Multifractality and Conformal Invariance at 2D Metal-Insulator Transition in the Spin-Orbit (Symplectic) Symmetry Class

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cf: Mirlin’s talk in this program
Plan of the talk

- Introduction
- Theoretical background
- Model
- Numerical results
Anderson localization

An electron moving in a random potential

P.W. Anderson (1958)

scaling theory  AALR (1979)

\[ \beta(g) = \frac{d \ln g}{d \ln L} = d - 2 - \mathcal{O}(1/g) \]

All the wave functions are localized in one and two dimensions!
Hikami-Larkin-Nagaoka


Spin-Orbit Interaction and Magnetoresistance in the Two Dimensional Random System

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3 symmetry classes (orthogonal, unitary, symplectic)

symplectic class: ○ time-reversal, × spin-rotation
spin-orbit interaction
anti-localization

\[
\frac{d \ln g}{d \ln L} = d - 2 + \frac{c}{g}, \quad c > 0
\]

Metal-insulator transition in 2D

critical point in 2D
Anderson transition (metal-insulator transition)

Continuous phase transition induced by disorder

localization length $\xi \to \infty$ $\xi \sim |E - E_c|^{-\nu}$

scale invariance universal critical exponents

delocalized

multifractal

critical point

metallic phase localizing phase
Multifractality (bulk case): statistics of (critical) wave functions

Critical wave function at a metal-insulator transition point

multifractal exponents $\tau_q$

$$L^d |\psi(r)|^{2q} \sim L^{-\tau_q}$$

$$\tau_q = d(q - 1) + \Delta_q$$

$$D_q = \frac{\tau_q}{q - 1}$$

fractal dim.

Continuous set of independent and universal critical exponents

$\Delta_q$: anomalous scaling dimensions

$$\Delta_q = 0 \text{ at } q = 0, 1$$

$$\frac{\Delta_q}{q(1-q)} = c(q)$$

singularity spectrum

$$f(\alpha) = q\alpha - \tau_q$$

$$\alpha = \frac{d\tau_q}{dq}$$

$\alpha > 0$

$$N_\alpha \sim L^{f(\alpha)}$$: measure of $r$ where $|\psi(r)|^2 \sim L^{-\alpha}$
Anderson transitions are continuous phase transitions induced by disorder

At ordinary continuous phase transition points without disorder,

- correlation length $\xi \to \infty$
- scale invariance $\rightarrow$ conformal invariance (Polyakov, 1970)
- 2D conformal field theory (BPZ, 1984)

Disorder-induced critical points in 2D:
- symplectic class, QHE (unitary class, class C, class D)

Conformal invariance at these critical points?

What kind of CFTs describe the disordered critical points in 2D?
To examine the presence of conformal invariance,

Consider disordered samples with open boundaries (surface)

Change the shape of samples

Surface multifractality
Surface critical phenomena
in conventional phase transitions

At conventional critical points:

**bulk:** $\ell \gg r$

$$\langle O(\vec{r}_1) O(\vec{r}_2) \rangle \sim r^{-2x_b}$$

**surface:** $r \gg \ell$

$$\langle O(\vec{r}_1) O(\vec{r}_2) \rangle \sim r^{-2x_s}$$

$x_s \neq x_b$

Surface critical exponents are different from bulk exponents

Boundary CFT (Cardy 1984)

$$w = z^{\pi/\theta}$$

$$x_\theta = \frac{\pi}{\theta} x_s$$

$O$: quasi-primary op.
Surface multifractality \textit{(Subramaniam et al., 2006)}

$\Delta_q^s \neq \Delta_q^b$

\[
L^d \langle |\psi(r)|^{2q} \rangle_{\text{bulk}} \sim L^{-\tau_q^b}
\]

\[
L^{d-1} \langle |\psi(r)|^{2q} \rangle_{\text{surface}} \sim L^{-\tau_q^s}
\]

\[
L^{d-2} \langle |\psi(r)|^{2q} \rangle_{\text{corner}} \sim L^{-\tau_q^c}
\]

cf. Gaussian theory

$\Delta_q = cq(1 - q), \quad \Delta_q^s = 2\Delta_q^b$

$\tau_q^b = d(q - 1) + \Delta_q^b$

$\tau_q^s = d(q - 1) + 1 + \Delta_q^s$

$\tau_q^c = d(q - 1) + 2 + \Delta_q^c$
Spin quantum Hall transition

class C  (Altland & Zirnbauer, 1997)

BdG quasiparticles at $E=0$ (p-h symmetry), × time reversal, ○ spin rotation

2D SQH plateau transition  ~ classical percolation  (Gruzberg, Ludwig, Read, 1999)

bulk multifractal exponents  \[ \Delta^b_2 = -\frac{1}{4}, \quad \Delta^b_3 = -\frac{3}{4} \]  
(Mirlin, Evers, Mildenberger, 2003)

surface multifractal exponents

\[ \Delta^s_2 = -\frac{1}{3}, \quad \Delta^s_3 = -1 \]  
(Subramaniam et al., 2006)
Multifractality vs field theory (Duplantier & Ludwig, 1991)

\[
\overline{O_r^q} \leftrightarrow \langle O_r^q \rangle \sim L^{-x_q}
\]

\(O_r\) loral random events at position \(r\)

\(O_r\) scaling operator in a field theory

\[
|\psi_r|^{2q} = \frac{O_r^q}{\left(\int_r O_r\right)^q} \sim \frac{O_r^q}{\left(\int_r O_r\right)^q} \sim L^{-d_q - (x_q - qx_1)}
\]

\(\Delta_q = x_q\)

Conformal mapping

\[
\Delta^c_q = \frac{\pi}{\theta} \Delta^s_q
\]

\(w = z^{\pi/\theta}\)
\[ \Delta_q^C = \frac{\pi}{\theta} \Delta_q^S \]

\[ \tau_q^S = 2(q - 1) + 1 + \Delta_q^S \]

\[ \tau_q^C = 2(q - 1) + 2 + \Delta_q^C \]

\[ = 2(q - 1) + 2 + \frac{\pi}{\theta} [\tau_q^S - 2(q - 1) - 1] \]

\[ \alpha_q^C - 2 = \frac{\pi}{\theta} (\alpha_q^S - 2) \]

\[ f_q^C(\alpha_q^C) = \frac{\pi}{\theta} \left[ f_q^S(\alpha_q^S) - 1 \right] \]
\[ \Delta_q^c = \frac{\pi}{\theta} \Delta_q^s \]

\[ \tau_q^s = 2(q - 1) + 1 + \Delta_q^s \]
\[ \tau_q^c = 2(q - 1) + 2 + \Delta_q^c \]
\[ = 2(q - 1) + 2 + \frac{\pi}{\theta} [\tau_q^s - 2(q - 1) - 1] \]

\[ \alpha_q^c - 2 = \frac{\pi}{\theta} \left( \alpha_q^s - 2 \right) \]

\[ f_c^c(\alpha_q^c) = \frac{\pi}{\theta} \left[ f_s^s(\alpha_q^s) - 1 \right] \]

\[ \alpha_q^c \geq 0 \quad \alpha_{q\theta}^c = 0 \]

\[ q < q_{\theta} \quad \Delta_q^c = \frac{\pi}{\theta} \Delta_q^s \]

\[ q > q_{\theta} \quad \alpha_q^c = 0, \quad \tau_q^c = \text{const.}, \]
\[ \Delta_q^c = \Delta_{q\theta}^c - 2(q - q_{\theta}) \]
Symmetry relations proposed by Mirlin et al. (Mirlin et al., PRL 2006)

LDOS distributions in NLσ model + universality

Symmetry relations of multifractal spectra

\[ \Delta_q^x = \Delta_{1-q}^x \quad \land \quad x = b, s, c \]

\[ \alpha_{1-q}^x - d = - \left( \alpha_q^x - d \right) \]

\[ f^x(\alpha_{1-q}^x) - \frac{\alpha_{1-q}^x}{2} = f^x(\alpha_q^x) - \frac{\alpha_q^x}{2} \]

Mirlin et al. confirmed these for 1d Power-law Random Banded Matrix Model

\[ \tau_q^x = dq - d_x + \Delta_q^x \]

\[ \tau_{1-q}^x = d(1 - q) - d_x + \Delta_{1-q}^x \]

\[ = d(1 - 2q) + \tau_q^x \]

\[ d_x = \begin{cases} 
  d, & \land \quad x = b \\
  d - 1, & \land \quad x = s \\
  d - 2, & \land \quad x = c 
\end{cases} \]
We will

- numerically study multifractal properties of critical wave functions in the 2D symplectic class
- compare bulk, surface, and corner exponents
- check the conformal invariance
  \[ \Delta^c_q = \frac{\pi}{\theta} \Delta^s_q \]
  \[ \alpha^c_q - 2 = \frac{\pi}{\theta} (\alpha^s_q - 2), \quad f^c(\alpha^c_q) = \frac{\pi}{\theta} \left[ f^s(\alpha^s_q) - 1 \right] \]
- check the symmetry relations conjectured by Mirlin et al.
SU(2) model for the symplectic class

Asada, Slevin, Ohtsuki, 2002

Tight-binding model on a 2D square lattice

\[ H = \sum_{i,\sigma} \epsilon_i c_{i,\sigma}^\dagger c_{i,\sigma} - \sum_{\langle i,j \rangle,\sigma,\sigma'} R(i, j)_{\sigma,\sigma'} c_{i,\sigma}^\dagger c_{j,\sigma'} \]

\[ R(i, j) = \begin{pmatrix} e^{i\alpha_{i,j}} \cos \beta_{i,j} & e^{i\gamma_{i,j}} \sin \beta_{i,j} \\ -e^{-i\gamma_{i,j}} \sin \beta_{i,j} & e^{-i\alpha_{i,j}} \cos \beta_{i,j} \end{pmatrix} \quad \epsilon_i \in [-W/2, W/2] \]

\[ \alpha, \gamma \in [0, 2\pi) \quad P(\beta) \, d\beta = \sin(2\beta) \, d\beta \quad 0 \leq \beta \leq \frac{\pi}{2} \]

\[ \nu = 2.73 \pm 0.02 \]
Our numerics

SU(2) model

system size \( L = 24 \sim 120 \)

# of samples \( 6 \times 10^4 \)

\( E = 1 \quad W_c = 5.95 \quad \psi \big|_{\text{surface}} = 0 \)

For each sample we keep only one eigenstate with \( E \) closest to 1.

\[ \text{Insulator} \]

\[ \text{Metal} \]

\[ E_F \]

\[ L = 24 \sim 120, \quad l = L/6, \quad w = 4, \quad h = 1 \]
Numerical Results

Probability distributions of wave function amplitudes

\[ \ln \left( |\psi|^2 L^2 \right) \]

\[ \text{Probability} \]

\[ \ln \left( |\psi|^2 L^2 \right) \]

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Numerical estimate of multifractal exponents:

\[ |\psi(r)|^{2q} \sim L^{-2q - \Delta^x_q} \quad (r \in x) \]

\[ \frac{|\psi(r)|^{2q} \left( |\psi(r)|^2 \right)^{-q}}{\sim L^{-\Delta^x_q}} \quad (\Delta^x_{q=1} = 0) \]

\[ \langle \langle \ln |\psi(r)|^2 \rangle \rangle_q \equiv \frac{|\psi(r)|^{2q} \ln |\psi(r)|^2}{|\psi(r)|^{2q}} \sim -\alpha^x_q \ln L \]

\[ \ln |\psi(r)|^{2q} \sim \left[ f^x(\alpha^x_q) - q\alpha^x_q - dx \right] \ln L \]

Bulk, Surface, Corner $\pi/2$
Bulk, surface, and corner multifractal spectra

Bulk, Surface, Corner \( \pi/2 \), Whole Cylinder

\[ f(\alpha): \text{bulk, surface, and corner} \]

Bulk, surface, and corner \( f(\alpha) \) are all different. Surface contributions dominate at large \(|q|\) in the whole cylinder \( f(\alpha) \).
Colored thin lines: $f^x(\alpha_{1-q}) = f^x(4 - \alpha_q) + \alpha_q - 2$

The bulk $f(\alpha)$ satisfies the symmetry relation suggested by Mirlin et al.

Numerical inaccuracy at large $|q|$?

We need calculations with larger $L$ and $N_{\text{sample}}$. 

Surface ?

Corner ??
\[ \theta = \frac{3\pi}{4}, \frac{\pi}{2}, \frac{\pi}{4} \]

Colored thin lines: Conformal Invariance !!

\[ \alpha_{q}^{\frac{3\pi}{4}} = 2.555 \pm 0.003 \]
\[ \alpha_{q}^{\frac{\pi}{2}} = 2.837 \pm 0.004 \]
\[ \alpha_{q}^{\frac{\pi}{4}} = 3.692 \pm 0.009 \]

\[ \alpha_{q}^\theta - 2 = \frac{\pi}{\theta} \left( \alpha_{q}^S - 2 \right) \]
\[ f^{\theta}(\alpha_q^\theta) = \frac{\pi}{\theta} \left[ f^{S}(\alpha_q^S) - 1 \right] \]

\[ q_{\pi/4} \]

rounding of cusps at \( q = q_\theta \)

\[ \alpha_{1-q}^x - 2 = - \left( \alpha_q^x - 2 \right) \]

finite-size effect
\[ \theta = \frac{3\pi}{4}, \frac{\pi}{2}, \frac{\pi}{4} \]

The inequality (Mirlin et al.) is violated.

\[ \alpha_0^{\frac{3\pi}{4}} = 2.555 \pm 0.003 \]
\[ \alpha_0^{\frac{\pi}{2}} = 2.837 \pm 0.004 \]
\[ \alpha_0^{\frac{\pi}{4}} = 3.692 \pm 0.009 \]

\( \alpha_q^\theta \) can exceed 4!

The inequality \( \alpha_q < 4 \) (Mirlin et al.) is violated.

cusps at \( q = q_\theta \); non-analytic
Bulk Anomalous Dimension $\Delta^b_q$

$\Delta_q \doteq \Delta_{1-q}$ (The symmetry relation suggested by Mirlin et al. holds.)

$$\frac{\Delta^b_q}{q(1-q)} \neq \text{const}$$

$f^b(\alpha)$ is not exactly parabolic.

Conformal mapping: 2D $\rightarrow$ cylinder (quasi-1D)

$$\frac{1}{\pi \delta_0} = \Lambda_c$$

(normalized localization length in quasi-1D, $\Lambda_c = 1.843$)

Same results obtained independently by Mildenberger and Evers, cond-mat/0608560.
$\Delta_q$ bulk, surface, and corner ($\theta = 3\pi/4, \pi/2, \pi/4$)

Good agreement at small $|q|$  

Cusps at $q = q_\theta$: “freezing transition”

Note that $f^\theta(\alpha) \leq 0$ (relatively poor statistics)  

Finite-size effects important  

$\Delta_q^c = \begin{cases} 
\frac{\pi}{\theta} \Delta_q^s, & q < q_\theta \\
\frac{\pi}{\theta} \Delta_q^s - 2(q - q_\theta), & q > q_\theta 
\end{cases}$
\[ \tau_q \quad \text{Bulk, Surface, } \theta = \pi/4, \text{ whole rhombus } \int_{r} |\psi_r|^{2q} \sim L^{-\tau_q^W} \]

\[ \tau^W_q = \min \{ \tau_q^b, \tau_q^s, \tau_q^{\pi/4}, \tau_q^{3\pi/4} \} \]

Corner exponent is observed in the multifractal exponent of the whole rhombi.
summary

- Conformal invariance \( \Delta^C_q = \frac{\pi}{\theta} \Delta^S_q \), \( f^C(\alpha^C_q) = \frac{\pi}{\theta} \left[ f^S(\alpha^S_q) - 1 \right] \)

- Symmetry relation proposed by Mirlin et al. confirmed in the bulk multifractality
  \( \Delta_q = \Delta^{1-q} \)

  violated in the corner \( (\alpha^C_q \text{ can exceed 4}) \)

- Surface (corner) may determine multifractal spectra of the whole sample (at large \(|q|\))