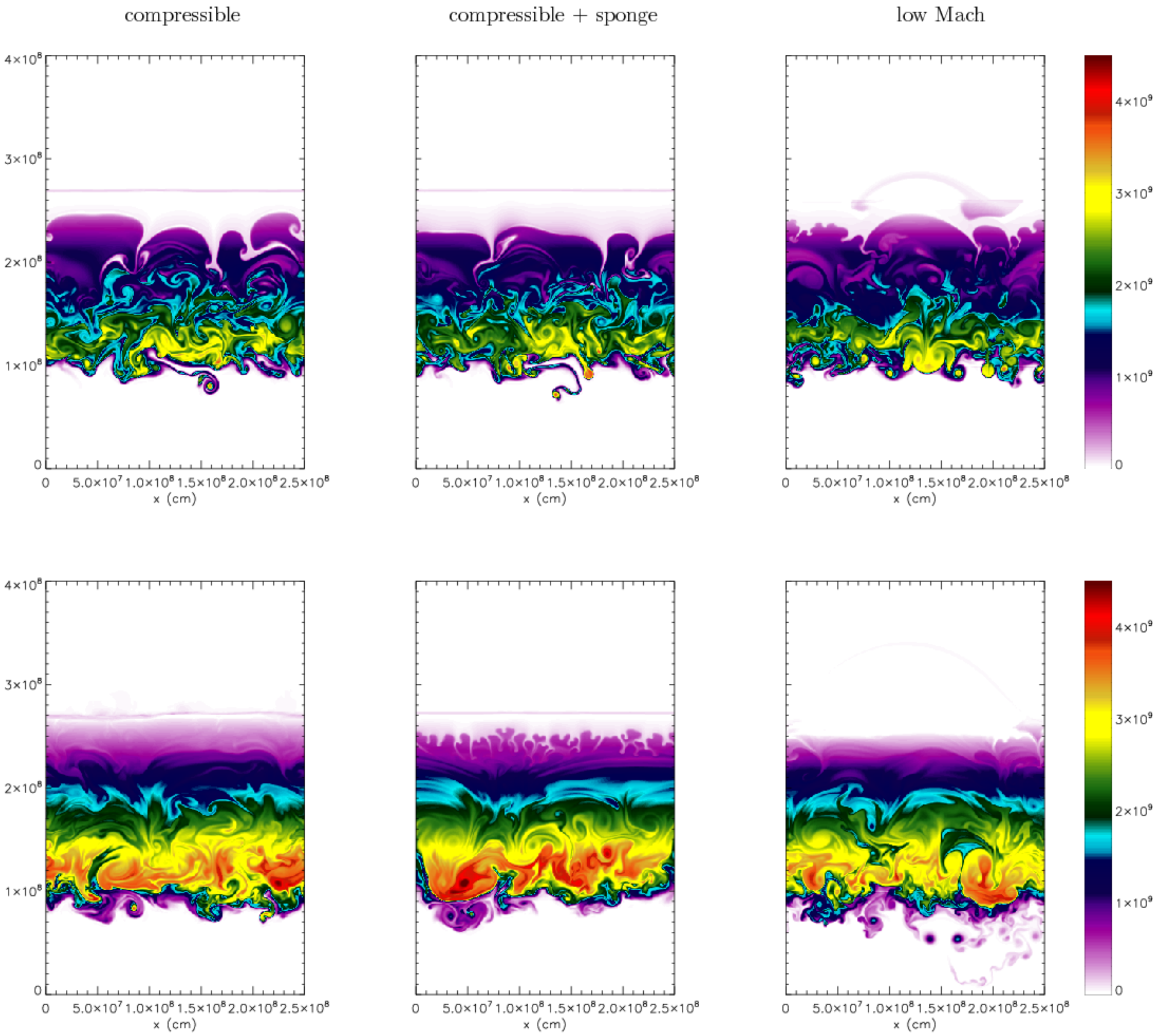
▶ Temperature (top) and ^{24}Mg mass fraction (bottom) at t = 2.5 s.

Long Time Evolution

(Almgren et al. 2007 submitted)

Large scale convection also shows strong agreement across the different algorithms.

In particular, we remain close to the equation of state.



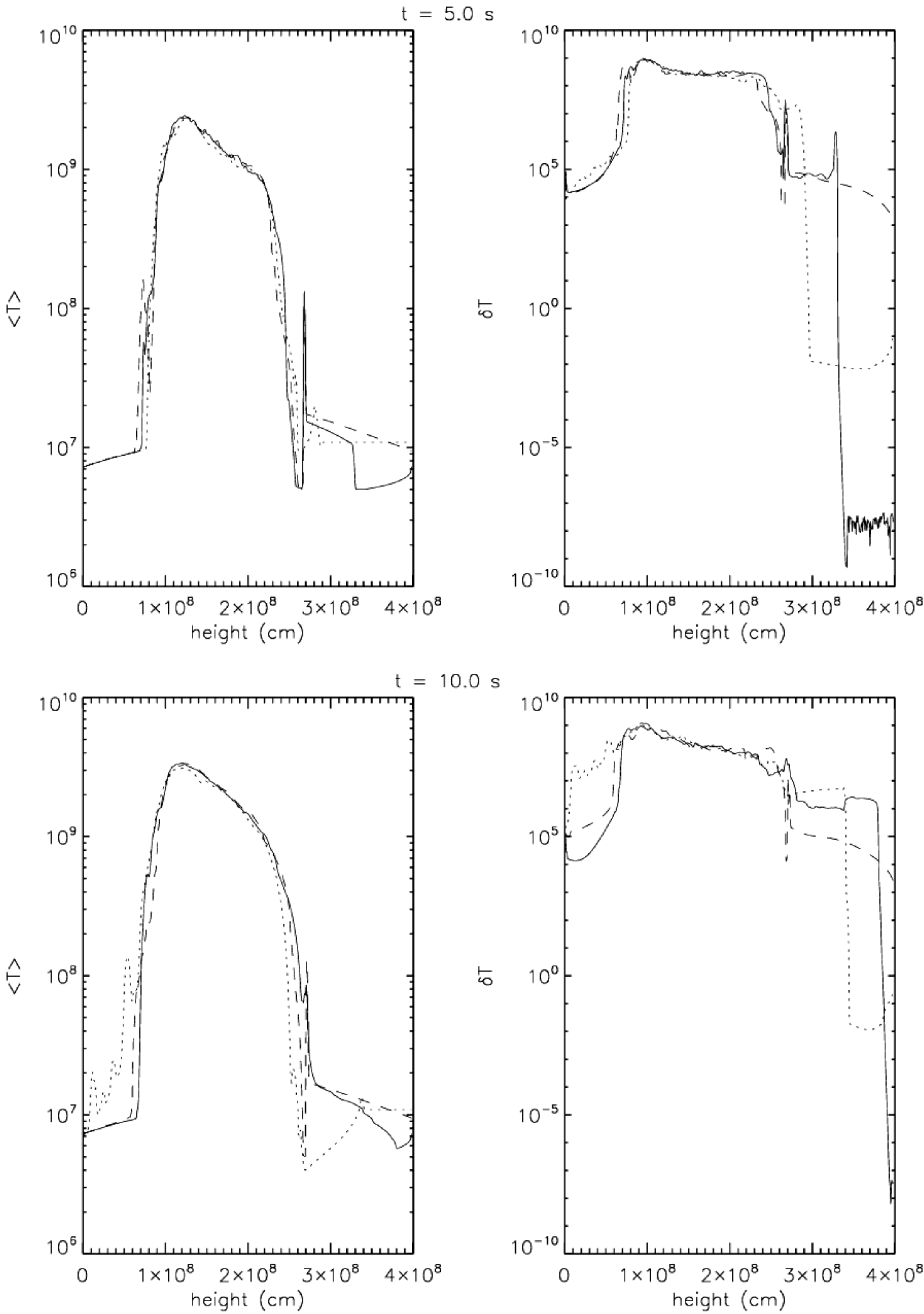
▶ Temperature shown for 3 different hydro algorithms at t = 5 s (top) and t = 10 s (bottom).

Long Time Evolution

(Almgren et al. 2007 submitted)

Diagnostics of these simulations demonstrate that both the average and deviation of the temperature agree well across codes.

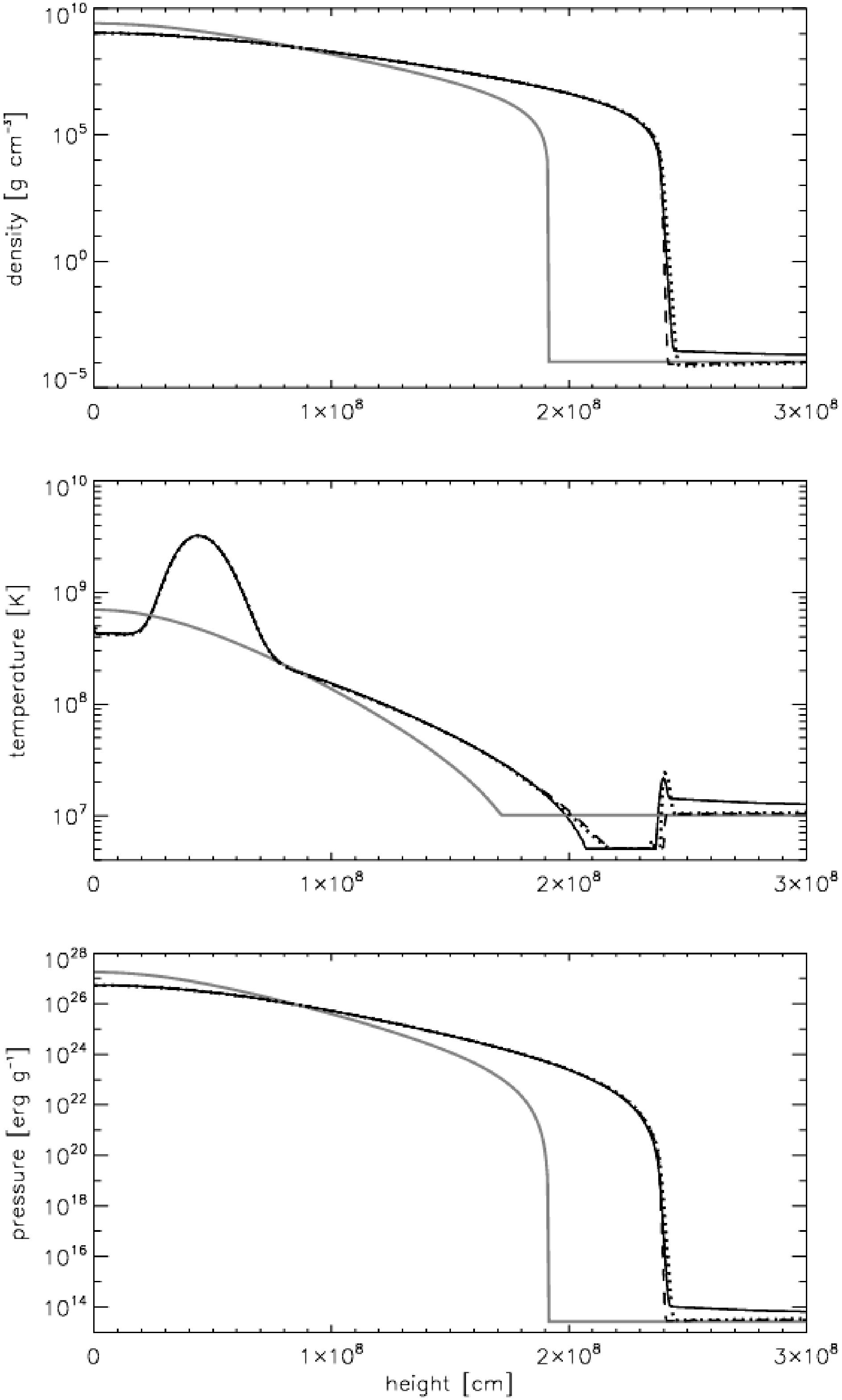
In particular, we seem to remain close to the equation of state.



▶ Fully compressible (solid line); Compressible + sponge (dashed line); low Mach (dotted line)

Onward to Full Stars...

(Almgren et al. 2007 submitted)



Evolving spherical self-gravitating stars leads to a different set of equations describing the base state evolution.

Here we show that we capture the expansion of a self-gravitating star due to heat release.

◀ 1-d comparison of compressible (solid), low Mach with CFL = 0.5 (dotted) and 0.1 (dashed)

Future Work

- Finish the implementation for full stars and begin convection studies.
- Embed the stratified algorithm into the parallel adaptive mesh framework.
- Incorporate long-wavelength acoustics to extend the range of validity of the method up to Mach 1

Conclusions

- The final algorithm should be applicable to SNe Ia ignition, Type I X-ray bursts, and Classical Novae
- No single algorithm is applicable to all of phases of these explosions, and that an accurate description must be built up by using the right tools for each piece.