Multifractality and Conformal Invarance at 2D Metal-Insulator Transition in the Spin-Orbit (Symplectic) Symmetry Class

cond-mat/0609161

cf: Mirlin's talk in this program

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Andreas Ludwig (UCSB)

Plan of the talk

- Introduction
- Theoretical background
- Model
- Numerical results

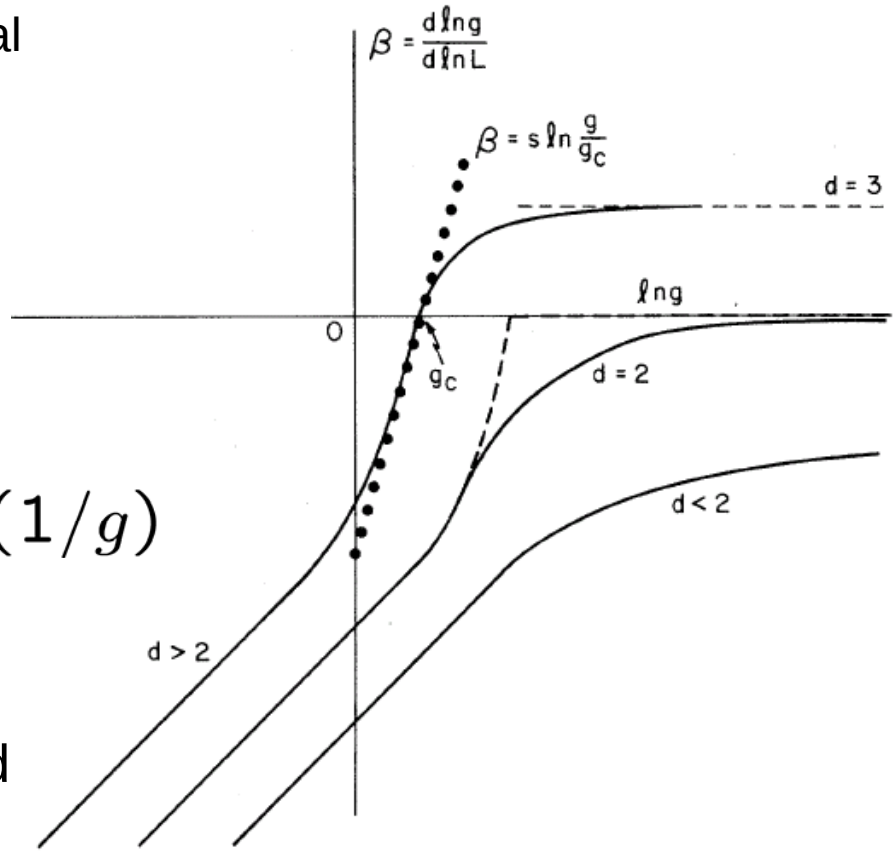
Anderson localization

An electron moving in a random potential

P.W. Anderson (1958)

scaling theory AALR (1979)

$$\beta(g) = \frac{d \ln g}{d \ln L} = d - 2 - \mathcal{O}(1/g)$$



All the wave functions are localized in one and two dimensions!

Hikami-Larkin-Nagaoka

Prog. Theor. Phys. Vol. 63, No. 2, February 1980, Progress Letters

Spin-Orbit Interaction and Magnetoresistance in the Two Dimensional Random System

Shinobu HIKAMI, Anatoly I. LARKIN*) and Yosuke NAGAOKA
Research Institute for Fundamental Physics
Kyoto University, Kyoto 606

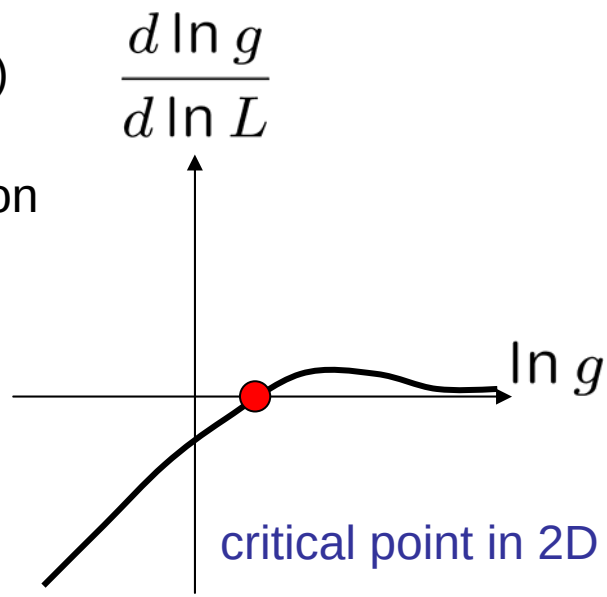
(Received November 5, 1979)

3 symmetry classes (orthogonal, unitary, symplectic)

symplectic class: ○ time-reversal, × spin-rotation
spin-orbit interaction

anti-localization

$$\frac{d\ln g}{d\ln L} = d - 2 + \frac{c}{g} \qquad c > 0$$



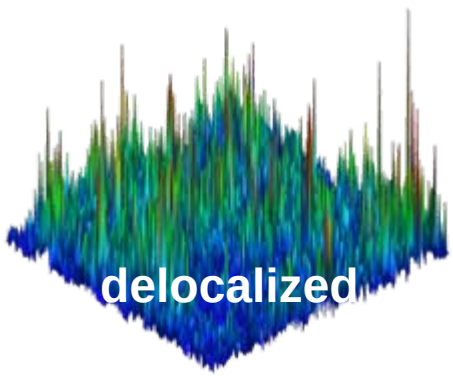
Metal-insulator transition in 2D

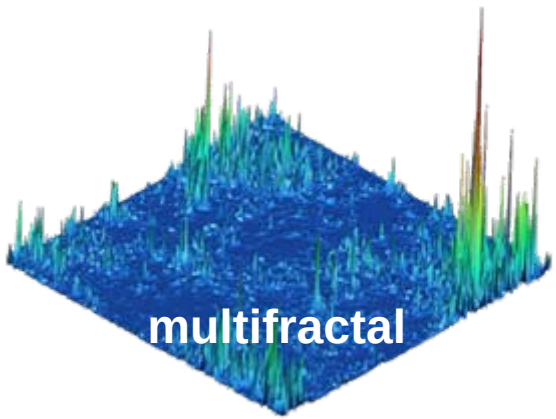
Anderson transition (metal-insulator transition)

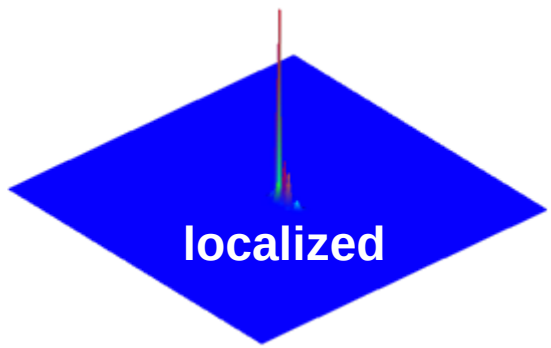
Continuous phase transition induced by disorder

localization length $\xi \to \infty$ $\xi \sim |E - E_c|^{-\nu}$

scale invariance universal critical exponents



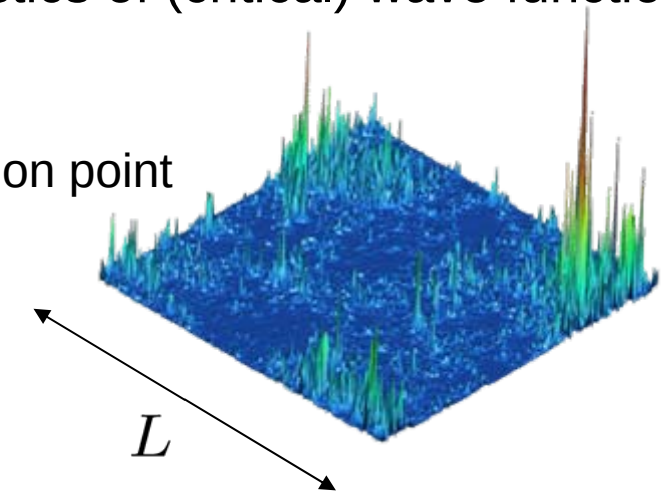




metallic phase critical point insulating phase

Multifractality (bulk case): statistics of (critical) wave functions

Critical wave function at a metal-insulator transition point



multifractal exponents τ_q

$$L^d \overline{|\psi(r)|^{2q}} \sim L^{-\tau_q}$$

$$\tau_q = d(q-1) + \underline{\Delta_q} \qquad D_q = \tau_q/(q-1)$$ fractal dim.

Continuous set of independent and universal critical exponents

Δ_q : anomalous scaling dimensions

$$\Delta_q = 0 \text{ at } q = 0, 1 \qquad \frac{\Delta_q}{q(1-q)} = c(q)$$

singularity spectrum $f(\alpha) = q\alpha - \tau_q \qquad \alpha = \frac{d\tau_q}{dq}$ $\qquad \alpha > 0$

$N_\alpha \sim L^{f(\alpha)}$: measure of *r* where $|\psi(r)|^2 \sim L^{-\alpha}$

Anderson transitions are continuous phase transitions induced by disorder

At ordinary continuous phase transition points without disorder,

correlation length $\xi \to \infty$

scale invariance $\Longrightarrow$ conformal invariance (Polyakov, 1970)

2D conformal field theory (BPZ, 1984)

Disorder-induced critical points in 2D:

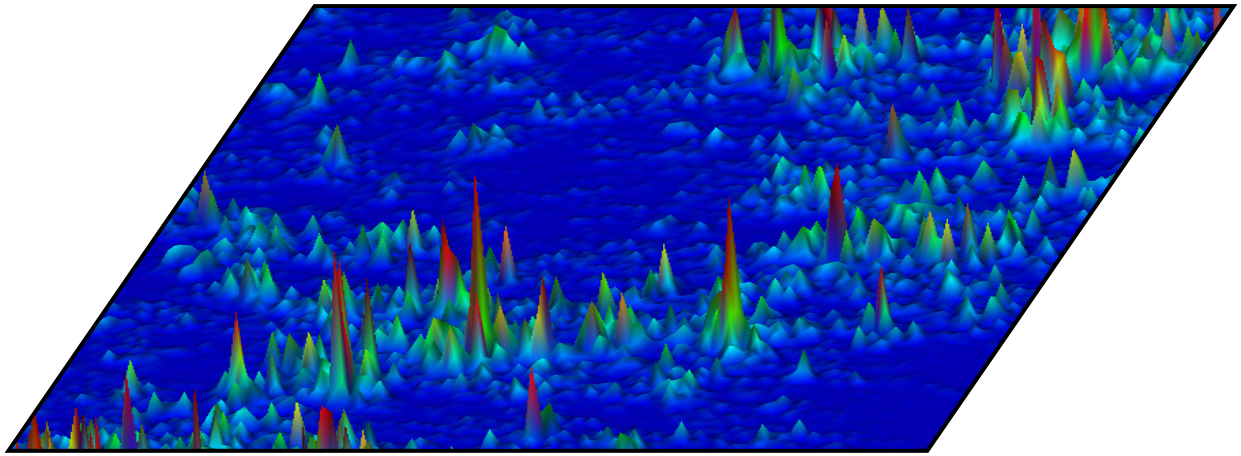
symplectic class, QHE(unitary class, class C, class D)

Conformal invariance at these critical points?

What kind of CFTs describe the disordered critical points in 2D?

To examine the presence of conformal invariance,

Consider disordered samples with open boundaries (surface)



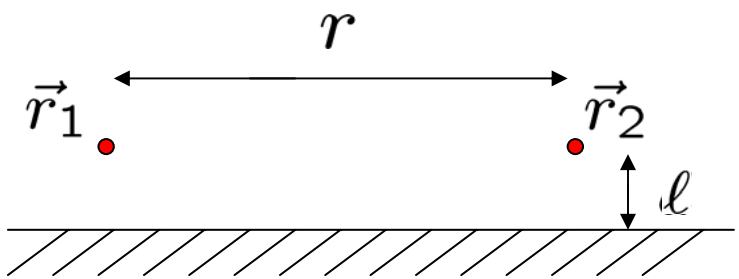
Change the shape of samples

Surface multifractality

Surface critical phenomena

in conventional phase transitions

At conventional critical points:



bulk: $\ell \gg r$

$$\langle O(\vec{r}_1) O(\vec{r}_2) \rangle \sim r^{-2x_b}$$

surface: $r \gg \ell$

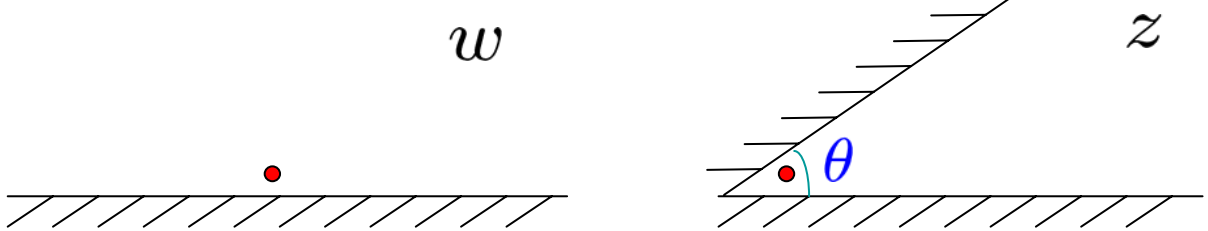
$$\langle O(\vec{r}_1) O(\vec{r}_2) \rangle \sim r^{-2x_s}$$

x: scaling dimension

$$x_s \neq x_b$$

Surface critical exponents are different from bulk exponents

Boundary CFT (Cardy 1984)



conformal mapping

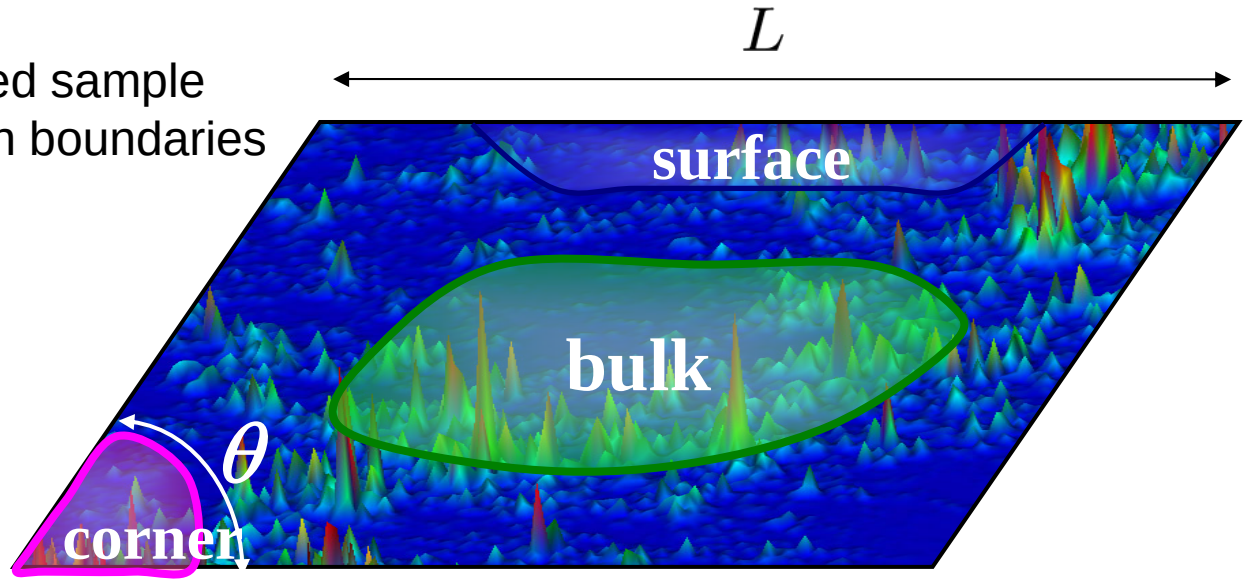
$$w = z^{\pi/\theta}$$

$$x_\theta = \frac{\pi}{\theta} x_s$$

O : quasi-primary op.

Surface multifractality (Subramaniam et al., 2006)

disordered sample
with open boundaries



$$\Delta_q^{\mathsf{s}} \neq \Delta_q^{\mathsf{b}}$$

$$L^d \overline{\langle |\psi(r)|^{2q} \rangle_{\mathsf{bulk}}} \sim L^{-\tau_q^{\mathsf{b}}} \qquad \tau_q^{\mathsf{b}} = d(q-1) + \Delta_q^{\mathsf{b}}$$

$$L^{d-1} \overline{\langle |\psi(r)|^{2q} \rangle_{\mathsf{surface}}} \sim L^{-\tau_q^{\mathsf{s}}} \qquad \tau_q^{\mathsf{s}} = d(q-1) + 1 + \Delta_q^{\mathsf{s}}$$

$$L^{d-2} \overline{\langle |\psi(r)|^{2q} \rangle_{\mathsf{corner}}} \sim L^{-\tau_q^{\mathsf{c}}} \qquad \tau_q^{\mathsf{c}} = d(q-1) + 2 + \Delta_q^{\mathsf{c}}$$

cf. Gaussian theory

$$\Delta_q = cq(1-q), \qquad \Delta_q^{\mathsf{s}} = 2\Delta_q^{\mathsf{b}}$$

Spin quantum Hall transition

class C (Altland & Zirnbauer, 1997)

BdG quasiparticles at E=0 (p-h symmetry), $\times$ time reversal, $\bigcirc$ spin rotation

2D SQH plateau transition $\sim$ classical percolation (Gruzberg, Ludwig, Read, 1999)

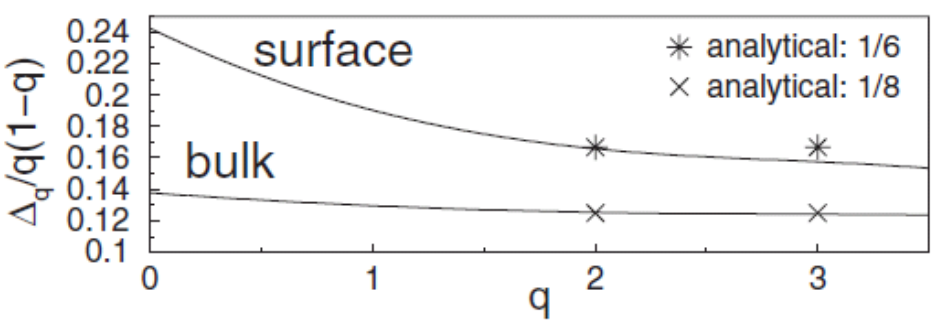
bulk multifractal exponents $\Delta_2^{\mathrm{b}} = -\frac{1}{4}, \quad \Delta_3^{\mathrm{b}} = -\frac{3}{4}$

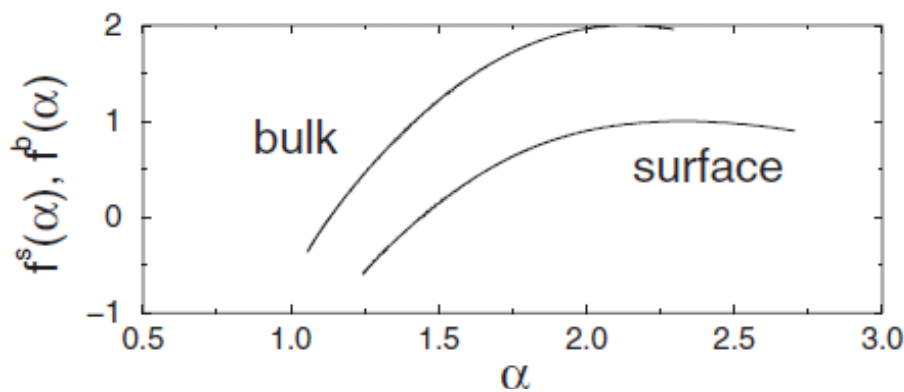
(Mirlin, Evers, Mildenberger, 2003)

surface multifractal exponents

$$\Delta_2^{\mathrm{s}} = -\frac{1}{3}, \quad \Delta_3^{\mathrm{s}} = -1$$

(Subramaniam et al., 2006)





Multifractality vs field theory (Duplantier & Ludwig, 1991)

$$\overline{O_r^q} \longleftrightarrow \langle \mathcal{O}_r^q \rangle \sim L^{-x_q}$$

O_r loral randam events at position r

$\mathcal{O}_r$ scaling operator in a field theory

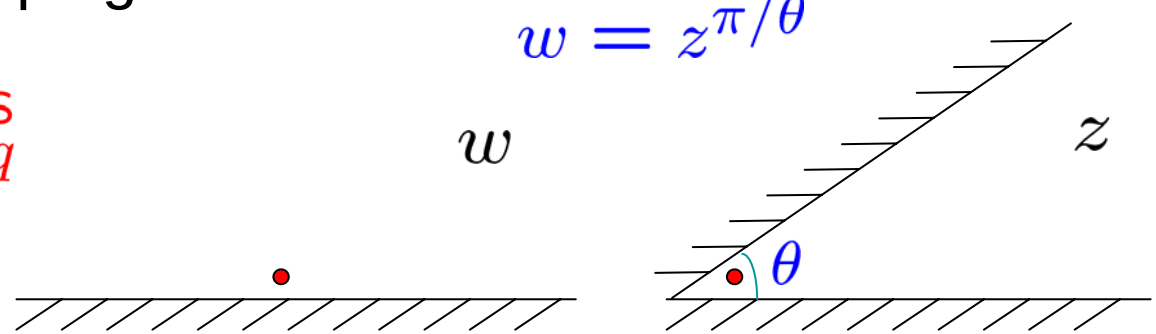
$$\overline{|\psi_r|^{2q}} = \frac{\overline{O_r^q}}{\left(\int_r O_r\right)^q} \sim \frac{\overline{O_r^q}}{\left(\int_r \overline{O_r}\right)^q} \sim L^{-dq-(x_q-qx_1)}$$

$$\Delta_q = x_q$$

Conformal mapping

$$\Delta_q^{\mathsf{c}} = \frac{\pi}{\theta}\Delta_q^{\mathsf{s}}$$

$$w = z^{\pi/\theta}$$



$$\Delta_q^{\mathsf{C}} = \frac{\pi}{\theta}\Delta_q^{\mathsf{S}}$$

$$\begin{aligned}\tau_q^{\mathsf{S}} &= 2(q-1)+1+\Delta_q^{\mathsf{S}}\\ \tau_q^{\mathsf{C}} &= 2(q-1)+2+\Delta_q^{\mathsf{C}}\\ &= 2(q-1)+2+\frac{\pi}{\theta}\left[\tau_q^{\mathsf{S}}-2(q-1)-1\right]\end{aligned}$$

$$\alpha_q^{\mathsf{C}} - 2 = \frac{\pi}{\theta}\left(\alpha_q^{\mathsf{S}} - 2\right)$$

$$f^{\mathsf{C}}(\alpha_q^{\mathsf{C}}) = \frac{\pi}{\theta}\left[f^{\mathsf{S}}(\alpha_q^{\mathsf{S}}) - 1\right]$$

$$\Delta_q^{\mathsf{c}} = \frac{\pi}{\theta}\Delta_q^{\mathsf{s}}$$

$$\begin{aligned}\tau_q^{\mathsf{s}} &= 2(q-1)+1+\Delta_q^{\mathsf{s}}\\ \tau_q^{\mathsf{c}} &= 2(q-1)+2+\Delta_q^{\mathsf{c}}\\ &= 2(q-1)+2+\frac{\pi}{\theta}\left[\tau_q^{\mathsf{s}}-2(q-1)-1\right]\end{aligned}$$

$$\alpha_q^{\mathsf{c}} - 2 = \frac{\pi}{\theta}\left(\alpha_q^{\mathsf{s}} - 2\right)$$

$$f^{\mathsf{c}}(\alpha_q^{\mathsf{c}}) = \frac{\pi}{\theta}\left[f^{\mathsf{s}}(\alpha_q^{\mathsf{s}}) - 1\right]$$

$$\alpha_q^{\mathsf{c}} \geq 0 \qquad \alpha_{q_\theta}^{\mathsf{c}} = 0$$

$\cdot\, q < q_\theta \quad \Delta_q^{\mathsf{c}} = \frac{\pi}{\theta}\Delta_q^{\mathsf{s}}$

$\cdot\, q > q_\theta \quad \alpha_q^{\mathsf{c}} = 0, \quad \tau_q^{\mathsf{c}} = \text{const.},$

$$\Delta_q^{\mathsf{c}} = \Delta_{q_\theta}^{\mathsf{c}} - 2(q - q_\theta)$$

Symmetry relations proposed by Mirlin et al.

(Mirlin et al., PRL 2006)

LDOS distributions in NLσ model + universality

$$q \Leftrightarrow 1-q$$

⇨ Symmetry relations of multifractal spectra

$$\Delta_q^{\mathrm{x}} = \Delta_{1-q}^{\mathrm{x}} \qquad \mathrm{x} = \mathrm{b}, \mathrm{s}, \mathrm{c}$$

$$\alpha_{1-q}^{\mathrm{x}} - d = -\left(\alpha_q^{\mathrm{x}} - d\right)$$

$$0 < \alpha_q^{\mathrm{x}} < 2d$$

$$f^{\mathrm{x}}(\alpha_{1-q}^{\mathrm{x}}) - \frac{\alpha_{1-q}^{\mathrm{x}}}{2} = f^{\mathrm{x}}(\alpha_q^{\mathrm{x}}) - \frac{\alpha_q^{\mathrm{x}}}{2}$$

Mirlin et al. confirmed these for 1d Power-law Random Banded Matrix Model

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$$\tau_q^{\mathrm{x}} = dq - d_{\mathrm{x}} + \Delta_q^{\mathrm{x}}$$

$$\tau_{1-q}^{\mathrm{x}} = d(1-q) - d_{\mathrm{x}} + \Delta_{1-q}^{\mathrm{x}}$$

$$= d(1-2q) + \tau_q^{\mathrm{x}}$$

$$d_{\mathrm{x}} = \begin{cases} d, & \mathrm{x} = \mathrm{b} \\ d-1, & \mathrm{x} = \mathrm{s} \\ d-2, & \mathrm{x} = \mathrm{c} \end{cases}$$

We will

- numerically study multifractal properties of critical wave functions in the 2D symplectic class

- compare bulk, surface, and corner exponents

- check the conformal invariance

$$\Delta_q^{\mathsf{C}} = \frac{\pi}{\theta}\Delta_q^{\mathsf{S}}$$

$$\alpha_q^{\mathsf{C}} - 2 = \frac{\pi}{\theta}\left(\alpha_q^{\mathsf{S}} - 2\right), \quad f^{\mathsf{C}}(\alpha_q^{\mathsf{C}}) = \frac{\pi}{\theta}\left[f^{\mathsf{S}}(\alpha_q^{\mathsf{S}}) - 1\right]$$

- check the symmetry relations conjectured by Mirlin et al.

SU(2) model for the symplectic class

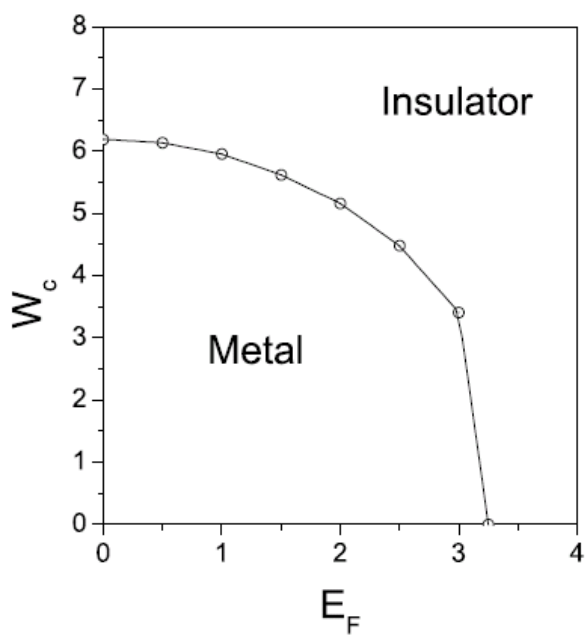
Asada, Slevin, Ohtsuki, 2002

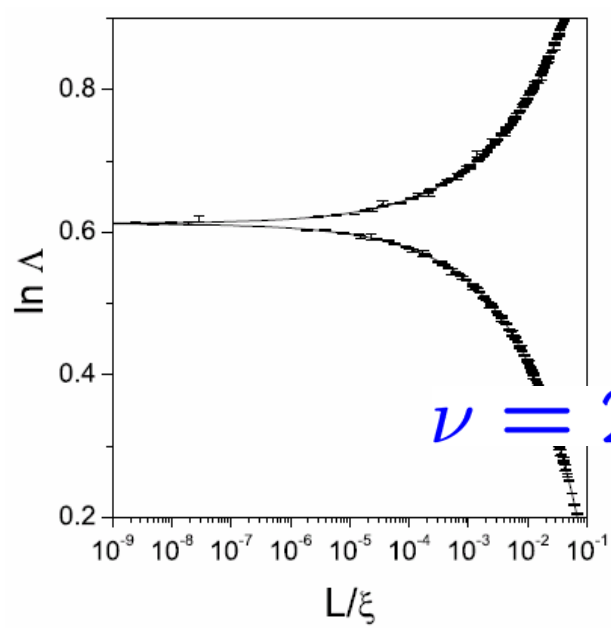
Tight-binding model on a 2D square lattice

$$H = \sum_{i,\sigma} \epsilon_i c^\dagger_{i,\sigma} c_{i,\sigma} - \sum_{\langle i,j \rangle, \sigma, \sigma'} R(i,j)_{\sigma,\sigma'} c^\dagger_{i,\sigma} c_{j,\sigma'}$$

$$R(i,j) = \begin{pmatrix} e^{i\alpha_{i,j}} \cos\beta_{i,j} & e^{i\gamma_{i,j}} \sin\beta_{i,j} \\ -e^{-i\gamma_{i,j}} \sin\beta_{i,j} & e^{-i\alpha_{i,j}} \cos\beta_{i,j} \end{pmatrix} \qquad \epsilon_i \in [-W/2, W/2]$$

$$\alpha, \gamma \in [0, 2\pi) \qquad P(\beta) d\beta = \sin(2\beta) d\beta \quad 0 \le \beta \le \frac{\pi}{2}$$





$$\nu = 2.73 \pm 0.02$$

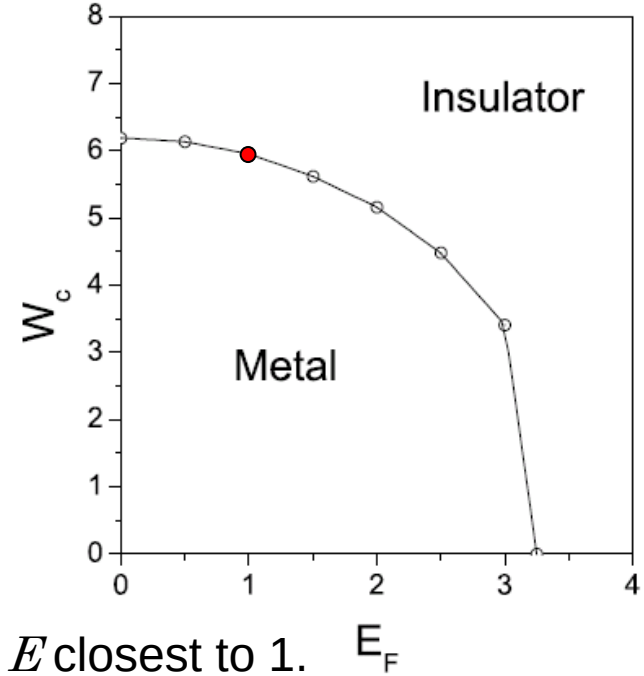
Our numerics

SU(2) model

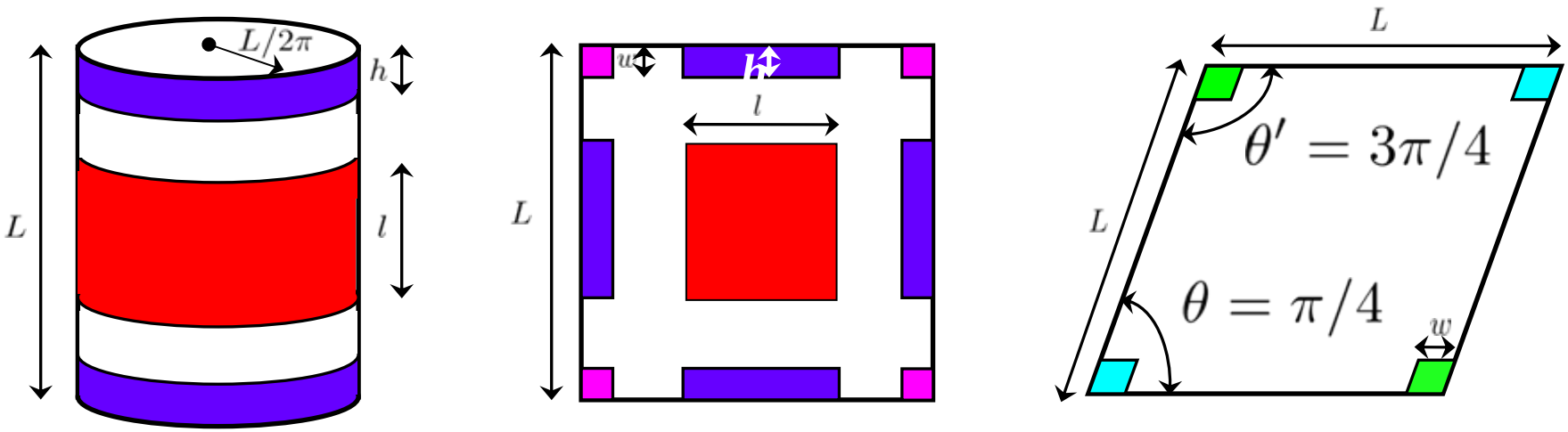
system size $L = 24 \sim 120$

# of samples 6×10^4

$E = 1 \quad W_c = 5.95 \qquad \psi|_{\text{surface}} = 0$



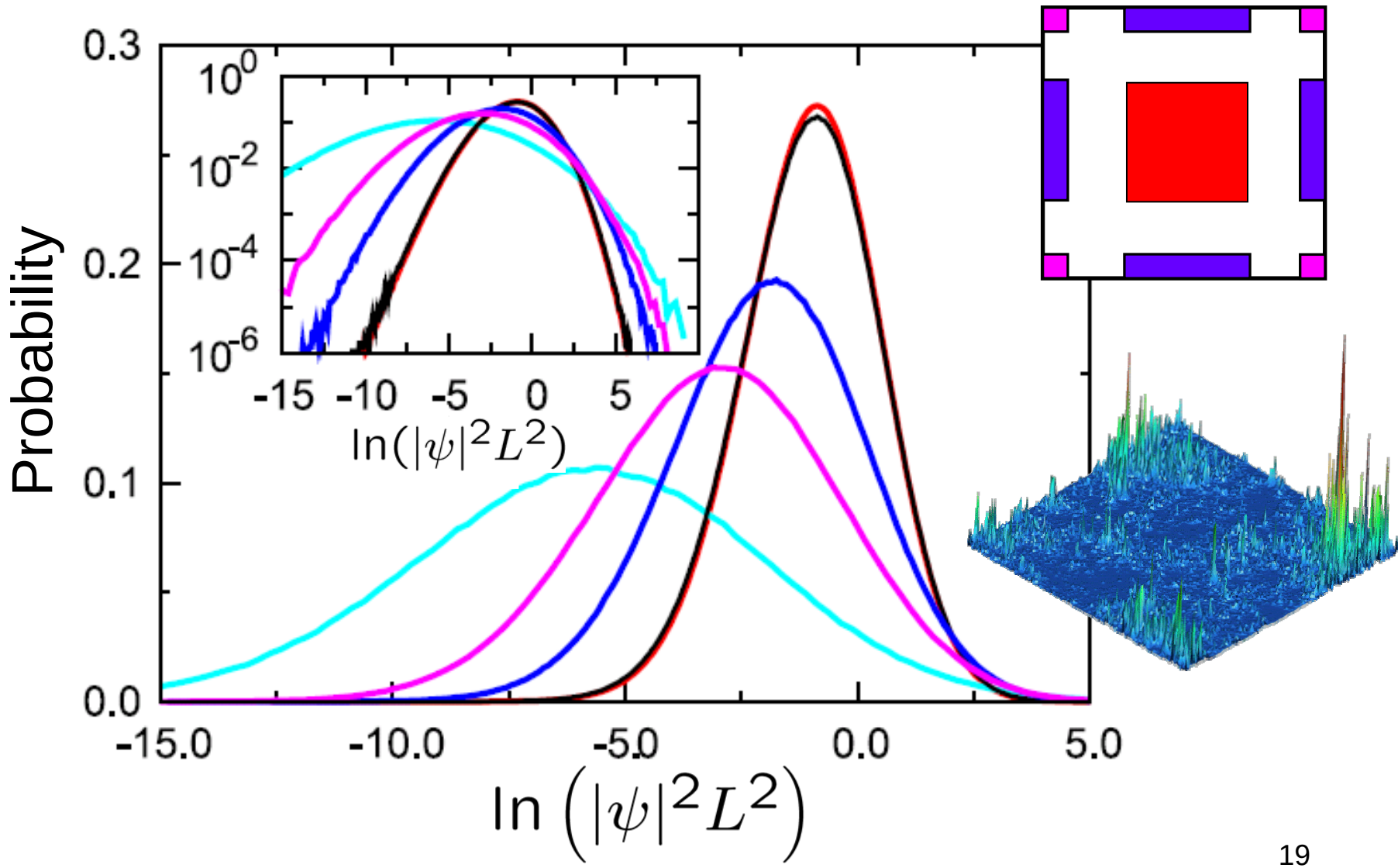
For each sample we keep only one eigenstate with E closest to 1.



$L = 24 \sim 120, \quad l = L/6, \quad w = 4, \quad h = 1$

Numerical Results

Probability distributions of wave function amplitudes



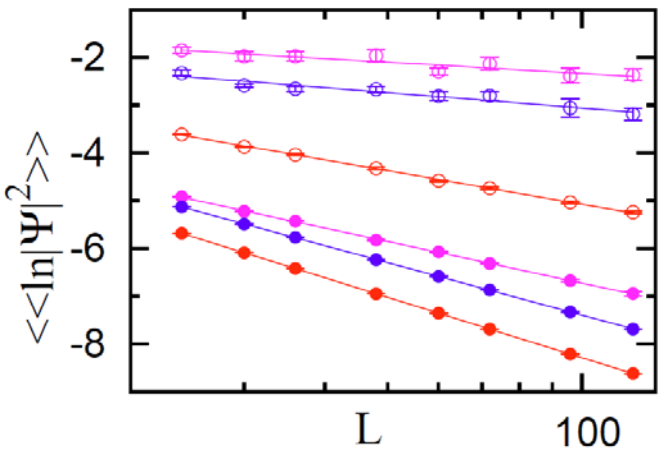
Numerical estimate of multifractal exponents:

$$\overline{|\psi(r)|^{2q}} \sim L^{-2q-\Delta_q^{\mathsf{x}}} \qquad (r \in \mathsf{x})$$

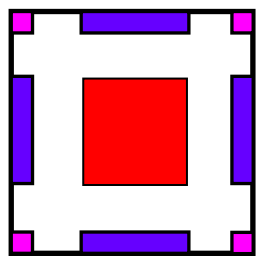
$$\Longrightarrow \quad \overline{|\psi(r)|^{2q}} \left(\overline{|\psi(r)|^2}\right)^{-q} \sim L^{-\Delta_q^{\mathsf{x}}} \qquad (\Delta_{q=1}^{\mathsf{x}} = 0)$$

$$\langle\langle \ln |\psi(r)|^2 \rangle\rangle_q \equiv \frac{\overline{|\psi(r)|^{2q} \ln |\psi(r)|^2}}{\overline{|\psi(r)|^{2q}}} \sim -\alpha_q^{\mathsf{x}} \ln L$$

$$\ln \overline{|\psi(r)|^{2q}} \sim \left[f^{\mathsf{x}}(\alpha_q^{\mathsf{x}}) - q\alpha_q^{\mathsf{x}} - d_{\mathsf{x}}\right] \ln L$$

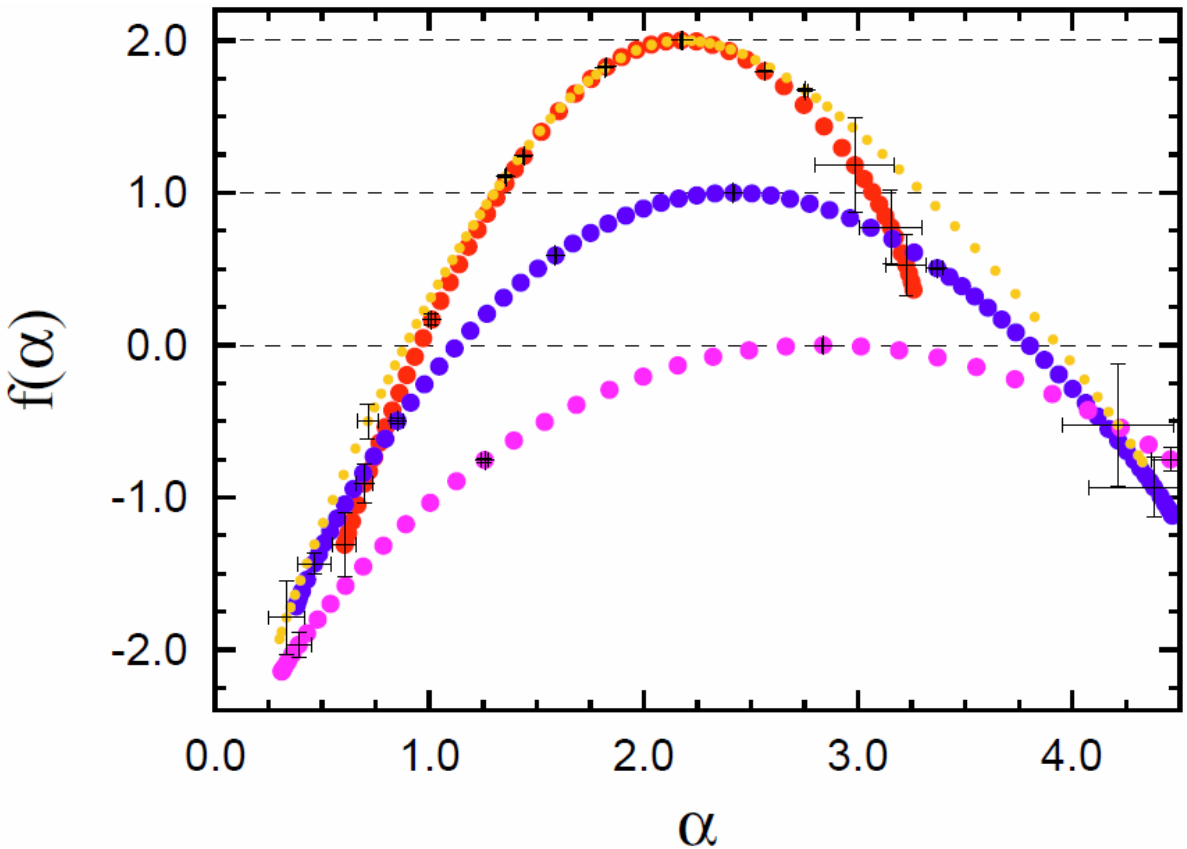


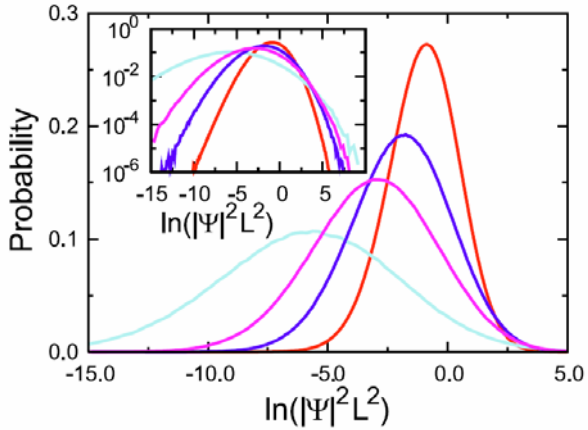
Bulk, Surface, Corner $\pi/2$)



$f(\alpha)$: bulk, surface, and corner multifractal spectra

Bulk, Surface, Corner ($\pi/2$), Whole Cylinder





$$P_x(|\psi|^2) \sim |\psi|^{-2} L^{f^x(\alpha)-d_x},$$
$$|\psi|^2 \sim L^{-\alpha}$$

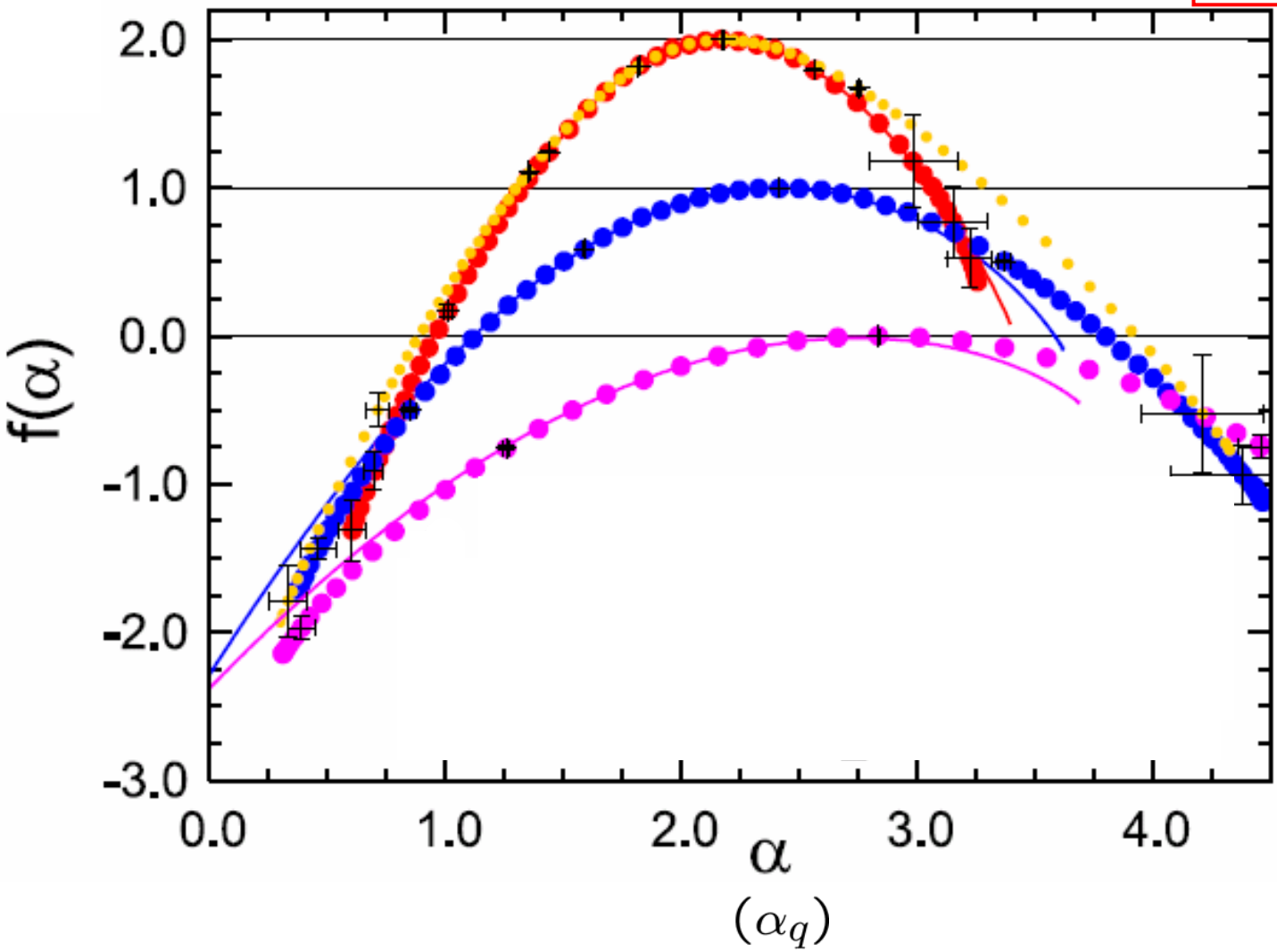
$$\alpha_0^b = 2.173 \pm 0.001$$
$$\alpha_0^s = 2.417 \pm 0.002$$
$$\alpha_0^{\pi/2} = 2.837 \pm 0.003$$

Bulk, surface, and corner $f(\alpha)$ are all different.
Surface contributions dominate at large |q| in the whole cylinder $f(\alpha)$.

Colored thin lines: $f^{x}(\alpha_{1-q}) = f^{x}(4-\alpha_q) + \alpha_q - 2$

The bulk $f(\alpha)$ satisfies the symmetry relation suggested by Mirlin et al.

Bulk, Surface, Corner $\pi/2$), Whole Cylinder

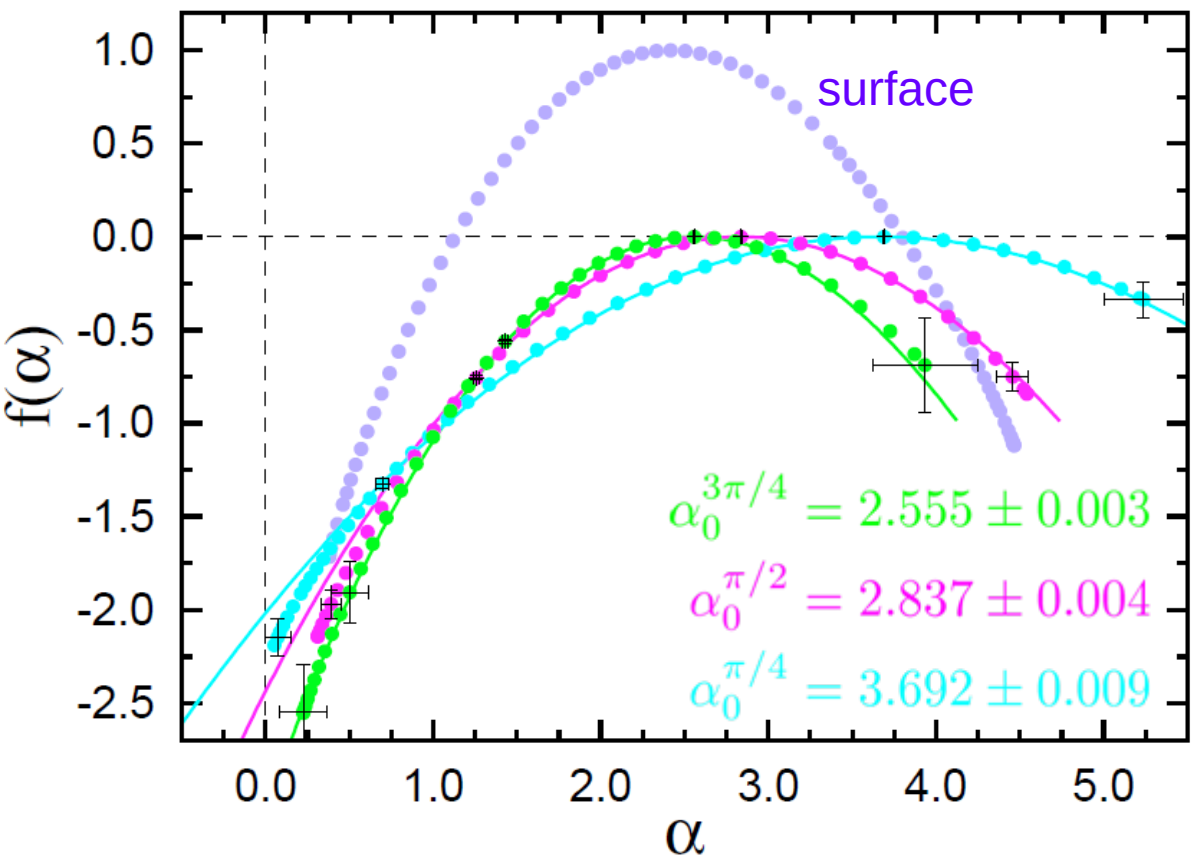


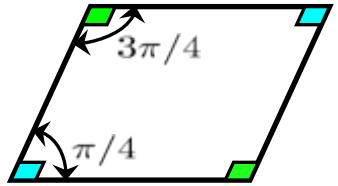
Surface ?
Corner ??

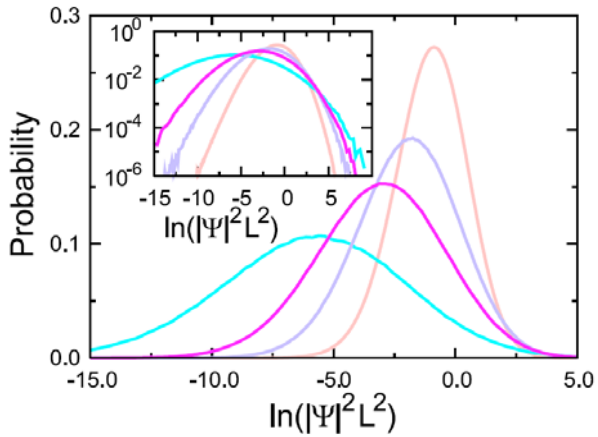
Numerical inaccuracy at large |q|??

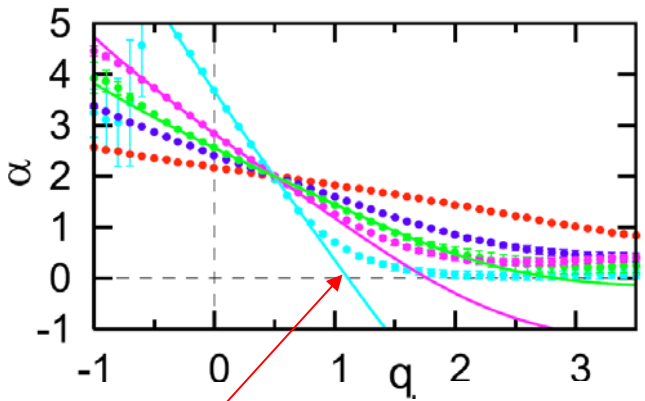
We need calculations with larger L and N_{sample}

$$\theta = 3\pi/4, \pi/2, \pi/4$$









$q_{\pi/4}$

Colored thin lines: Conformal Invariance !!

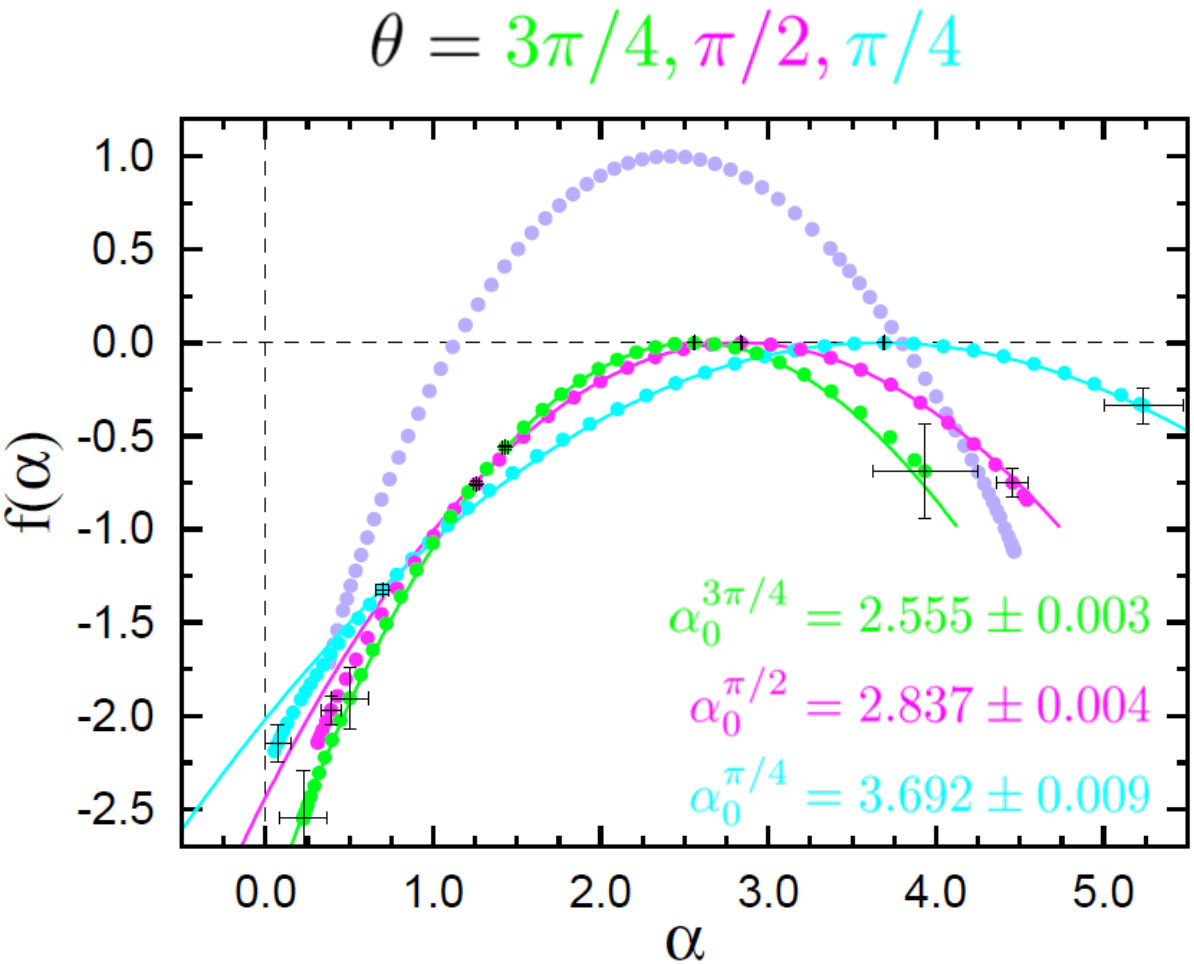
$$\alpha_q^\theta - 2 = \frac{\pi}{\theta}(\alpha_q^S - 2)$$

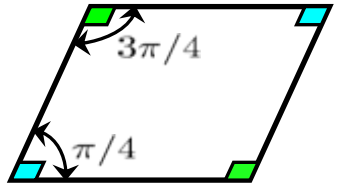
$$f^\theta(\alpha_q^\theta) = \frac{\pi}{\theta}[f^S(\alpha_q^S) - 1]$$

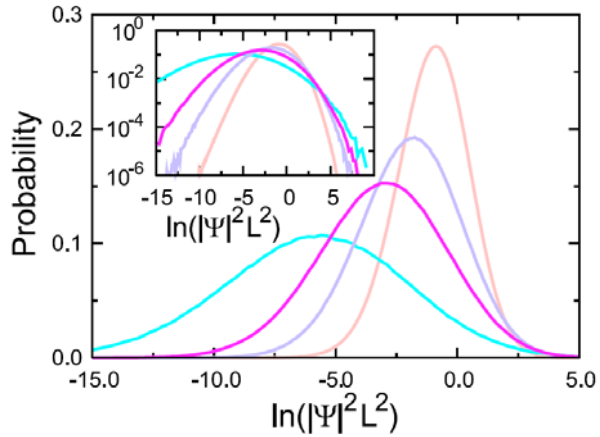
rounding of cusps at $q = q_\theta$

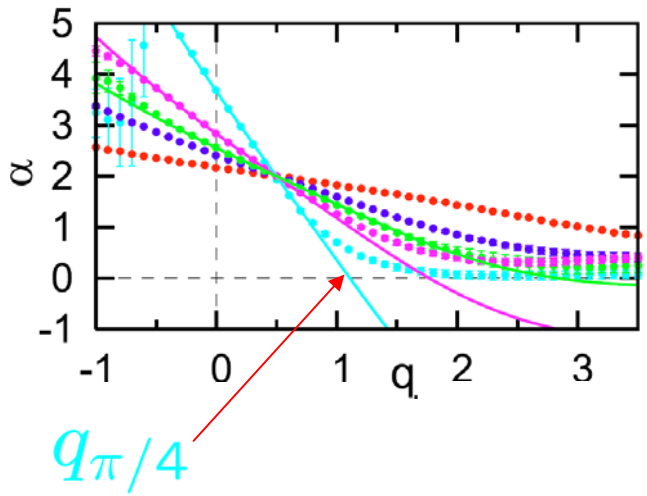
⟺ finite-size effect

$$\alpha_{1-q}^X - 2 = -\left(\alpha_q^X - 2\right)$$







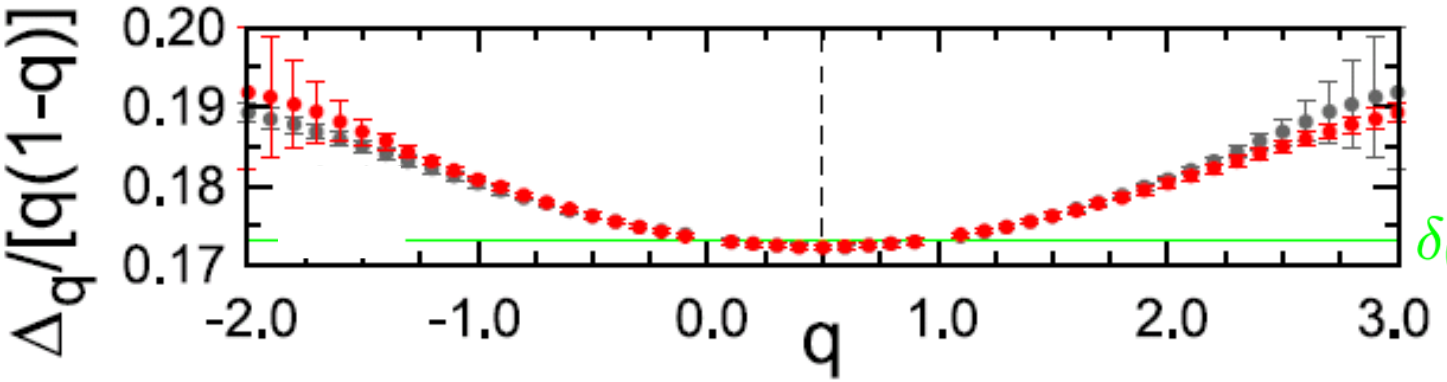


α_q^θ can exceed 4!

The inequality $\alpha_q < 4$ (Mirlin et al.) is violated.

cusps at $q = q_\theta$; non-analytic

Bulk Anomalous Dimension Δ_q^{b}



$\delta_0 = 0.173(1)$

$$\Delta_q \doteqdot \Delta_{1-q}$$ (The symmetry relation suggested by Mirlin et al. holds.)

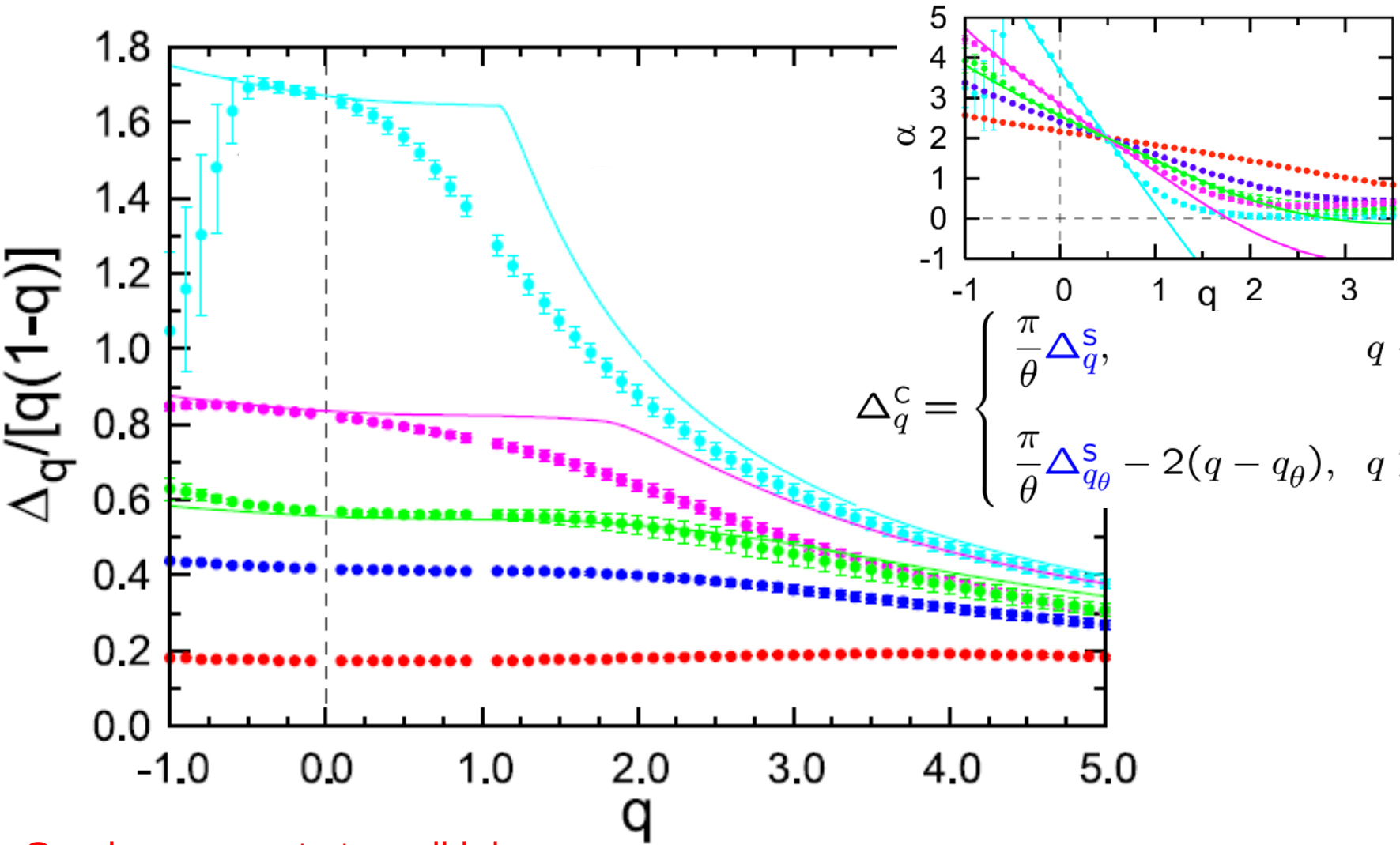
$$\frac{\Delta_q^{\mathrm{b}}}{q(1-q)} \neq \mathrm{const}$$ $f^{\mathrm{b}}(\alpha)$ Is not exactly prabolic.

Conformal mapping: 2D ⇨ cylinder (quasi-1D)

$$\frac{1}{\pi\delta_0} = \Lambda_c$$ (normalized localization length in quasi-1D, $\Lambda_c = 1.843$)

Same results obtained independently by Mildenberger and Evers, cond-mat/0608560.

Δ_q bulk, surface, and corner ($\theta = 3\pi/4, \pi/2, \pi/4$)



$$\Delta_q^c = \begin{cases} \frac{\pi}{\theta}\Delta_q^s, & q < q_\theta \\ \frac{\pi}{\theta}\Delta_{q_\theta}^s - 2(q - q_\theta), & q > q_\theta \end{cases}$$

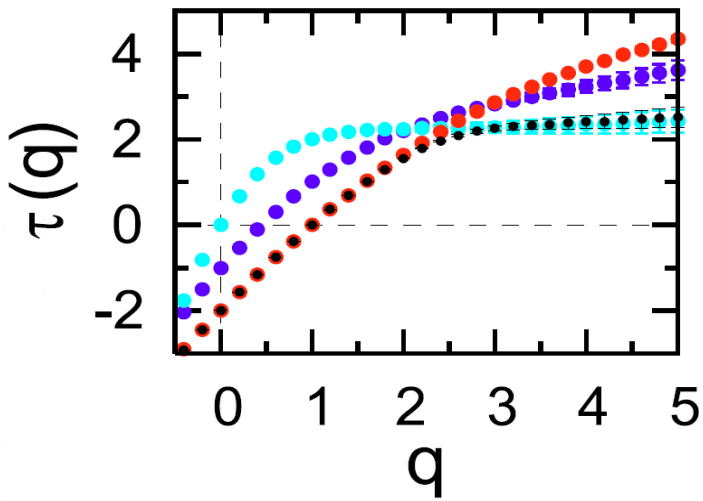
Good agreement at small |q|

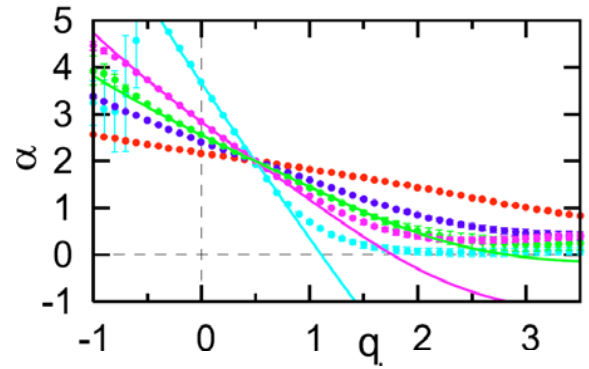
Cusps at $q = q_\theta$: "freezing transition"

Note that $f^\theta(\alpha) \leq 0$ (relatively poor statistics)

Finite-size effects important

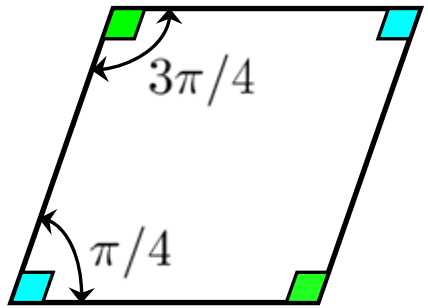
τ_q Bulk, Surface, $\theta = \pi/4$, whole rhombus $\int_r |\psi_r|^{2q} \sim L^{-\tau_q^{\mathrm{W}}}$





$\alpha = \frac{d\tau_q}{dq} = 0$

$$\tau_q^{\mathrm{W}} = \min\{\tau_q^{\mathrm{b}}, \tau_q^{\mathrm{s}}, \tau_q^{\pi/4}, \tau_q^{3\pi/4}\}$$



Corner exponent is observed in the multifractal exponent of the whole rhombi.

summary

- Conformal invariance $\Delta_q^{\mathsf{C}} = \frac{\pi}{\theta}\Delta_q^{\mathsf{S}}, \quad f^{\mathsf{C}}(\alpha_q^{\mathsf{C}}) = \frac{\pi}{\theta}\left[f^{\mathsf{S}}(\alpha_q^{\mathsf{S}}) - 1\right]$
- Symmetry relation proposed by Mirlin et al. $\Delta_q = \Delta_{1-q}$

  confirmed in the bulk multifractality

  violated in the corner (α_q^{C} can exceed 4)

- Surface (corner) may determine multifractal spectra of the whole sample (at large |q|)