Generic Singularities in Cosmological Spacetimes

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- Singularities in general relativity.

- Spatially homogeneous cosmologies.

- Spatially inhomogeneous cosmologies with 1 or 2 symmetry directions.

- Open questions.

See BKB, *Liv. Rev. Rel.* (www.livingreviews.org)

- **Singularity theorems**: Regular, generic initial data for reasonable matter will evolve to yield a pathological behavior if gravity becomes sufficiently strong.
- The nature of the pathology is not predicted and various types are known in special cases.

- **Cosmic censorship hypotheses**:

– (1) Generically, singularities will be hidden inside the horizons of black holes. Naked singularities do not occur in nature.

– (2) Time-like singularities will not occur generically even inside a horizon. An observer will only detect a singularity by hitting it.

• Singularity theorems: Regular, generic initial data for reasonable matter will evolve to yield a pathological behavior if gravity becomes sufficiently strong.

• The nature of the pathology is not predicted and various types are known in special cases.

Note that in this context "quantum" matter can be "unreasonable."

• Cosmic censorship hypotheses:

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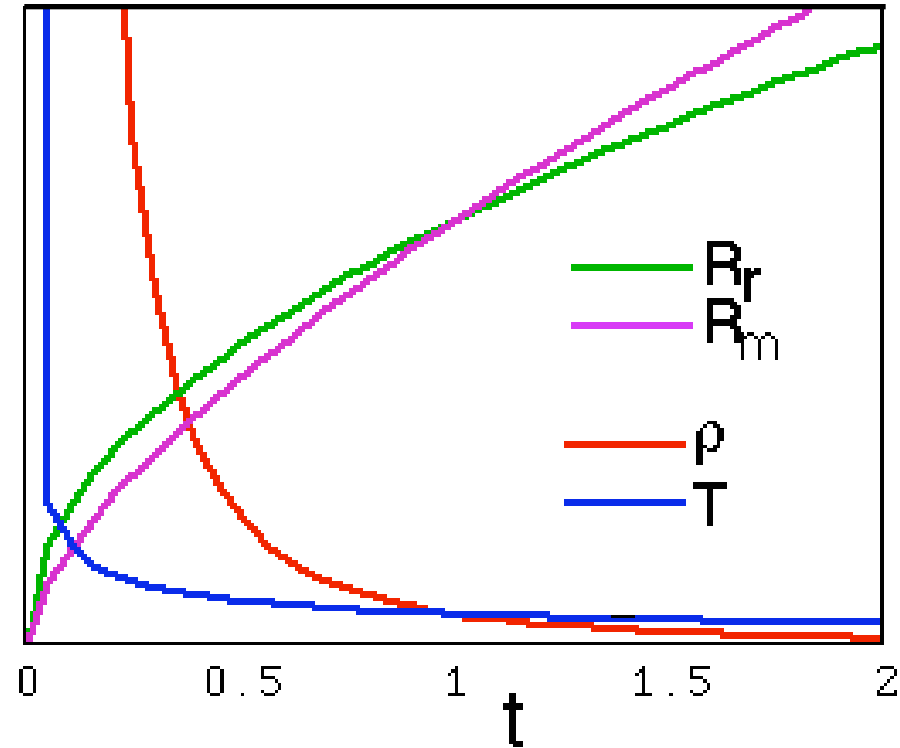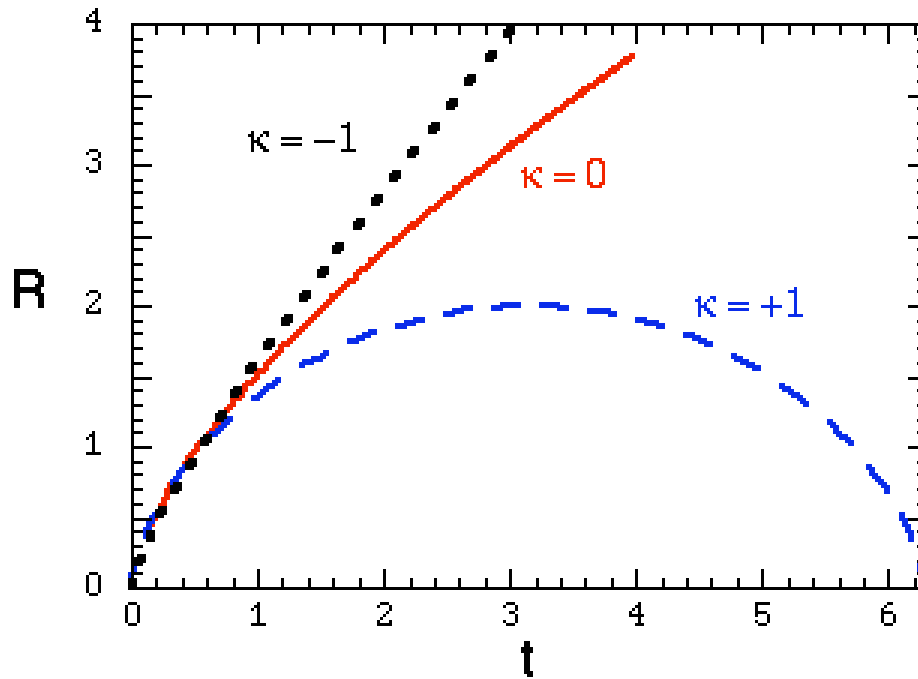
Good!

spacelike singularity

Schwarzschild

Bad!

timelike singularity

IH

CH

EH

$\mathcal{I}^+$

$\mathcal{I}^-$
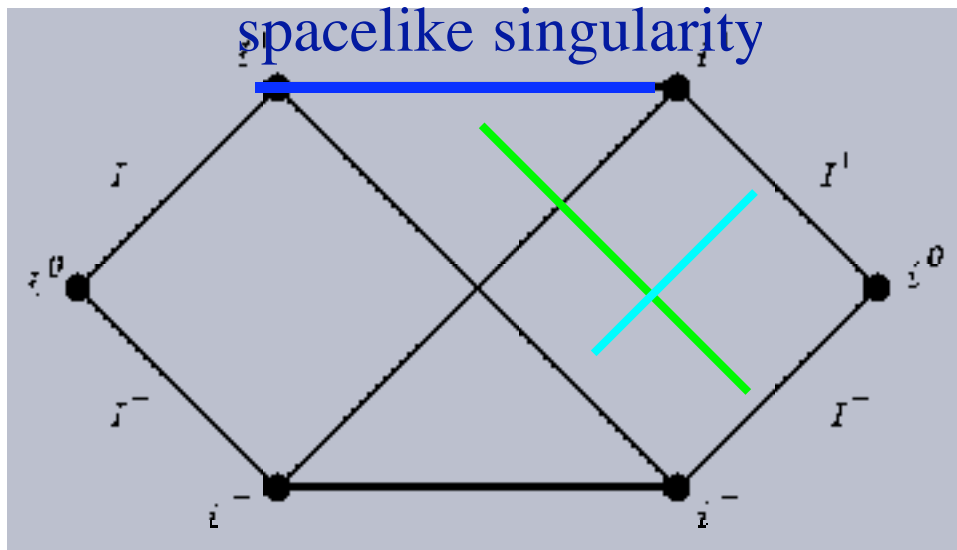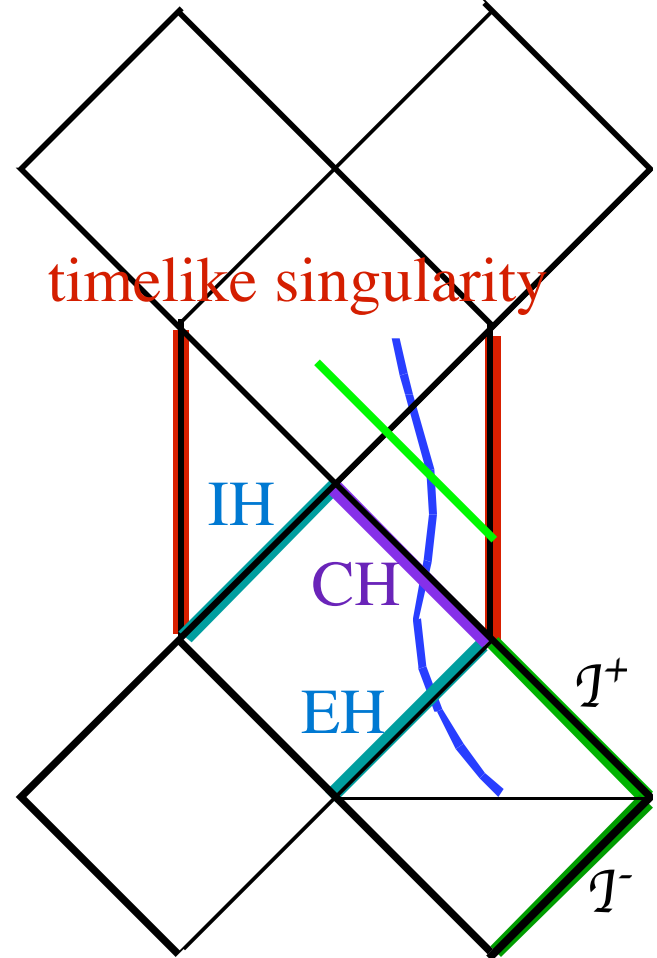
Kerr or Reissner-Nordstrom

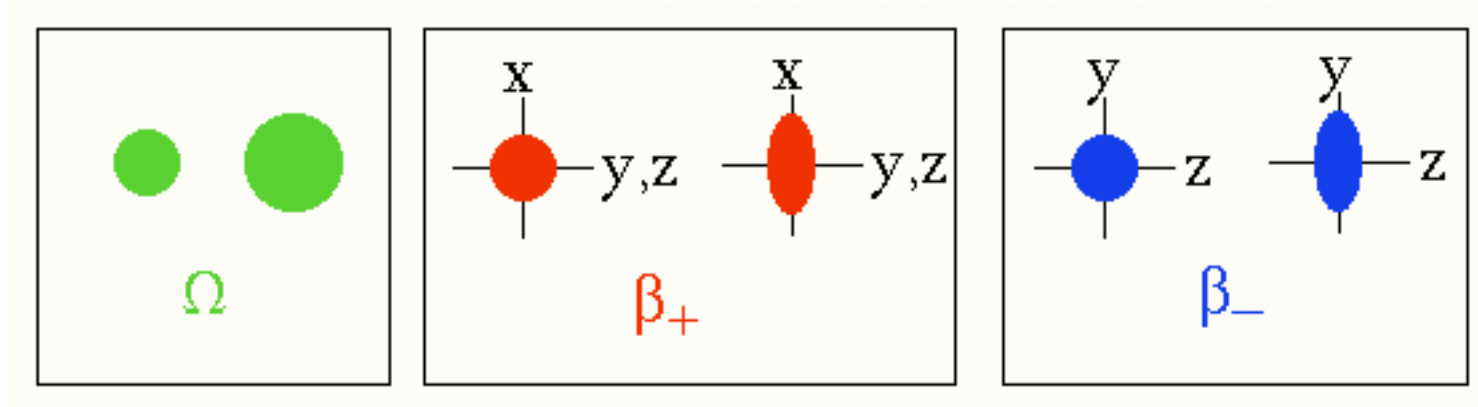A cosmological spacetime lacks an asymptotically flat region.

The singularity in the FRW cosmology is spacelike and characterized by "curvature blowup."

What happens in more general cosmological spacetimes with anisotropic expansion and/or spatial inhomogeneities?

Cosmological spacetimes can have anisotropic expansion and no matter:

$$R \rightarrow (R_x, R_y, R_z) \rightarrow (\Omega, \beta_+, \beta_-)$$
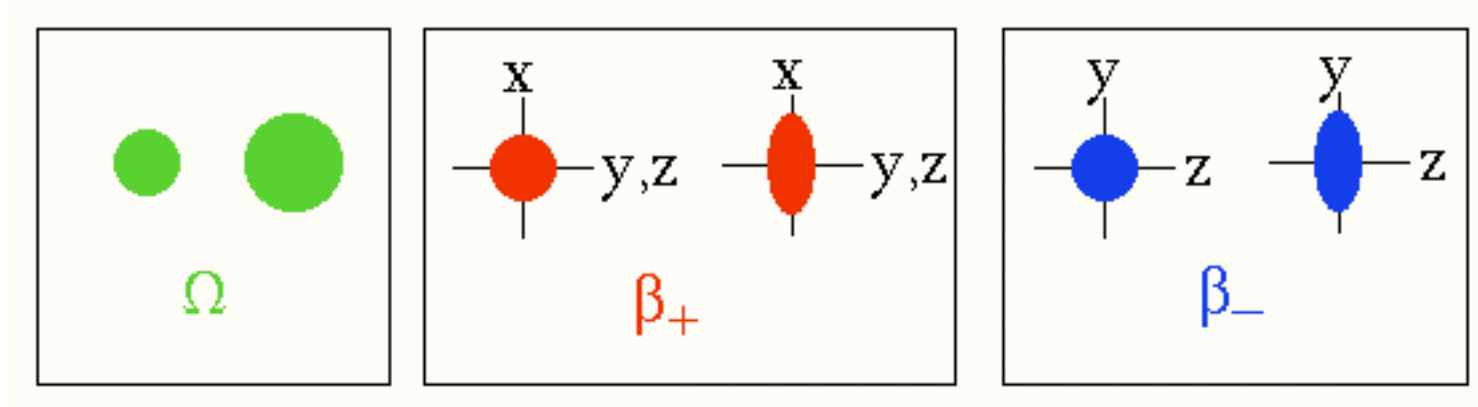

$$\dot{\Omega}^2 - V(\Omega, \beta_+, \beta_-) = 8\pi G(\rho_{matter} + \rho_{anis})$$

$$\rho_{anis} \propto \dot{\beta}_+^2 + \dot{\beta}_-^2$$

Anisotropy "energy" can act as a source for expansion. Spatial scalar curvature V will act as a "potential" for the dynamics in "minisuperspace."

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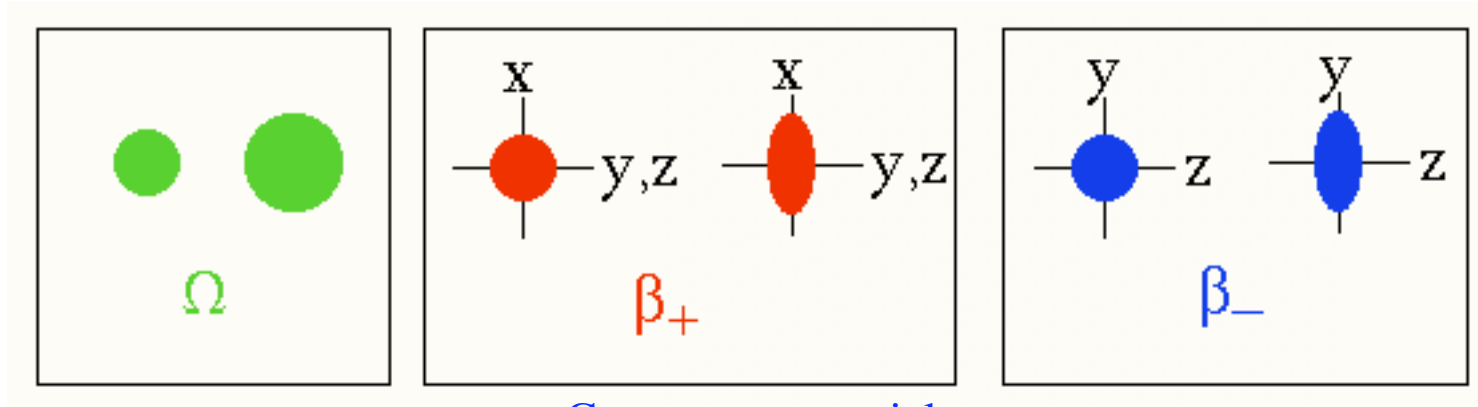
Hubble parameter

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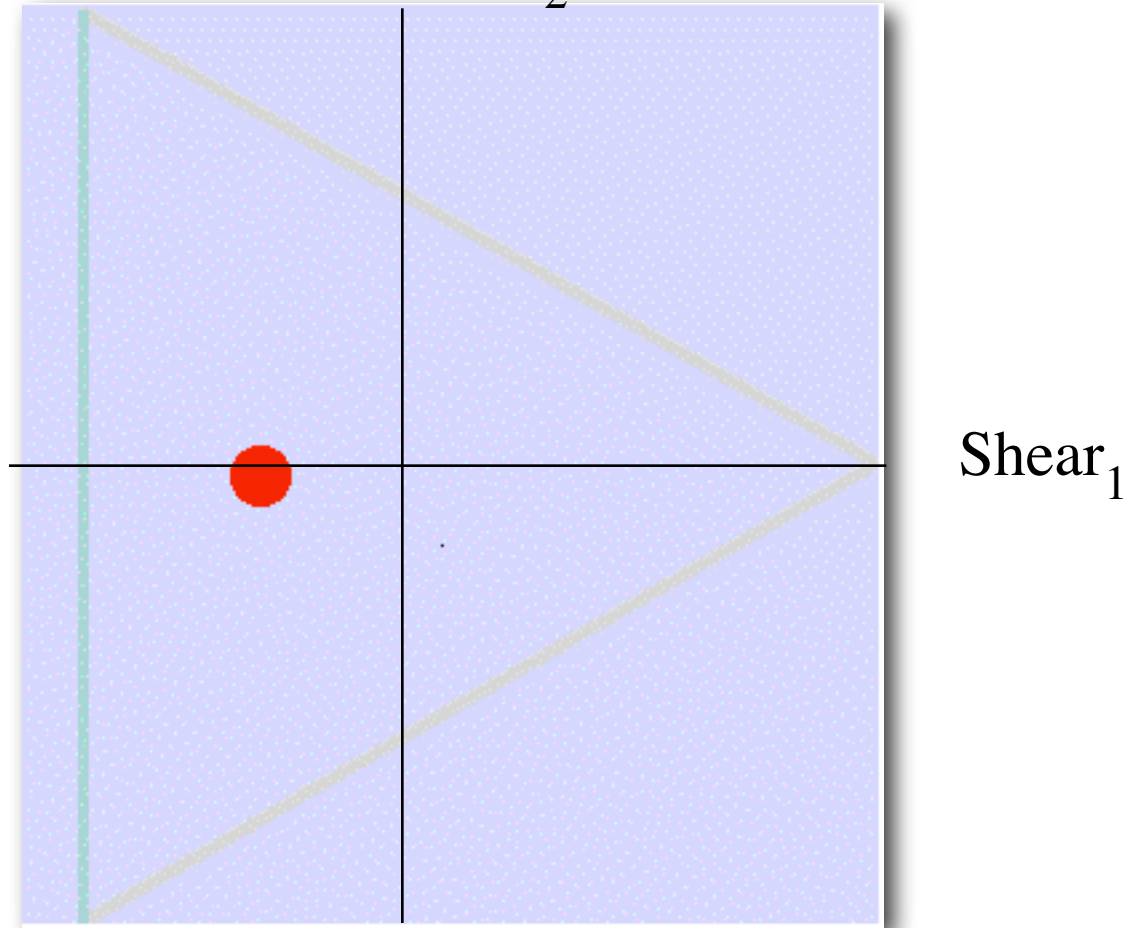

Curvature potential

$$\dot{\Omega}^2 - V(\Omega, \beta_+, \beta_-) = 8\pi G(\rho_{matter} + \rho_{anis})$$

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Anisotropy "energy" can act as a source for expansion. Spatial scalar curvature V will act as a "potential" for the dynamics in "minisuperspace."

The Minisuperspace Picture:

$Shear_2$

$Shear_1$

The anisotropy parameters of a spatially homogeneous universe at a given time define a point in MSS. The trajectory in MSS is determine by Einstein's equations. Spatial scalar curvature and rotation provide "walls."
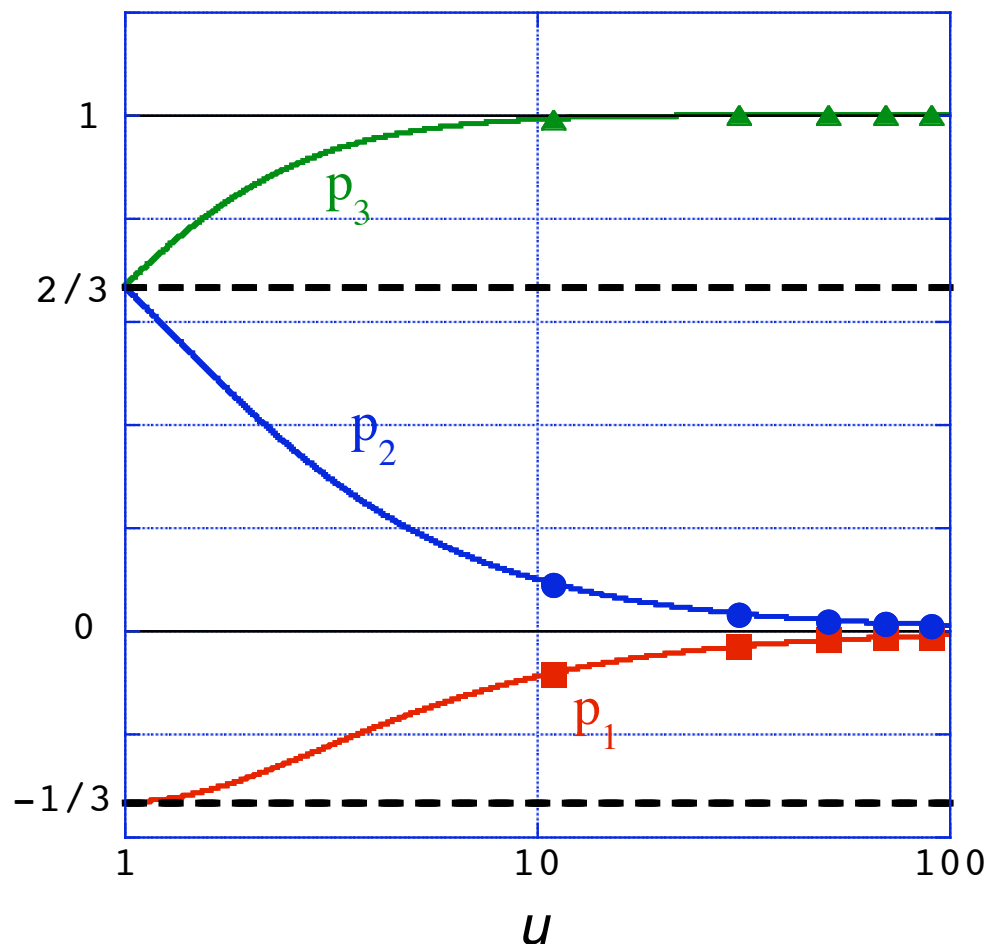
The Kasner Spacetime (vacuum, Bianchi Type I):

$$ds^2 = -dt^2 + t^{2p_1}dx^2 + t^{2p_2}dy^2 + t^{2p_3}dz^2$$

$$\text{where} \quad \sum_{i=1}^{3} p_i = 1 = \sum_{i=1}^{3} p_i^2$$

The three Kasner indices may be parametrized by a single variable u (introduced by BKL):

$$p_1 = \frac{-u}{u^2 + u + 1} \quad ; \quad p_2 = \frac{u+1}{u^2 + u + 1} \quad ; \quad p_3 = \frac{u(u+1)}{u^2 + u + 1}$$
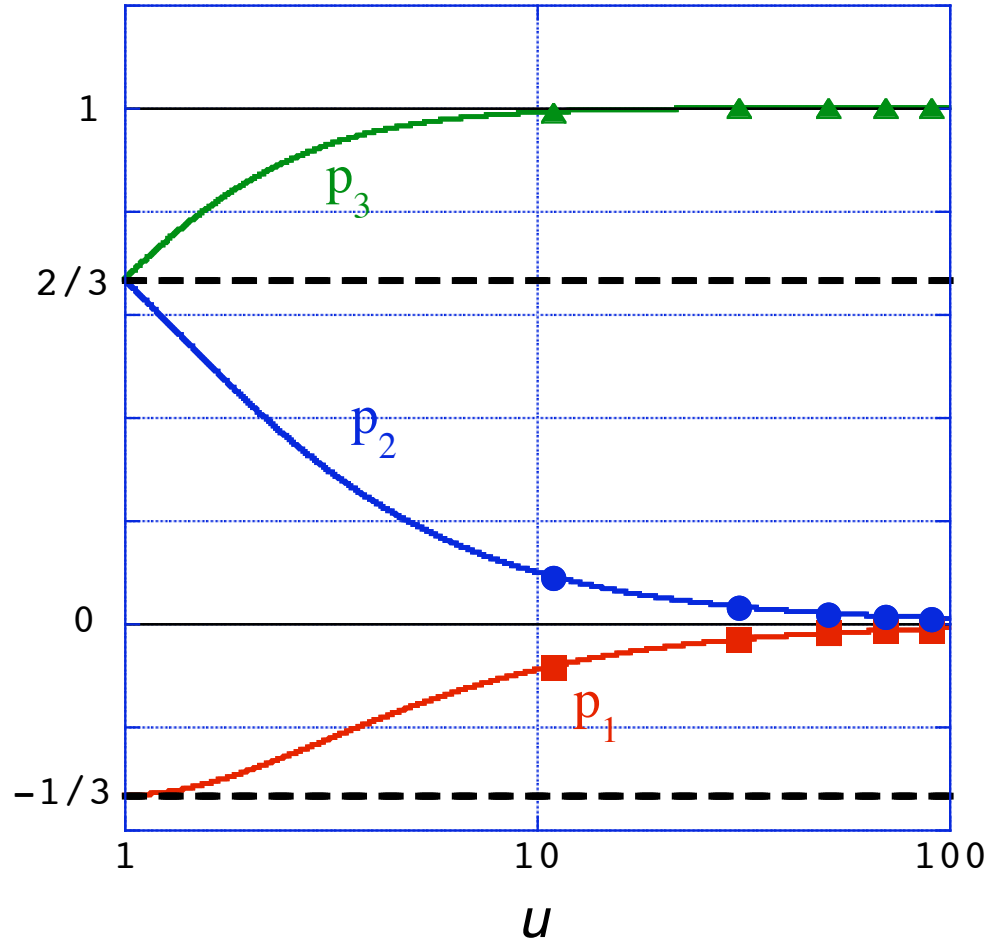
The Kasner Spacetime (vacuum, Bianchi Type I):

Each *u*-value in $[1, \infty]$ indicates a distinct Kasner evolution.

A set of measure zero: The (1,0,0) Kasner $u = \infty$ is the Minkowski spacetime in different coordinates.

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"Milne universe"

The Kasner Singularity:

Note that in the collapse (expansion) direction, one Kasner axis is expanding (collapsing). However,

$$\sqrt{^{3}g} = t$$

and the first non-zero curvature invariant blows up as $t \rightarrow 0$ unless $u = \infty$:

$$\kappa = R^{\mu\nu\rho\sigma} \, R_{\mu\nu\rho\sigma} = \frac{16}{t^4} \; \frac{u^2 \, (u+1)^2}{(u^2 + u + 1)^3}$$

This is a spacelike, curvature blowup singularity just as for FRW.

The Kasner Spacetime is a "free particle" in minisuperspace:

In terms of $d\tau = e^{-3\Omega}\,dt$ and the momenta conjugate to the MSS variables, Einstein's equations may be obtained by variation of the Hamiltonian constraint
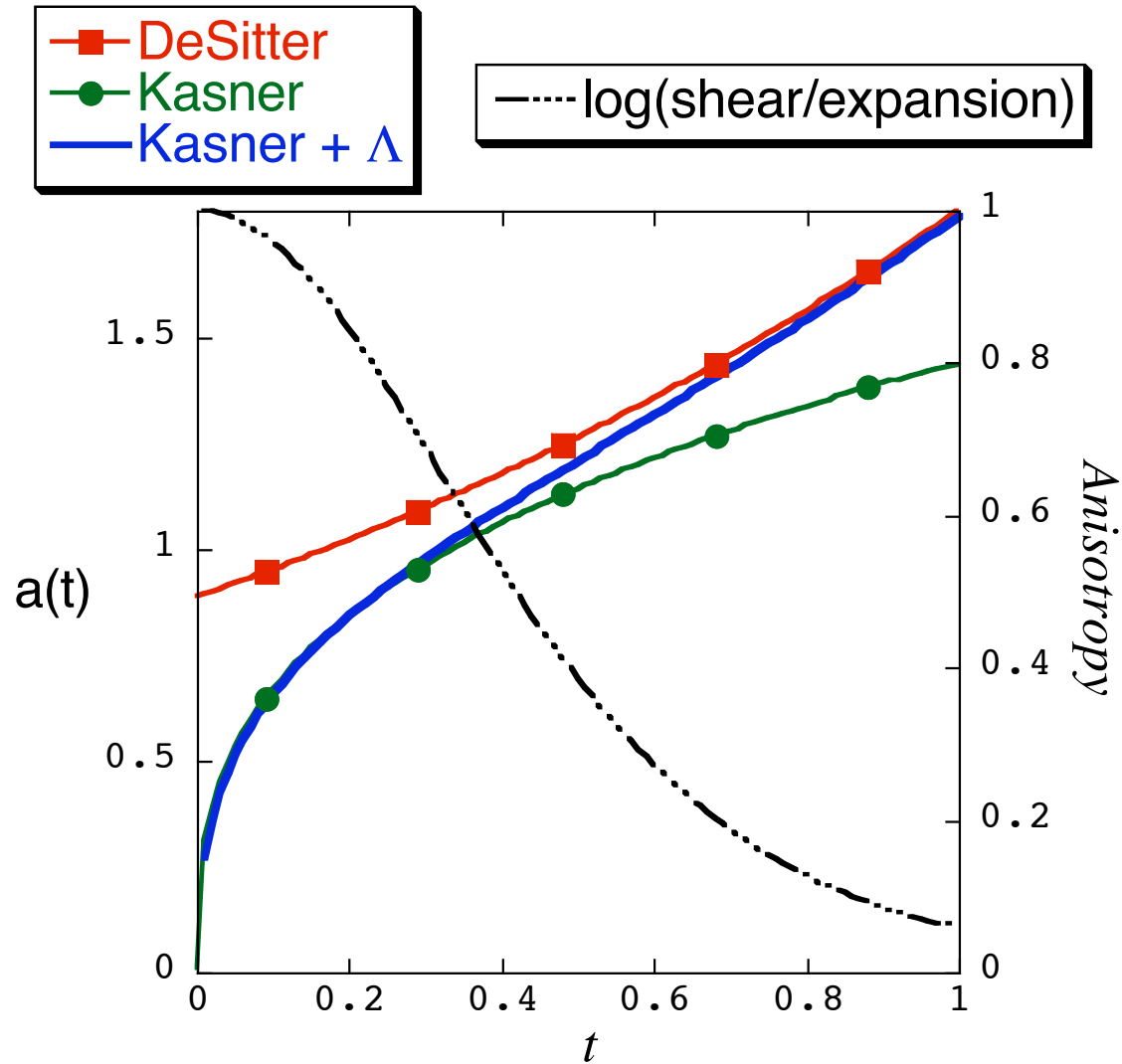
$$H = -p_\Omega^2 + p_+^2 + p_-^2 = 0$$

to yield

$$\left(\frac{p_+}{p_\Omega}\right)^2 + \left(\frac{p_-}{p_\Omega}\right)^2 = v_+^2 + v_-^2 = 1$$

$$\beta_\pm = v_\pm\,|\Omega|$$

Note that the straight line trajectory in MSS may be described by a single angle θ which may be shown to be equivalent to *u*.

The Kasner singularity is "velocity term dominated" (VTD).

Aside on the role of matter (or effective matter):

Taub spacetime (vacuum Bianchi Type II):

For these models, the Hamiltonian constraint

$$H = -p_\Omega^2 \; + \; p_+^2 \; + \; p_-^2 \; + \; e^{-8\beta_+ + 4\Omega} = 0$$

to yield

$$\frac{p_+^2}{p_\Omega^2} \; + \; \frac{p_-^2}{p_\Omega^2} \; + \; \frac{e^{-8\beta_+ + 4\Omega}}{p_\Omega^2} = 1$$

This can be treated exactly as scattering off an exponential potential relating the initial Kasner epoch to the final Kasner epoch.

This singularity is "asymptotically velocity term dominated" (AVTD) because there is a last "bounce" off the potential.

Conservation of momentum can be used to develop "bounce laws" to relate asymptotically constant variables before (e.g. u_{in}) and after (e.g. u_{out}) the bounce off the potential:
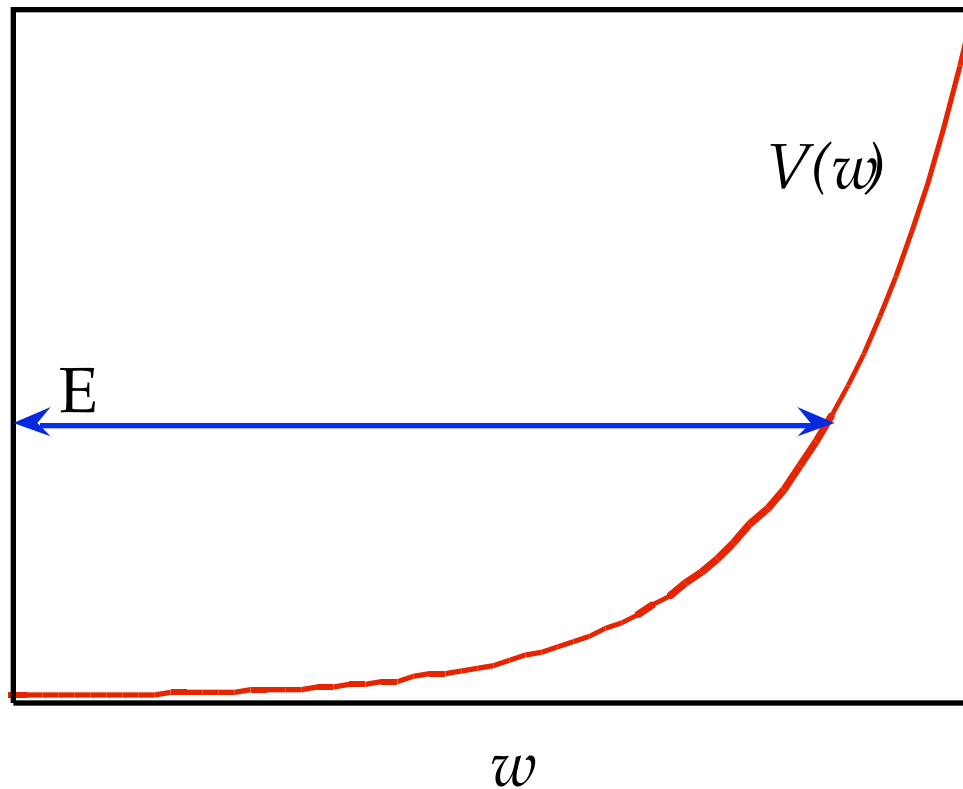
Typical trajectory in minisuperspace:
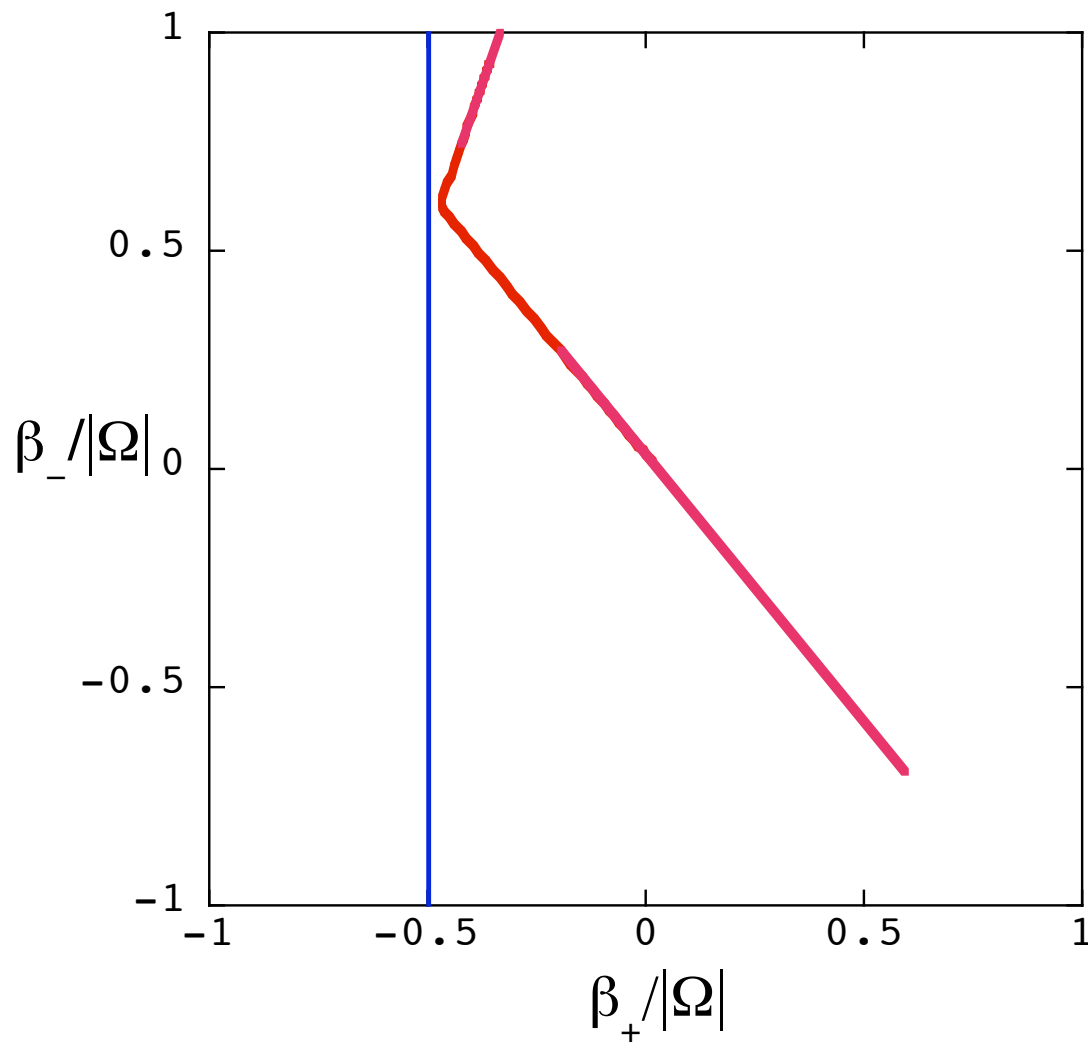
(The single "wall" could be oriented at any angle.)

Method of Consistent Potentials   (Grubisic, Moncrief) :

1. Neglect all terms arising from spatial dependence and solve the truncated Einstein equations (ODE's). This yields the Velocity Term Dominated   solution as $\tau \rightarrow \infty$ at each spatial point. ■
Click here

2. Substitute the VTD solution into the full Einstein equations. If all previously neglected terms are exponentially small as $\tau \rightarrow \infty$, we predict that the full solution is Asymptotically VTD .

3. Terms which are not exponentially small act as potentials which then dominate the dynamics.

4. Compare the prediction to numerical simulations of the full Einstein equations.

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Note that spatial scalar curvature in spatially homogeneous cosmologies arises from spatial derivatives.

Application of the MCP to the Taub spacetime:
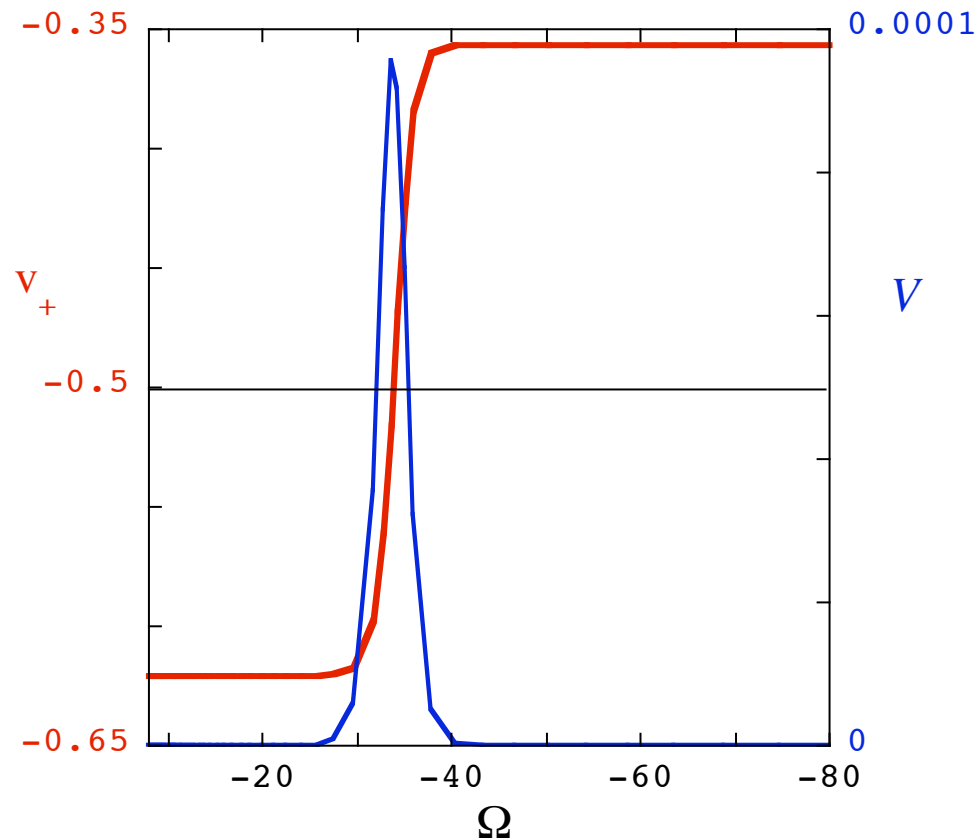
If $\beta_\pm = v_\pm |\Omega|$, then

$$V = e^{-8\beta_+ + 4\Omega} \longrightarrow e^{-8(v_+ + \frac{1}{2})|\Omega|}.$$

For $\Omega \to -\infty$, V grows to cause a bounce only if $-1 \leq v_+ < -\frac{1}{2}$.

The "most general" homogeneous cosmology is (non-diagonal) Bianchi IX:

- Mixmaster models (diagonal)

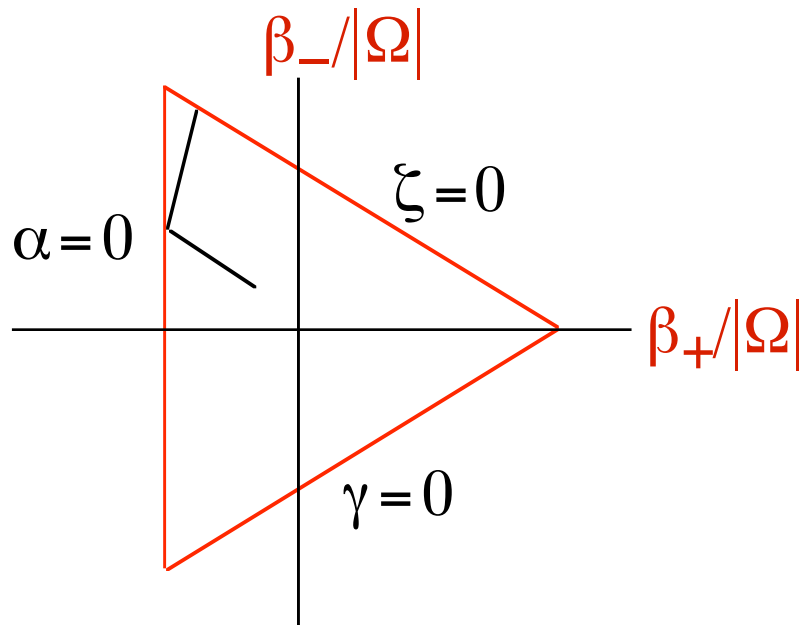$$ds^2 = -N^2 \, d\tau^2 + e^{2\Omega} \left(e^{2\beta}\right)_{ij} \sigma^i \sigma^j$$

where
$$d\sigma^i = \varepsilon^i_{jk} \, \sigma^j \wedge \sigma^k$$

In minisuperspace,

$$2H = -p_\Omega^2 + p_+^2 + p_-^2 + e^{4\alpha} + e^{4\zeta} + e^{4\gamma} + \ldots = 0$$

Mixmaster dynamics is an infinite sequence of Kasner epochs.

One Kasner epoch changes to another in a bounce off one of the exponential potentials.

$$\alpha = \Omega - 2\beta_+$$
$$\zeta = \Omega + \beta_+ + \sqrt{3}\beta_-$$
$$\gamma = \Omega + \beta_+ - \sqrt{3}\beta_-$$

$$2H = -p_\Omega^2 + p_+^2 + p_-^2 + e^{4\alpha} + e^{4\zeta} + e^{4\gamma} + \dots$$

If $\beta_\pm = v \pm |\Omega|$, one of α, ζ, or γ must grow.

From a numerical simulation:
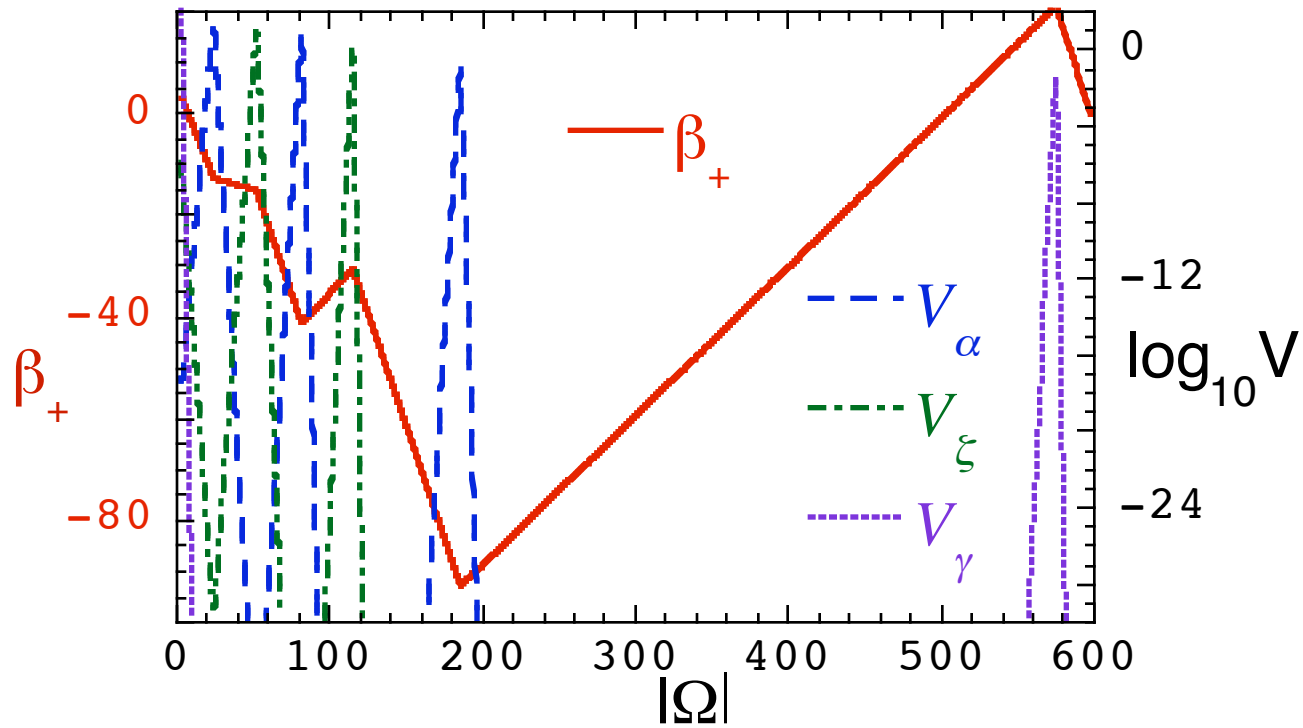
BKB, D. Garfinkle, E. Strasser, CQG **14**, L29 (1996).

Evolution of *u* from a typical Mixmaster simulation follows the (chaotic) BKL map:

$$u_{n+1} = \begin{cases} u_n - 1 & 2 \leq u \\[2em] \dfrac{1}{(u_n - 1)} & 1 \leq u \leq 2 \end{cases}$$

toward singularity $\longrightarrow$
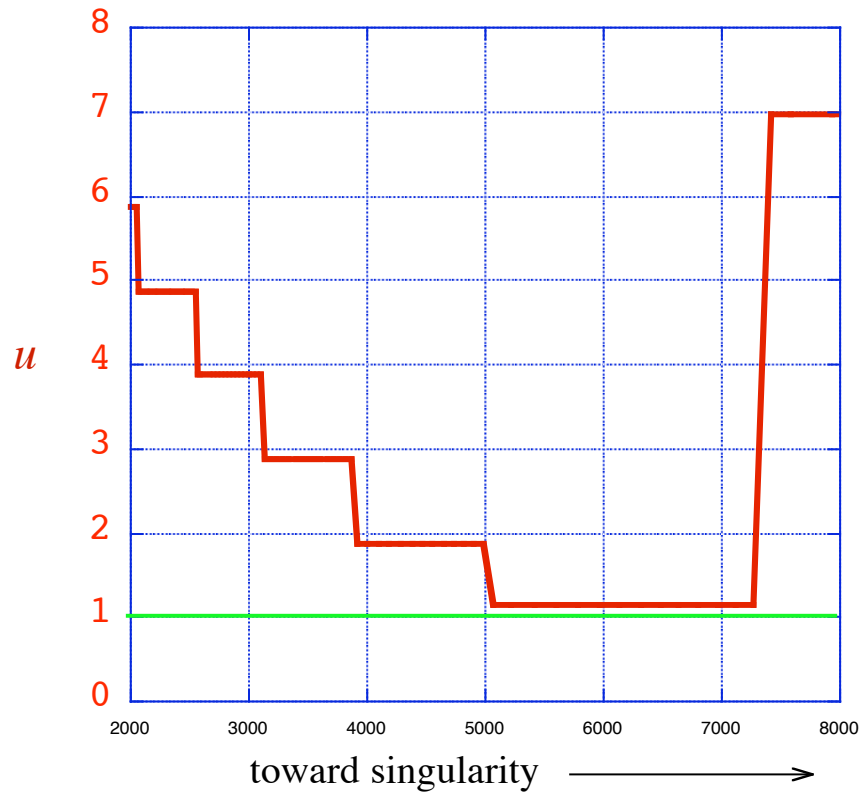
How well does a Mixmaster simulation obey the *u*-map?

Onset of qualitative deviation from u-map (computed with Mathematica) for double and quadruple precision ODE solutions.
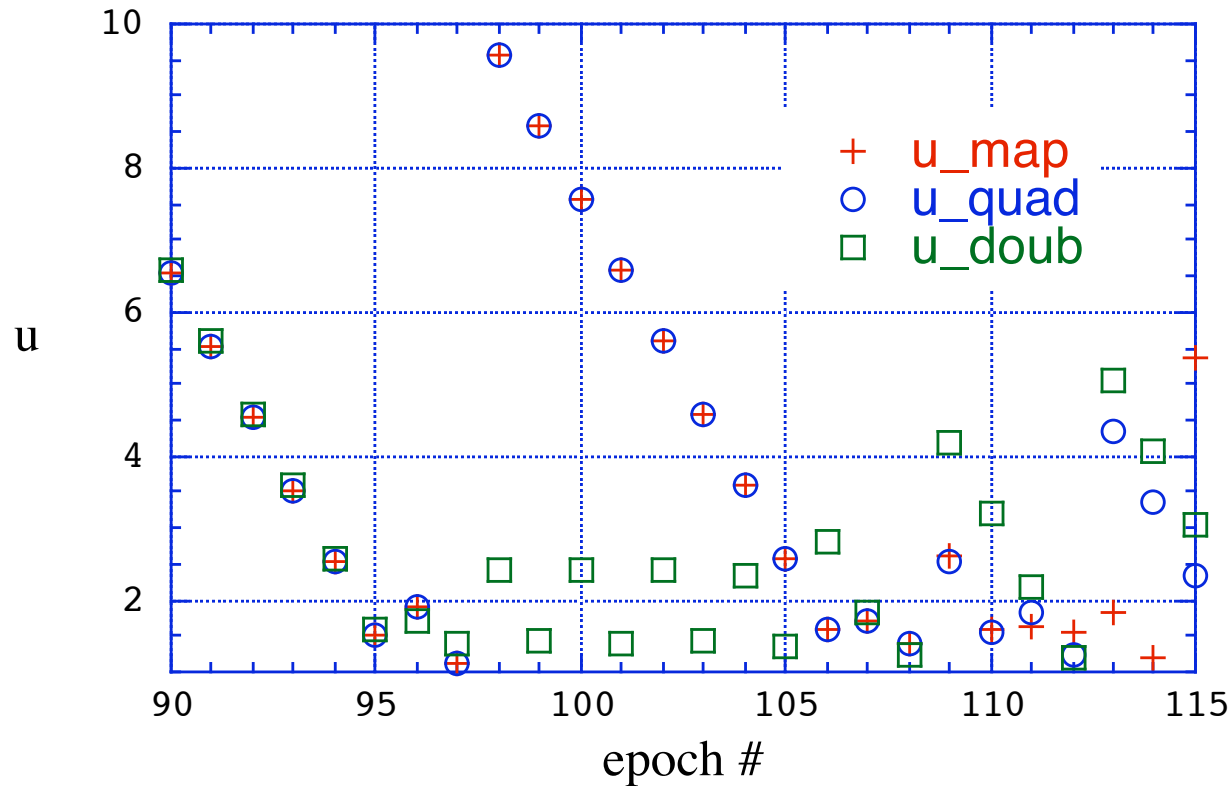
Exceptional case: Any rational *u*-value will eventually yield $u = \infty$.

Except for Taub initial data, usually negligible terms in the potential will restore Mixmaster dynamics.

A Mixmaster simulation with > 250 bounces:
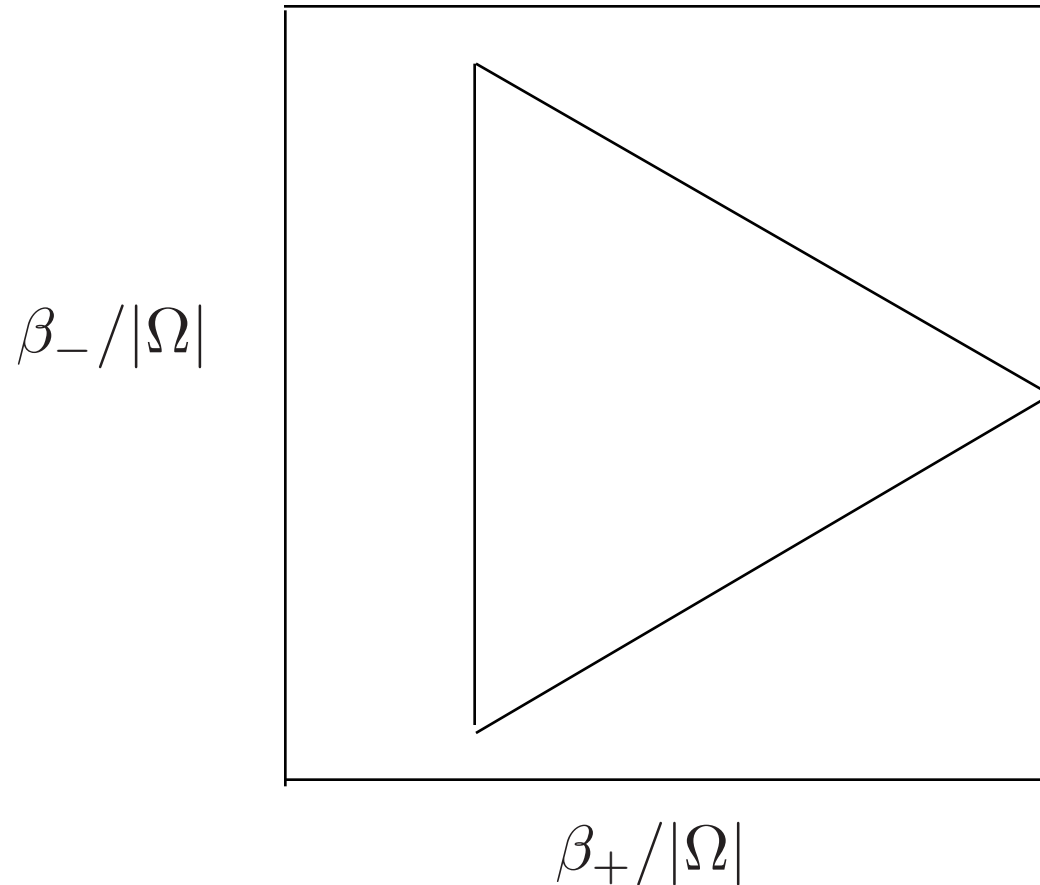
$\beta_-/|\Omega|$

$\beta_+/|\Omega|$

Ringström has proven that the Mixmaster singularity for non-Taub initial data is of the curvature blow-up type.

H. Ringström, Class.Quant.Grav. 17 (2000) 713-731.

## A Mixmaster simulation with > 250 bounces:

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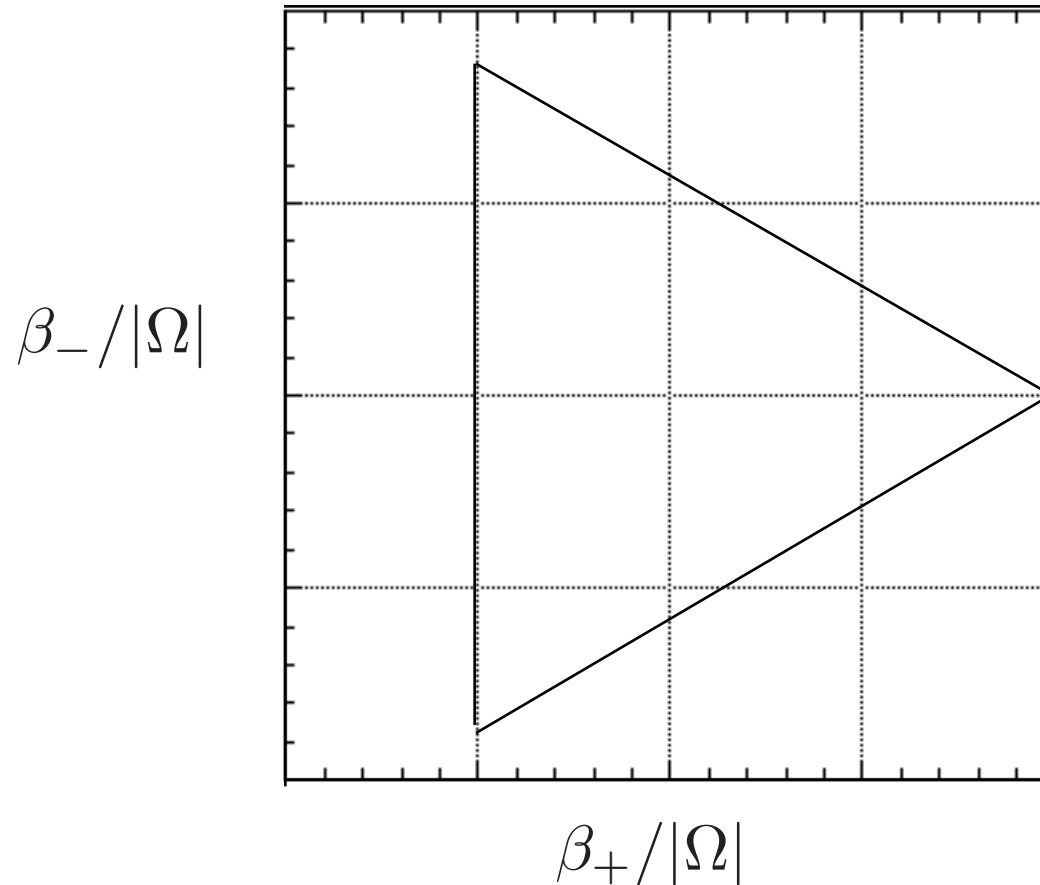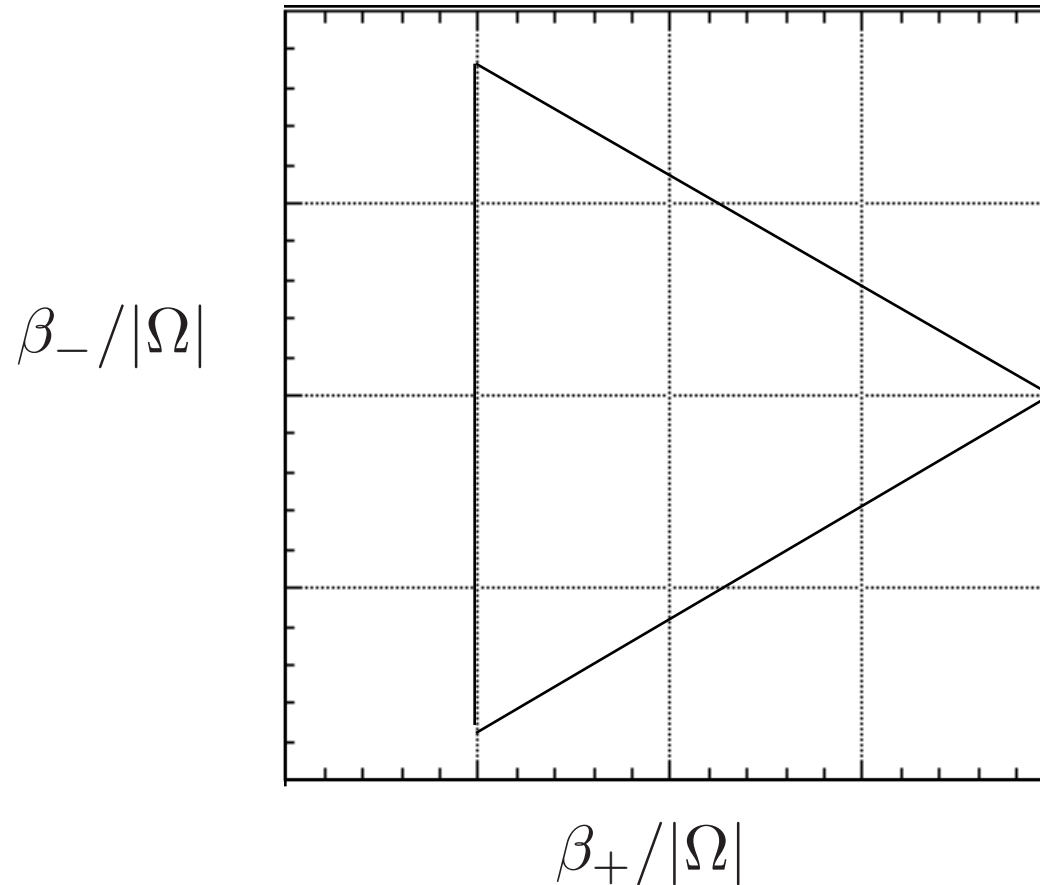
Ω is a logarithmic time coordinate. The ratio of the Planck time to the Hubble time gives $\Delta\Omega \approx 1000$. However, 250 bounces requires $\Delta\Omega \approx 10^{60}$.

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H. Ringström, Class.Quant.Grav. 17 (2000) 713-731.

Is the Mixmaster singularity generic?
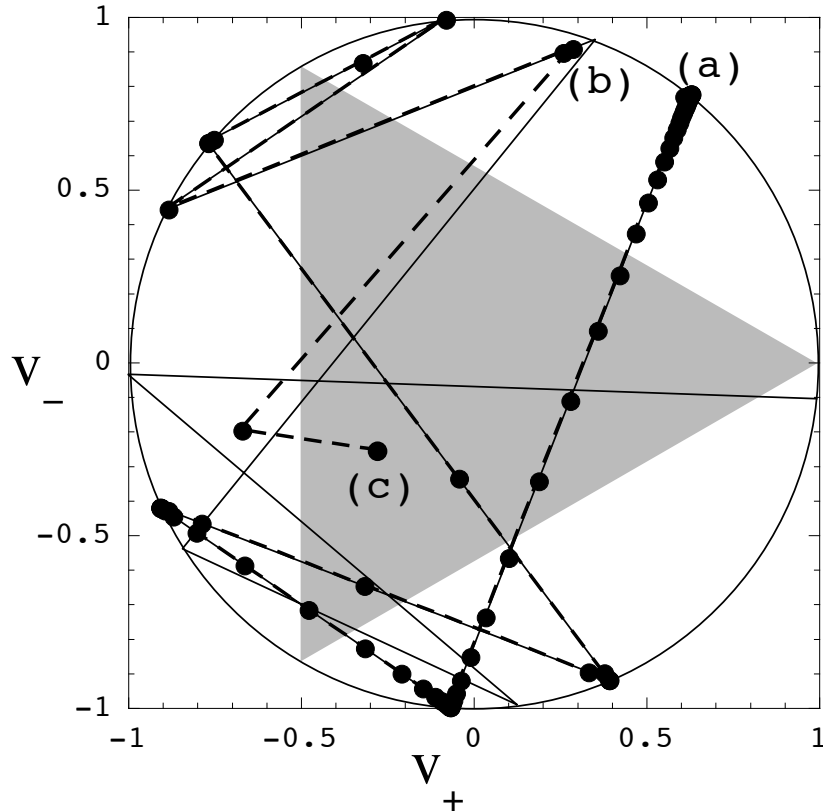
Minimally coupled
scalar field destroys
Mixmaster oscillations
because $v_+^2 + v_-^2 < 1$.

$$H = -p_\Omega^2 + p_+^2 + p_-^2 + \textcolor{red}{p_\varphi^2} + \textcolor{blue}{e^{4\Omega} V(\beta_+, \beta_-)} + \textcolor{red}{e^{6\Omega} V(\varphi)} = 0$$

Scalar fields and extra dimensions can cause a final "bounce."
Additional fields (e.g. magnetic) can restore Mixmaster dynamics by
adding walls.

Spatially inhomogeneous cosmological spacetimes:

BKL claim that sufficiently close to the singularity, spatial derivatives become dynamically irrelevant compared to time derivatives so that each spatial point evolves as a separate universe with either an AVTD or Mixmaster singularity.

toward
the
singularity
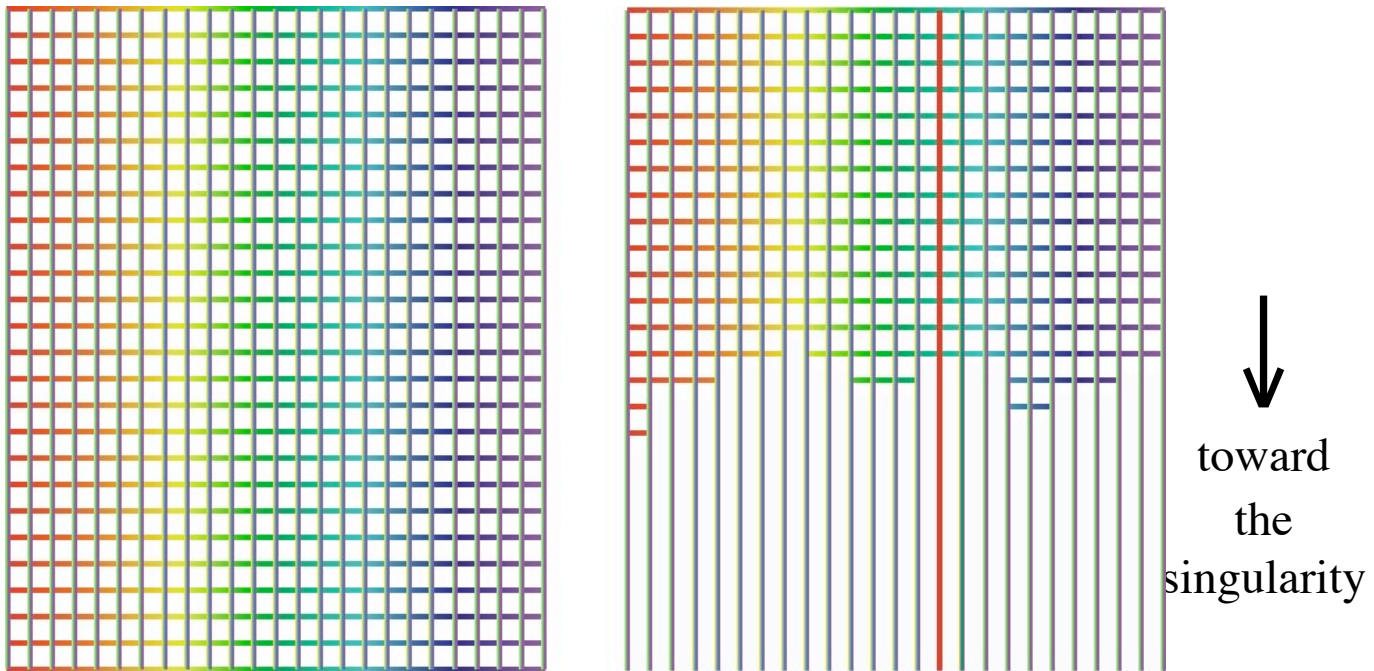
"Qualitative" studies of collapsing U(1) symmetric spatially inhomogeneous cosmologies with T^3 spatial topology:
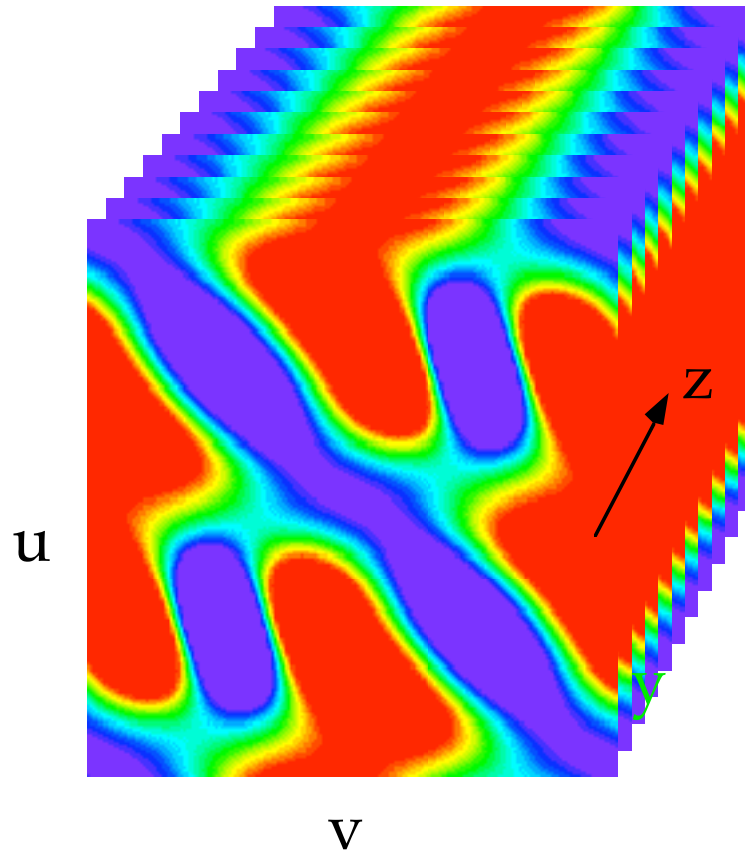
The BKL conjecture implies that eventually in a collapsing cosmology, we can consider an independent minisuperspace at every spatial point.

Do U(1) symmetric cosmologies exhibit local Mixmaster dynamics?

$$ds^2 = e^{-2\varphi}\,e^{-2\Lambda}d\tau^2 + e^{\Lambda}e_{ab}(x,z)dx^a dx^b + e^{2\varphi}\left(dx^3 + \beta_a dx^a\right)^2$$

Perform a canonical transformation: $(e^a, \beta_a) \rightarrow (r, \omega)$

Five degrees of freedom x, z, Λ, φ, ω depend on spatial variables u, v and time τ.

Problematical 2 + 1 numerics (for us) means that only qualitative signatures are reliable.

If these models exhibit LMD as claimed by BKL, what would it look like?

Berger, B. K. and Moncrief, V., Phys. Rev. D **58**, 064023 (1998).
Berger, B. K. and Moncrief, V., Phys. Rev. D **62**, 123501 (2000).

Mixmaster logarithmic scale factors
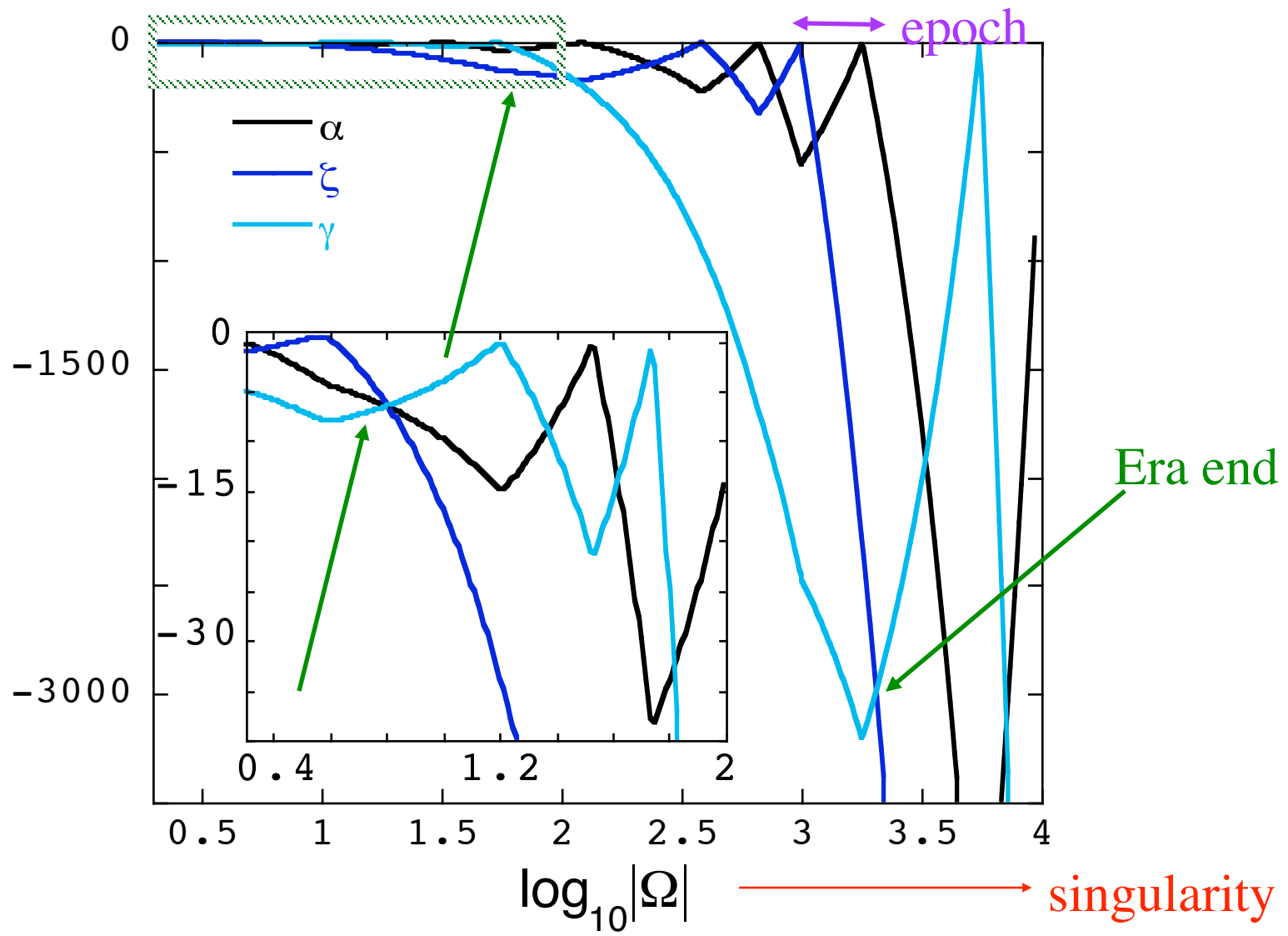
Mixmaster logarithmic scale factors
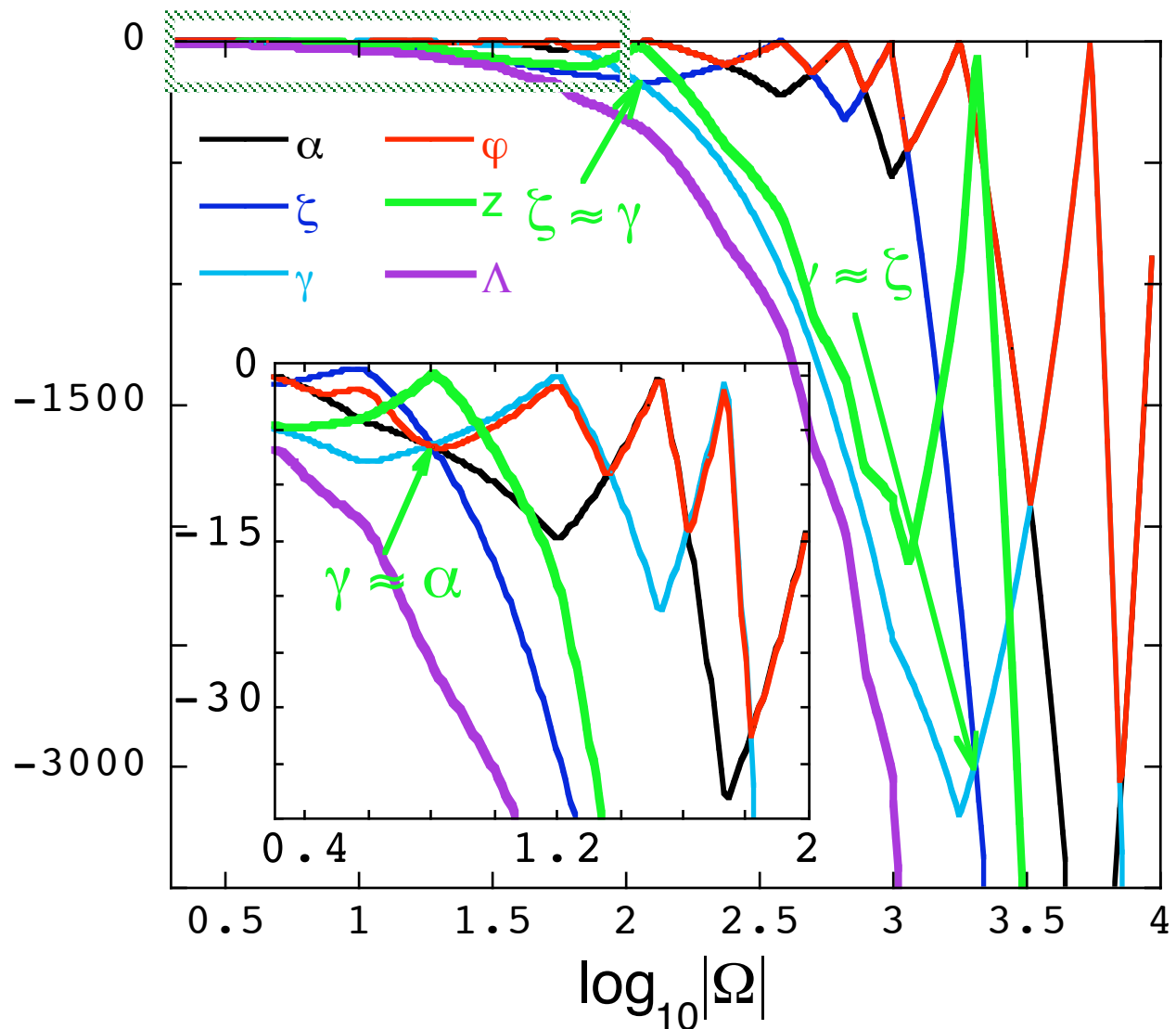
A robust signature for local Mixmaster dynamics in simulations of U(1) symmetric cosmologies:

- The variable φ should oscillate with bounces at different times at different spatial points.

- The variable z should decay monotonically most of the time but will occasionally grow at some spatial point when a Mixmaster era ends.

- In the U(1) variables, most simulations validate the MCP view that φ oscillates due to bounces off two competing terms in the Hamiltonian (constraint).

φ bounces in a local minisuperspace

$$V_2 \approx \tfrac{1}{2}(\nabla \omega)^2 \, e^{\Lambda - 2z - 4\varphi}$$

$$V_1 = \tfrac{1}{2} p_\omega^2 \, e^{4\varphi}$$

$$\dot{\varphi}^2$$

$$\Lambda - 2z - 4\varphi$$

$$\varphi$$

$$\perp \text{ directions}$$

$$\Lambda, z$$
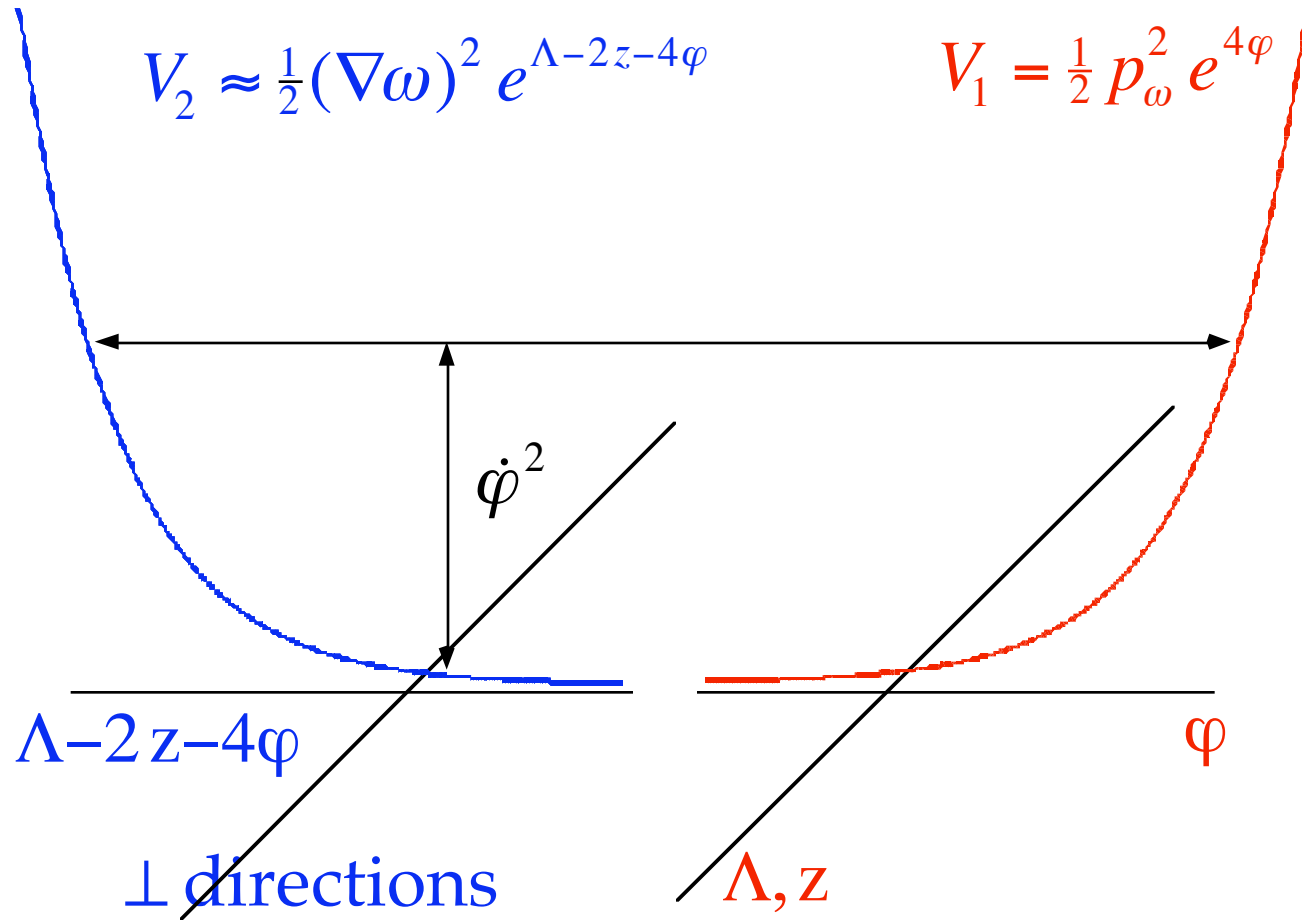
Note that z, Λ, and φ are involved.
Local bounce rules may be developed using conservation
of momentum.

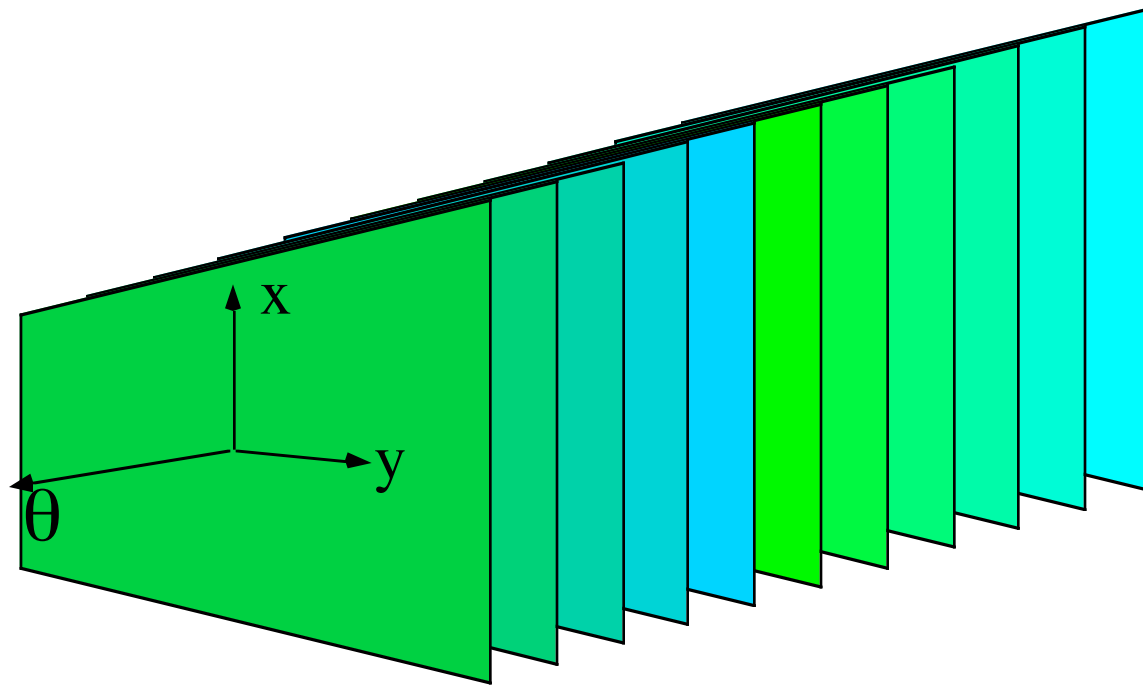At a typical spatial point in a U(1) simulation
(no approximations):
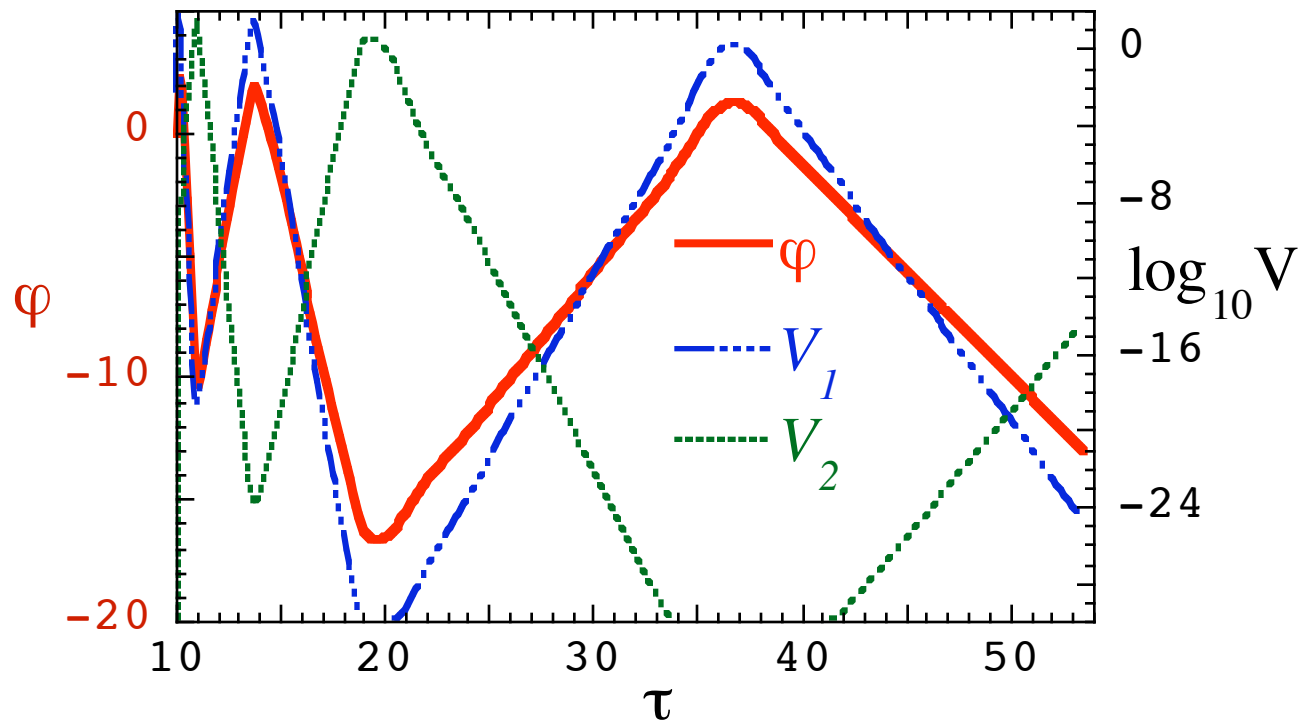
Example of z-bounces in a U(1) symmetric cosmology.
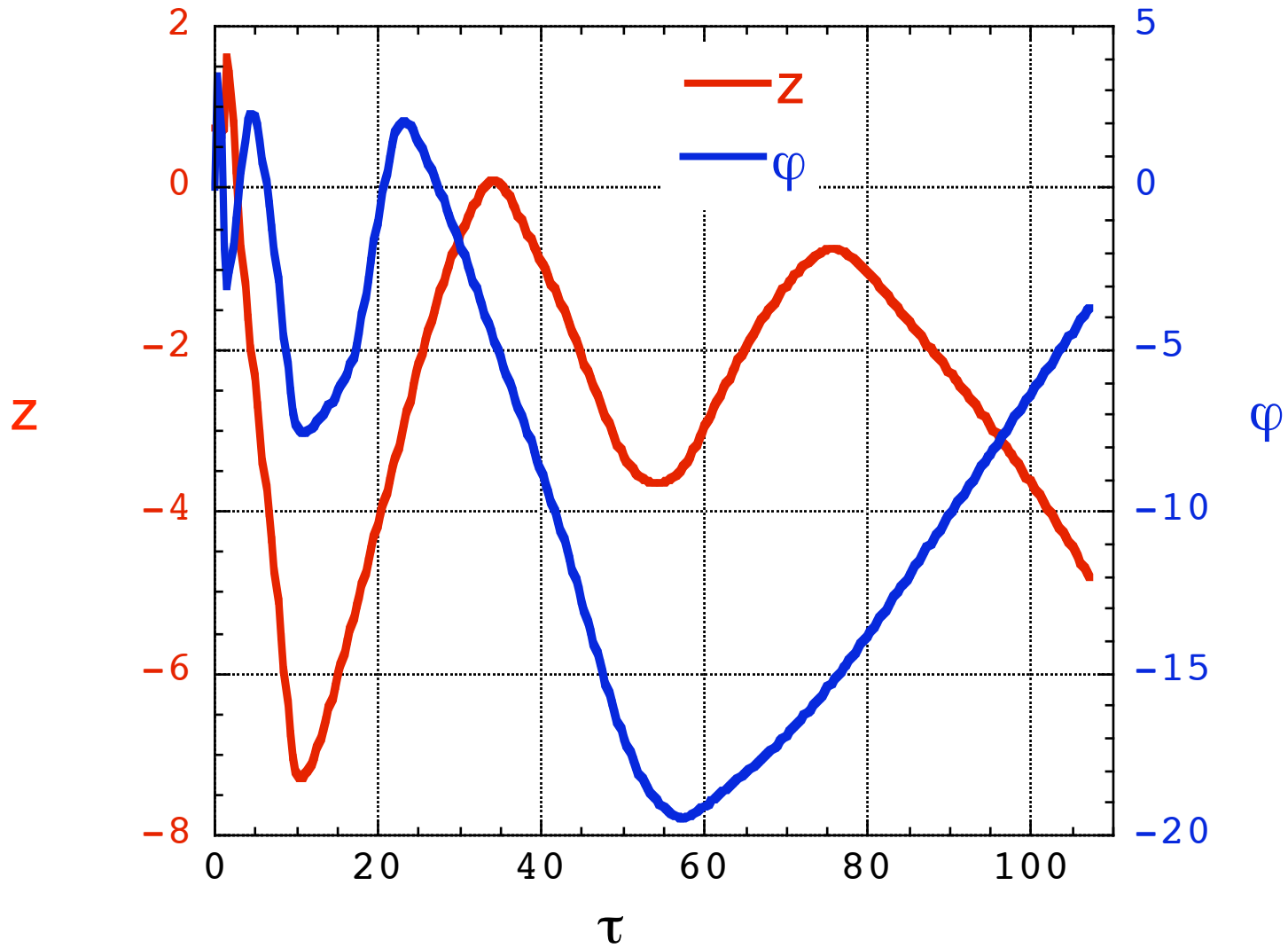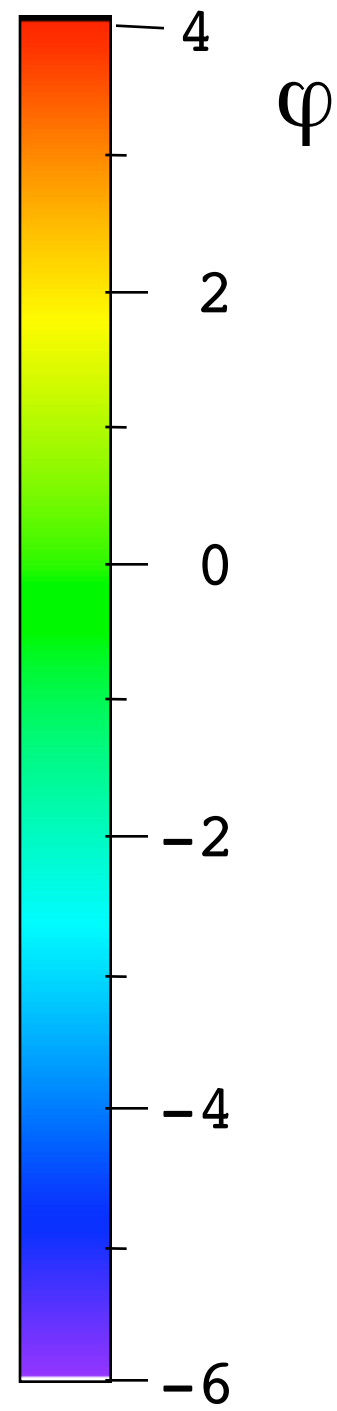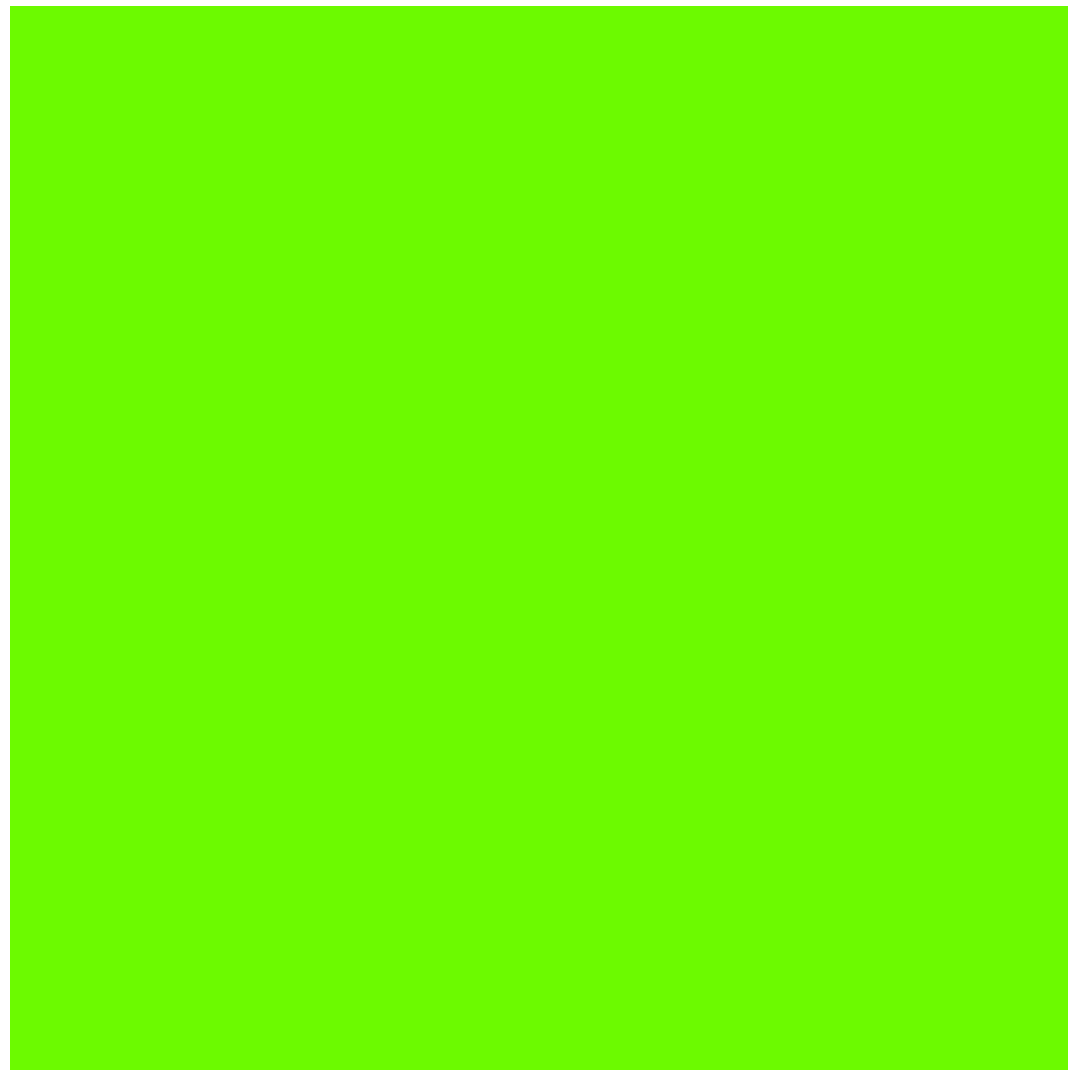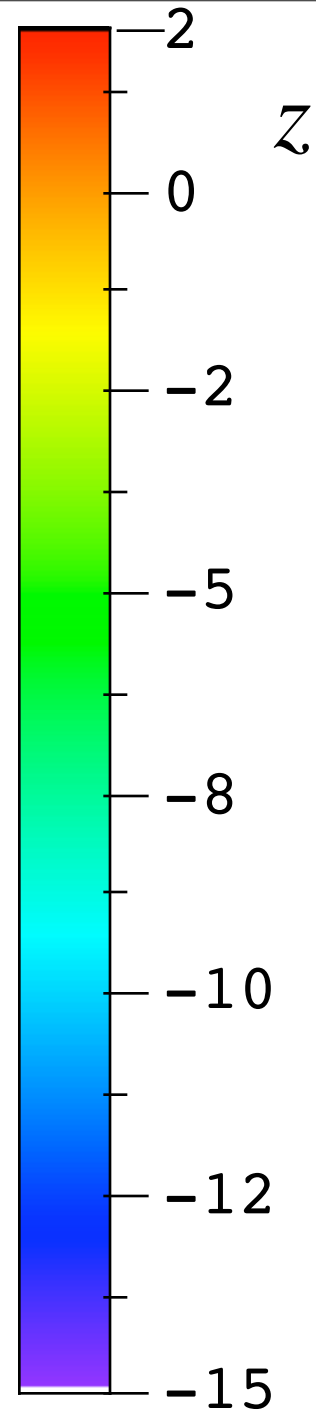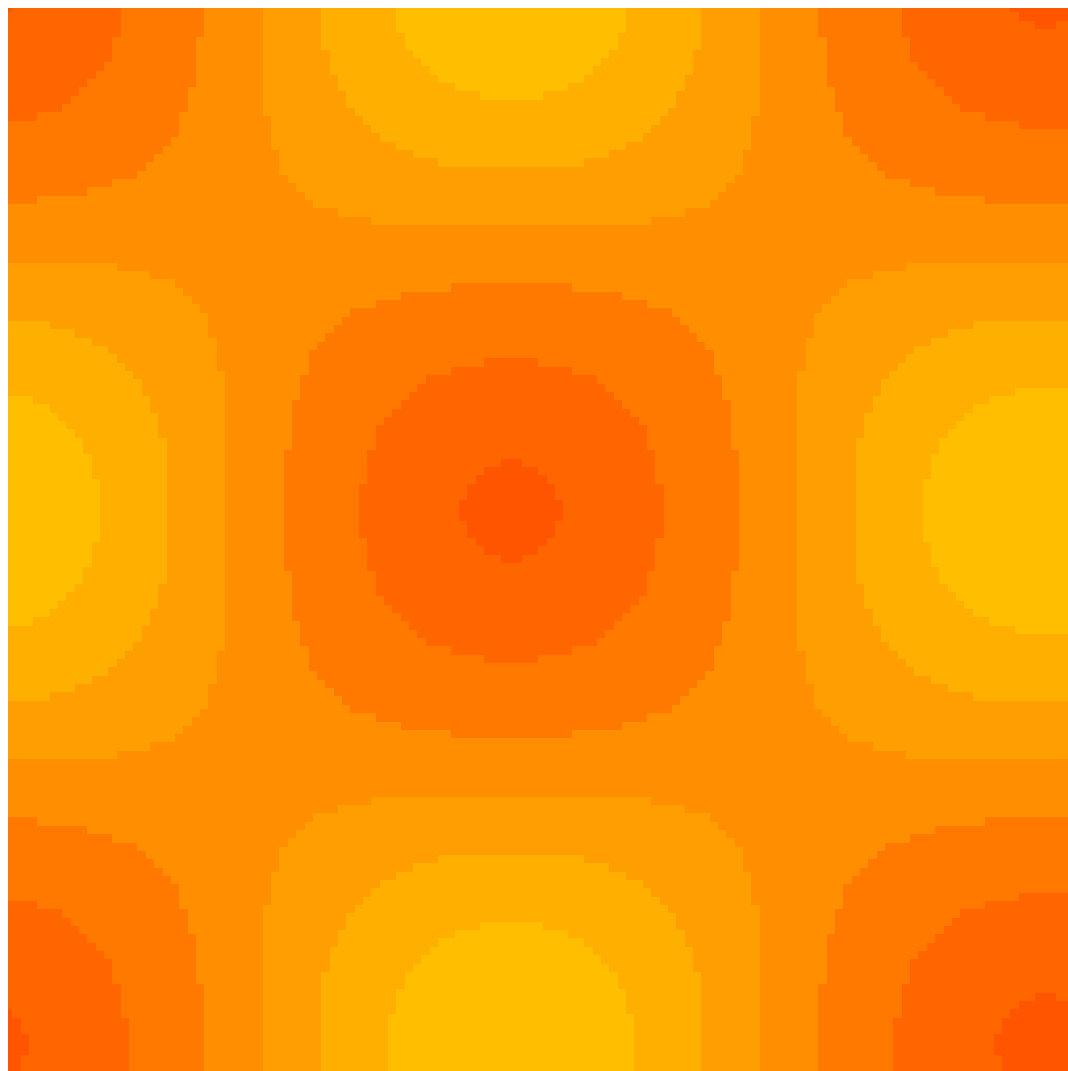
"Quantitative" studies of T^2 symmetric spatially inhomogeneous cosmologies with T^3 spatial topology:

Polarized Gowdy: spatial axes are fixed

Generic Gowdy: orientation of x and y axes changes with time

General T^2 symmetric: orientation of all spatial axes depends on time

$$ds^2 = e^{(\lambda+\tau)/2} \left(-e^{-2\tau} d\tau^2 + d\theta^2 \right) + e^{P-\tau} \left(d\sigma + Q d\delta \right)^2 + e^{-P-\tau} d\delta^2$$

Einstein's equations consist of wave equations for P and Q and constraints which may be solved for λ. The wave equations may be obtained by variation of

$$2\mathcal{H} = \pi_P^2 + e^{-2P} \pi_Q^2 + e^{-2\tau} P_{,\theta}^2 + e^{2(P-\tau)} Q_{,\theta}^2$$

where $\mathcal{H} \neq 0$.

As $\tau \to \infty$, the VTD solution (neglect spatial derivatives) is

$$P(\theta,\tau) \to v(\theta)\,\tau \quad , \quad \pi_P(\theta,\tau) \to v(\theta)$$

$$Q(\theta,\tau) \to Q^0(\theta) \quad , \quad \pi_Q(\theta,\tau) \to \pi_Q^0(\theta)$$

Terms in the Hamiltonian act as potentials. For AVTD behavior of the model, these potentials must decay exponentially.

$$V_1 = e^{-2P}\, \pi_Q^2 \rightarrow e^{-2v\tau}\left(\pi_Q^0\right)^2$$

requires $v > 0$ for consistency.

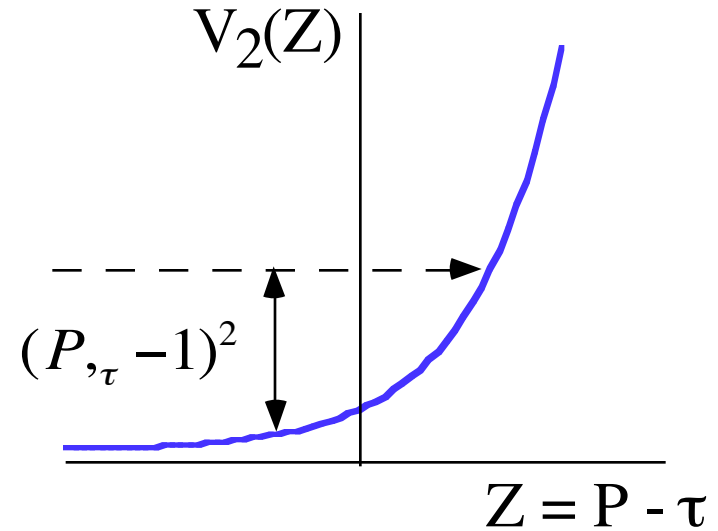$$V_2 = e^{2(P-\tau)}\left(Q_{,\theta}\right)^2 \rightarrow e^{2(v-1)\tau}\left(Q_{0,\theta}\right)^2$$

requires $v < 1$ for consistency.

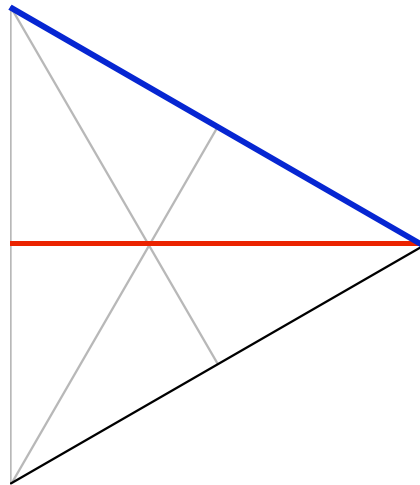Thus the MCP predicts that the singularity is AVTD (at any spatial point) only if $0 \leq v < 1$.

Competing Potentials in Equation for P

$V_1(P)$

$P,_\tau^2$

P

$V_2(Z)$

$(P,_\tau - 1)^2$

Z = P - τ
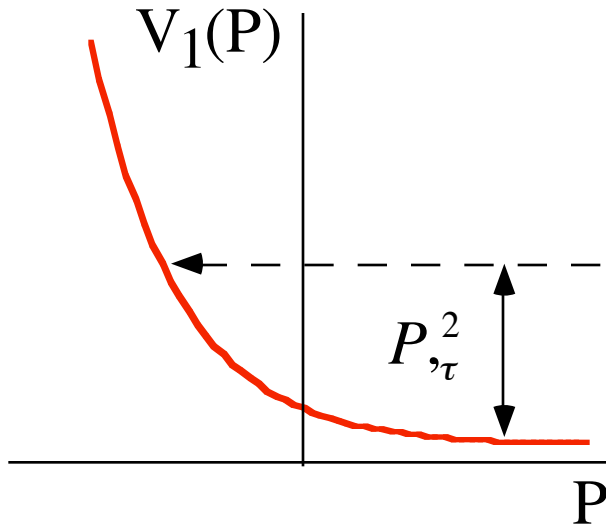
$V_1 = e^{-2P}\pi_Q^2$

$V_2 = e^{2(P-\tau)}Q,_\theta^2$

Gowdy walls as
Mixmaster MSS

Numerical simulations show how v is driven into the range (0,1) by bounces off the potentials. A typical single spatial point is shown.
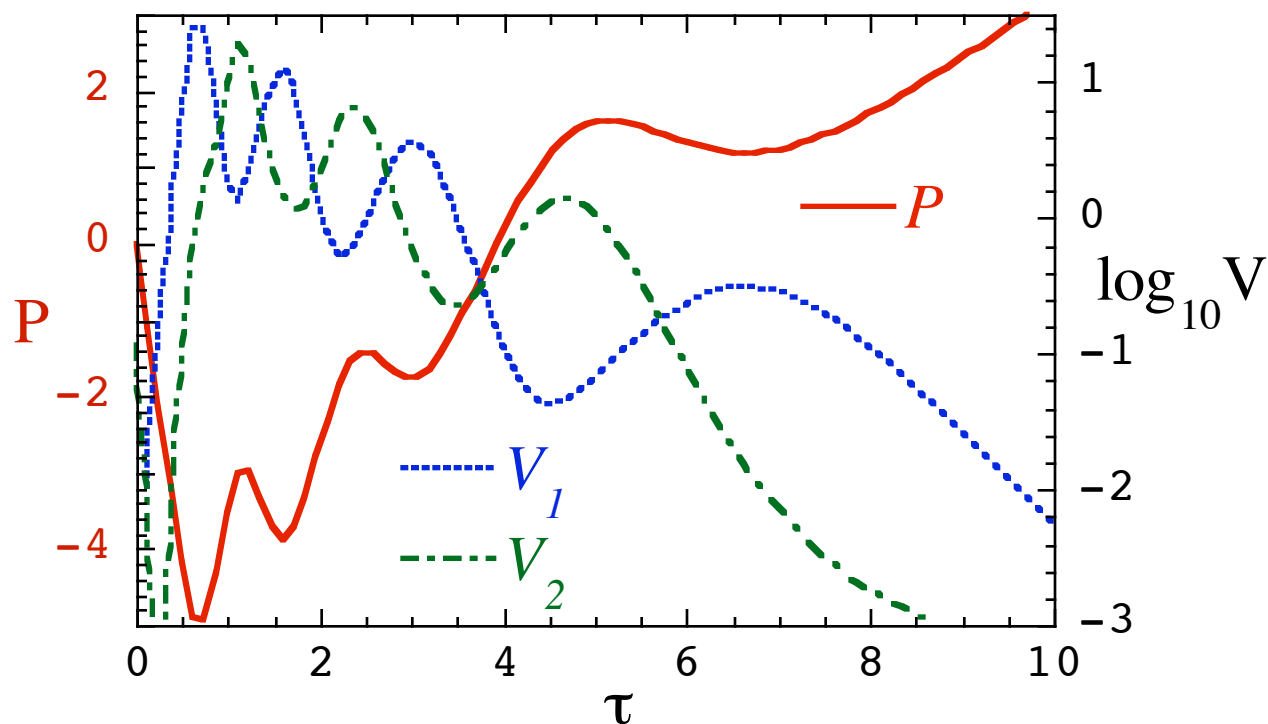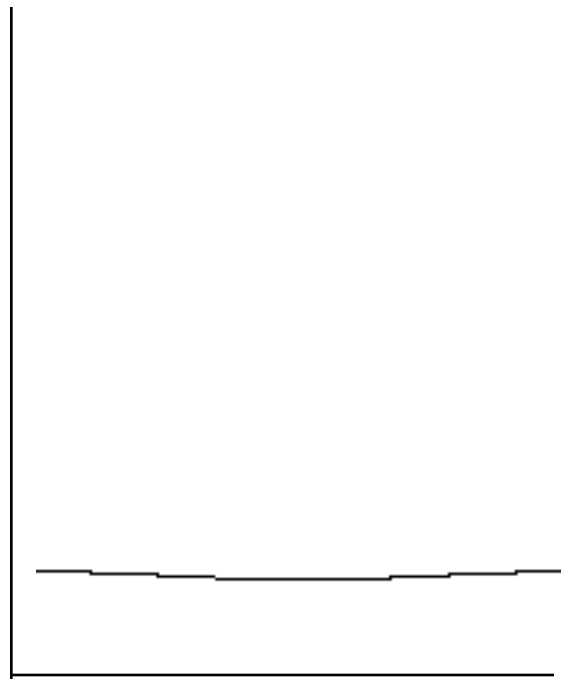

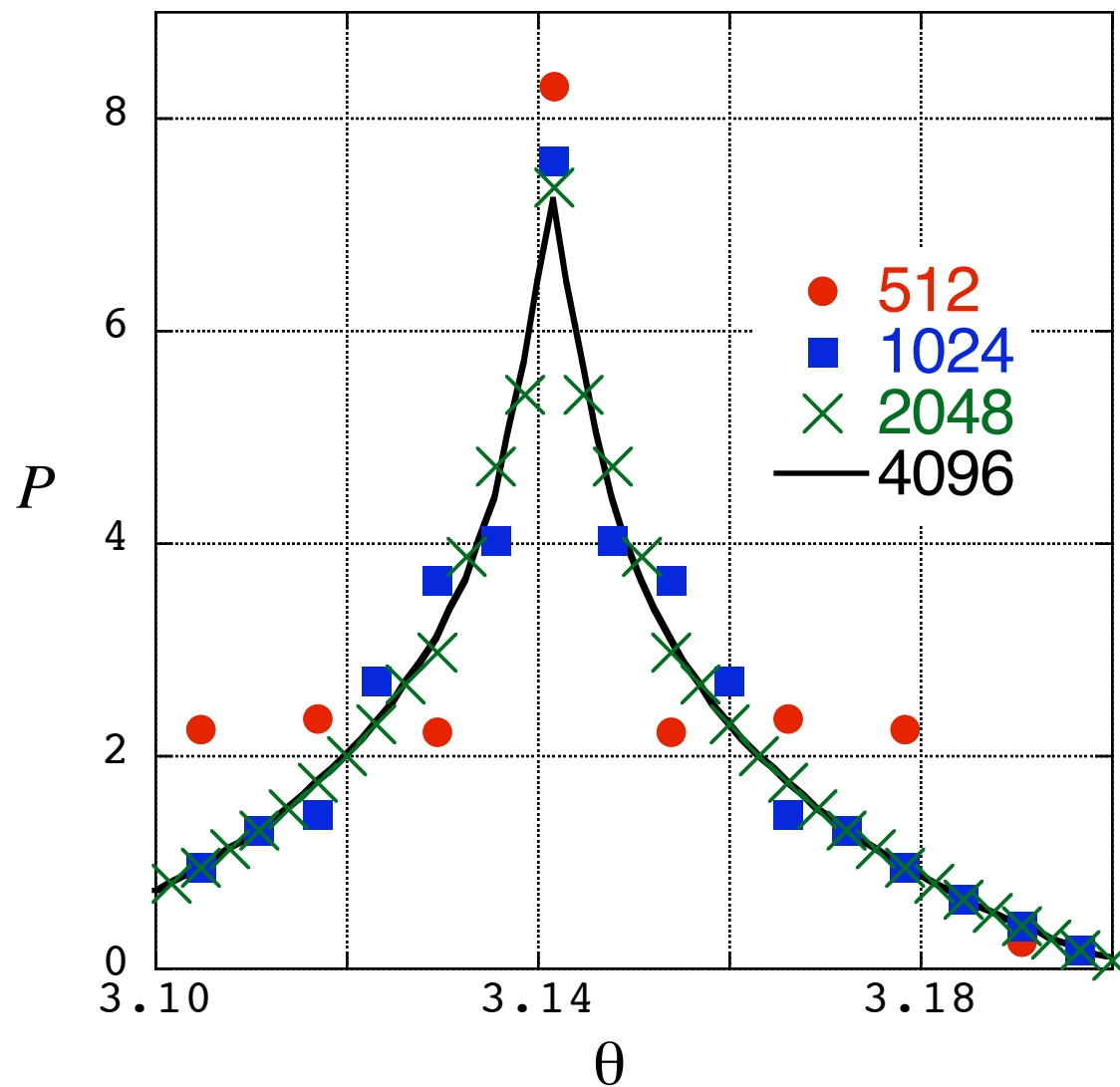B.K. Berger, D. Garfinkle, Phys. Rev. D **57**, 4767 (1998).

P

θ

Q

θ
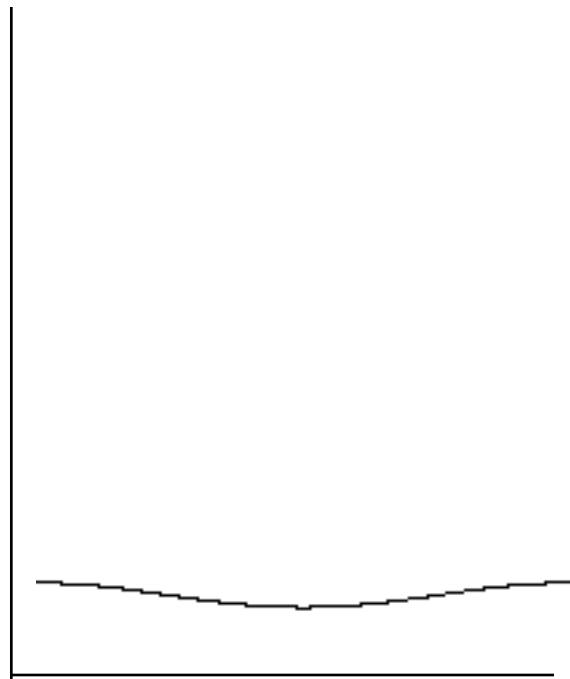
The spiky features offer a challenging code test:

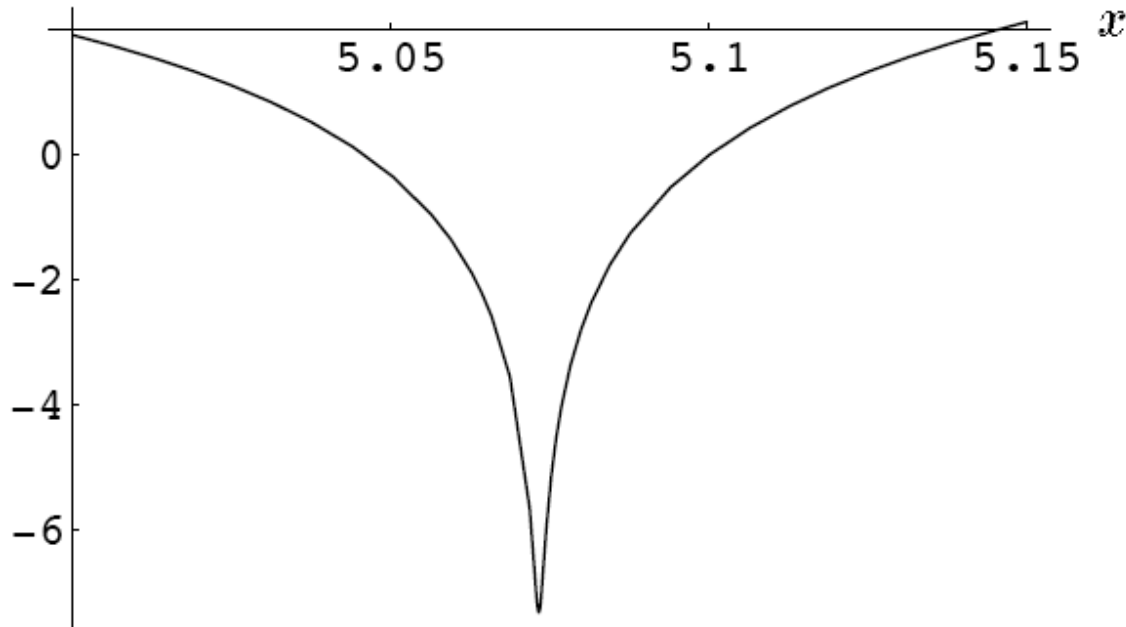## Not quite resolved feature

But are understood mathematically:

Figure 1: P_1 in a neighbourhood of x_{spike} at small t.

A.D. Rendall, M. Weaver, "Manufacture of Gowdy Spacetimes with Spikes," CQG **18**, 2959 (2001)

General T^2 symmetric spacetime: Gowdy: $\kappa = 0$, $\pi_\lambda = \frac{1}{2}$

$$ds^2 = -e^{(\lambda - 3\tau)/2}d\tau^2 + e^{(\lambda + \mu + \tau)/2}d\theta^2$$

$$+ e^{P-\tau}[d\sigma + Qd\delta + \left(\int^\tau (Q\Theta) - Q \int^\tau \Theta\right)d\theta]^2$$

$$+ e^{-P-\tau}[d\delta - (\int^\tau \Theta) d\theta]^2$$

where $\Theta = e^{(\lambda + 2P + 3\tau)/2}e^{\mu/4}\kappa$

Hamiltonian formulation:

$$H = H_0 + H_{small} + H_{kin} + H_{curv} + H_{twist}$$

$$H = \frac{\pi_P^2}{4\pi_\lambda} + \frac{P,_\theta^2\, e^{-2\tau}}{4\pi_\lambda} + \frac{\pi_Q^2 e^{-2P}}{4\pi_\lambda}$$

$$+ \frac{Q,_\theta^2\, e^{2(P-\tau)}}{4\pi_\lambda} + \sigma\kappa^2\pi_\lambda e^{(\lambda + 2P + 3\tau)/2}$$

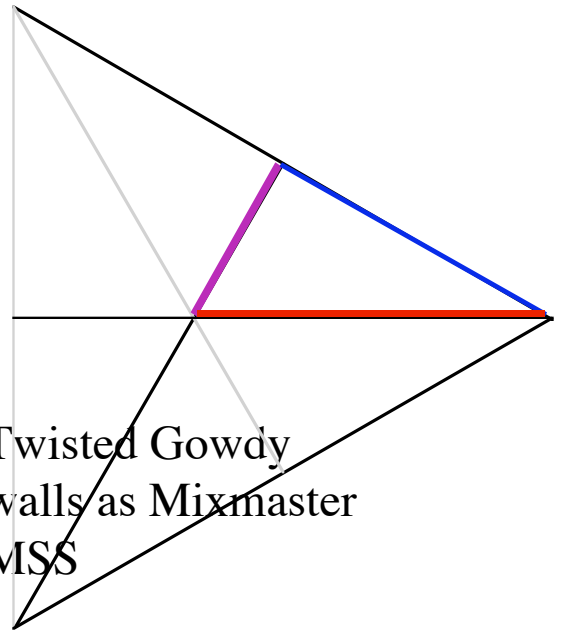B.K. Berger, J. Isenberg, M. Weaver, PRD **64**, 084006 (2001)

$$H = H_K + H_C + H_G$$

$$= \frac{1}{4\pi_\lambda}\left(\pi_P^{\ 2} + e^{-2P}\pi_Q^{\ 2}\right)$$

$$+ \frac{1}{4\pi_\lambda}\left(e^{-2\tau}P_{,\theta}^{\ 2} + e^{2(P-\tau)}Q_{,\theta}^{\ 2}\right)$$

$$+ \pi_\lambda\, e^{(\lambda+2P+3\tau)/2}\, K^2$$

twist potential

Twisted Gowdy
walls as Mixmaster
MSS

$$w = \frac{v}{2\pi_\lambda^0}$$

The VTD solution is

$$P(\theta, \tau) \rightarrow w(\theta)\,\tau \quad , \quad \pi_P(\theta, \tau) \rightarrow w(\theta)$$

$$Q(\theta, \tau) \rightarrow Q^0(\theta) \quad , \quad \pi_Q(\theta, \tau) \rightarrow \pi_Q^0(\theta)$$

$$\lambda(\theta, \tau) \rightarrow -w(\theta)^2\tau \quad , \quad \pi_\lambda(\theta, \tau) \rightarrow \pi_\lambda^0(\theta)$$

The Gowdy potentials restrict the range of w.
However, if $-1 < w < 3$

$$V_3 = \kappa^2\,\pi_\lambda\,e^{\lambda+2P+3\tau} \rightarrow \kappa^2\,\pi_\lambda\,e^{-(w+1)(w-3)\tau}$$

will grow exponentially.

P(θ)

$\kappa = 0$

P

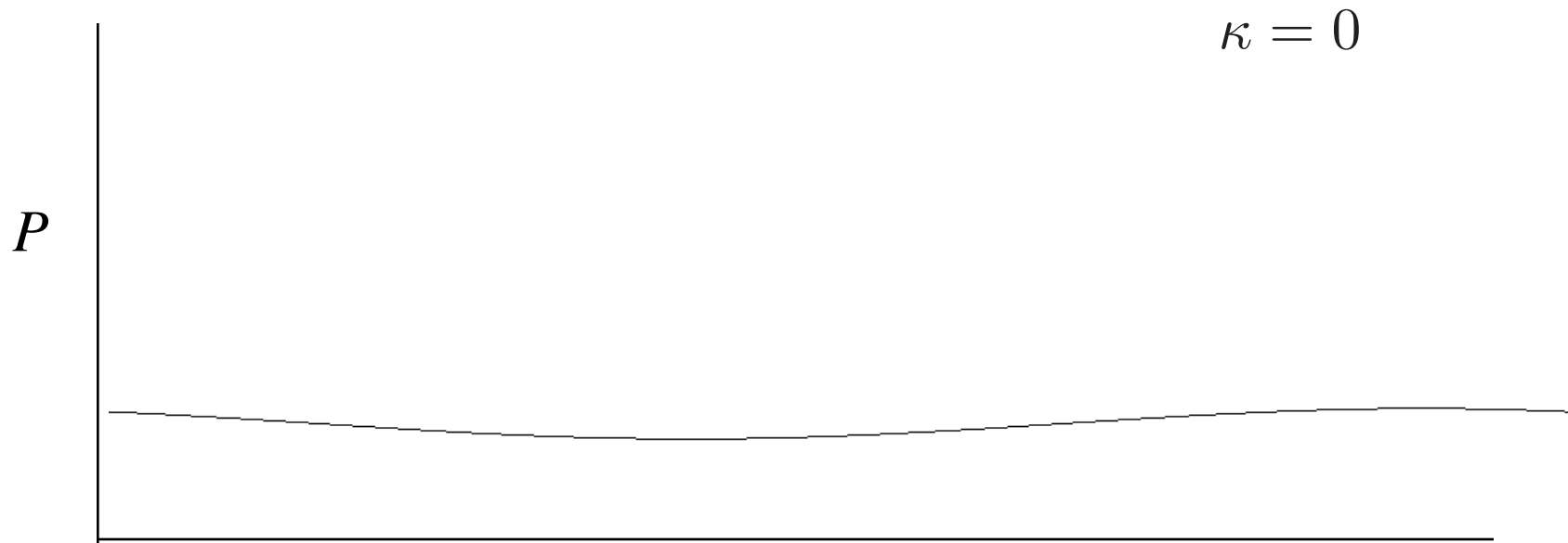θ

$\kappa = 1$

P

θ

$\kappa = 0$

P

θ

$\kappa = 1$
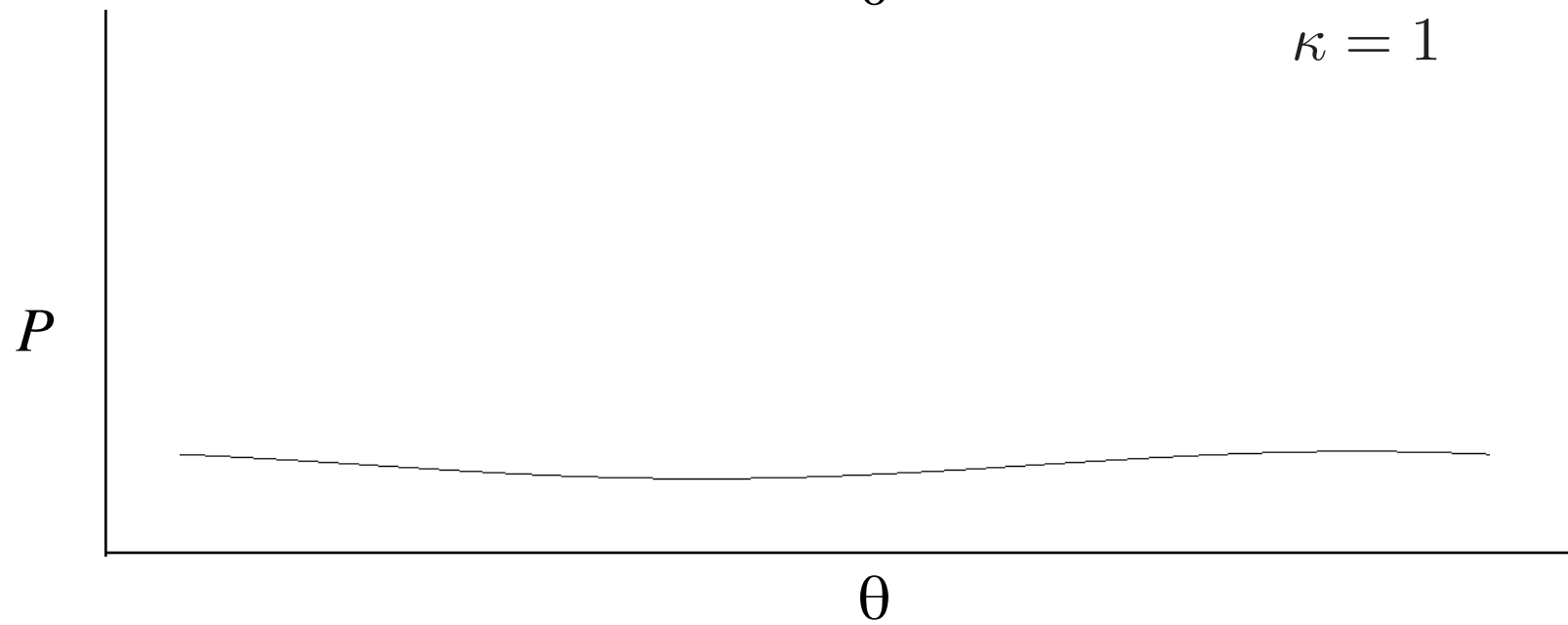
P

θ

P

$\kappa = 0$

θ

P

$\kappa = 1$

θ
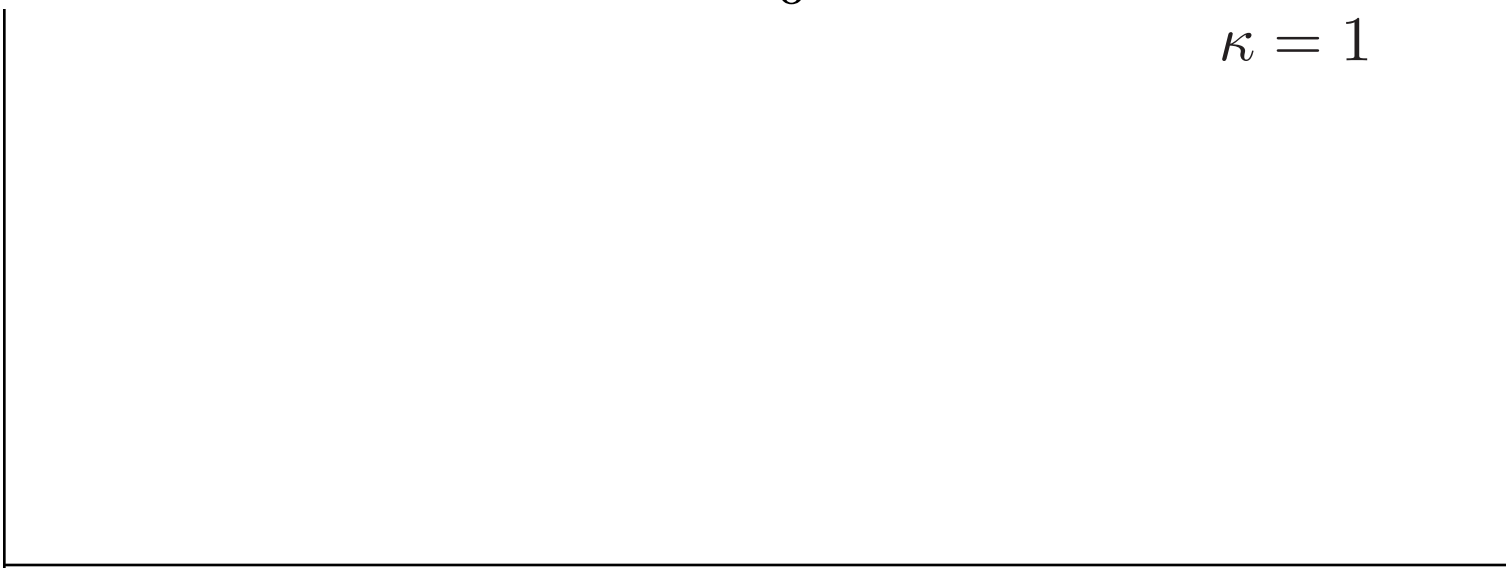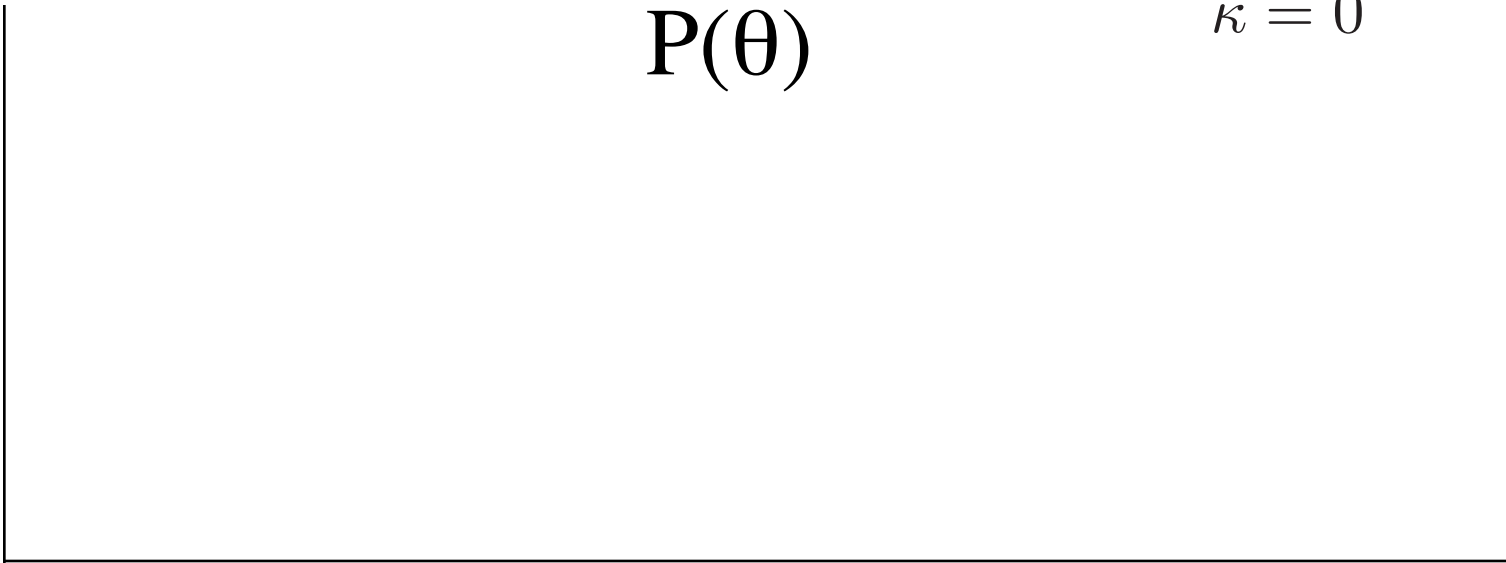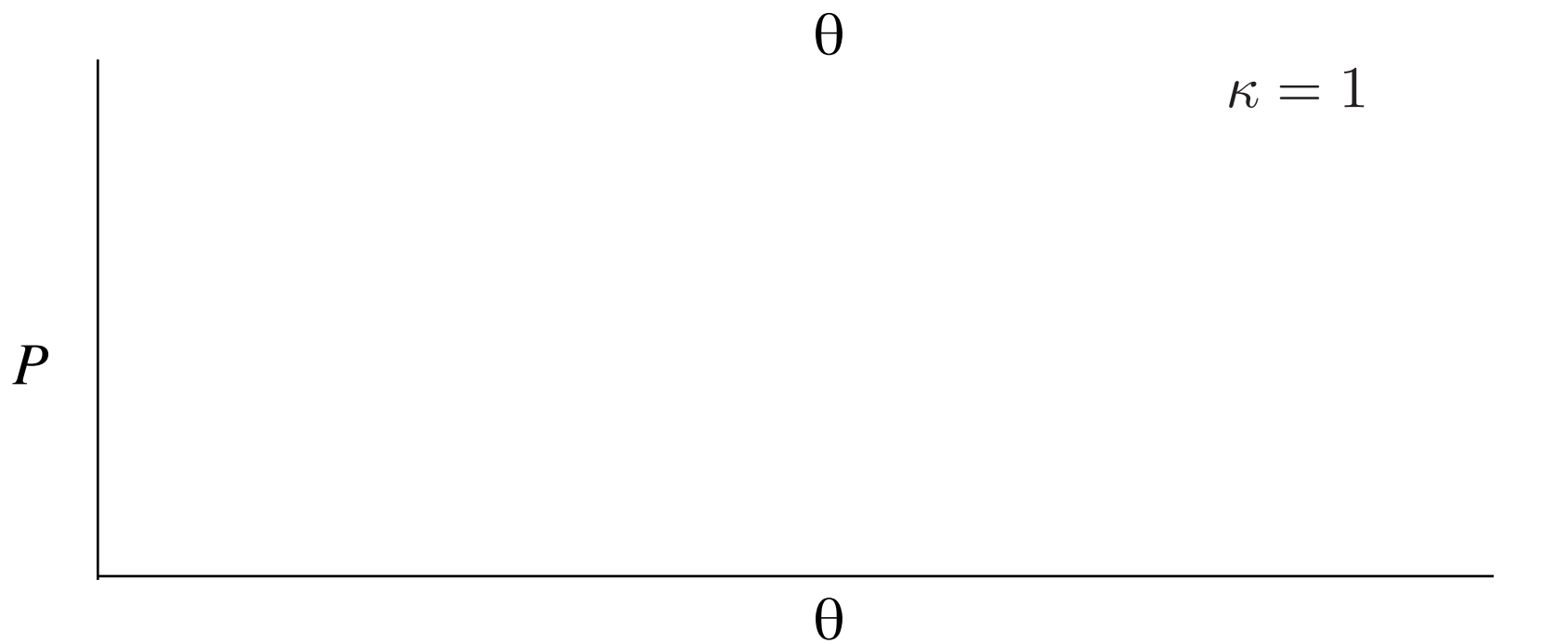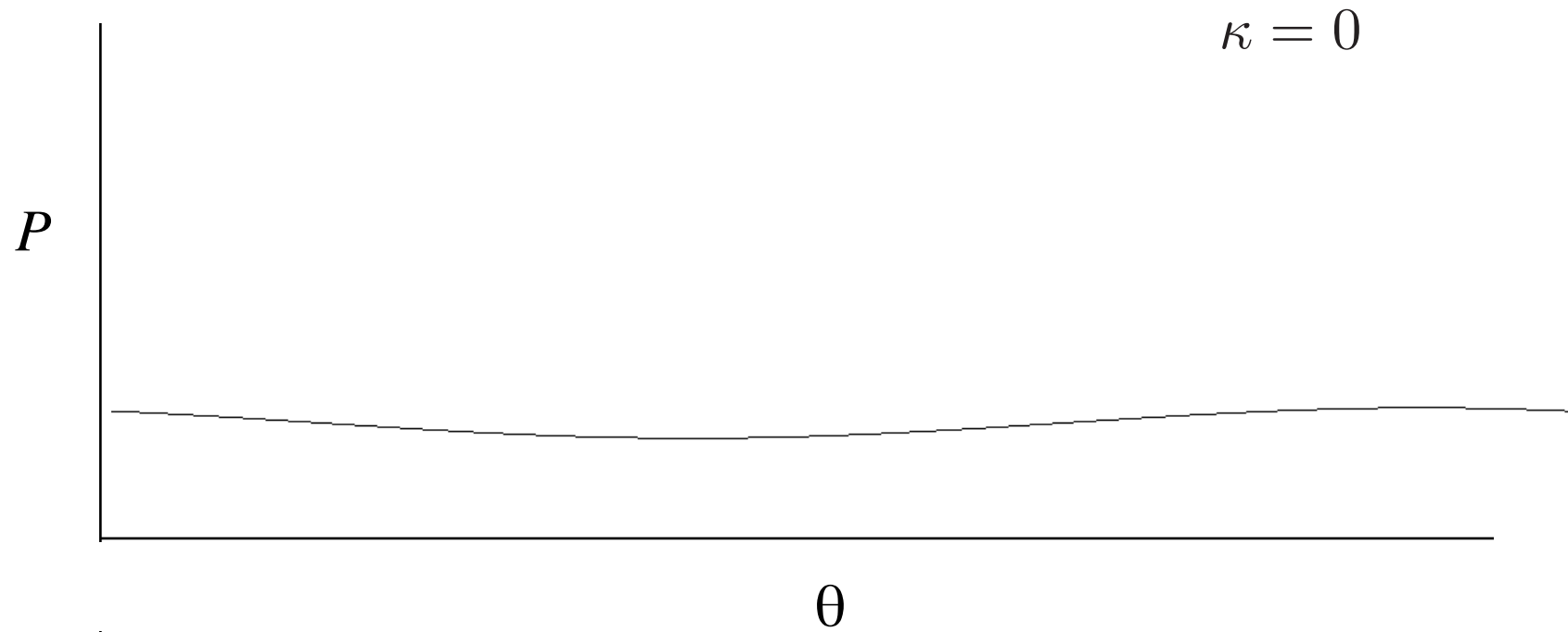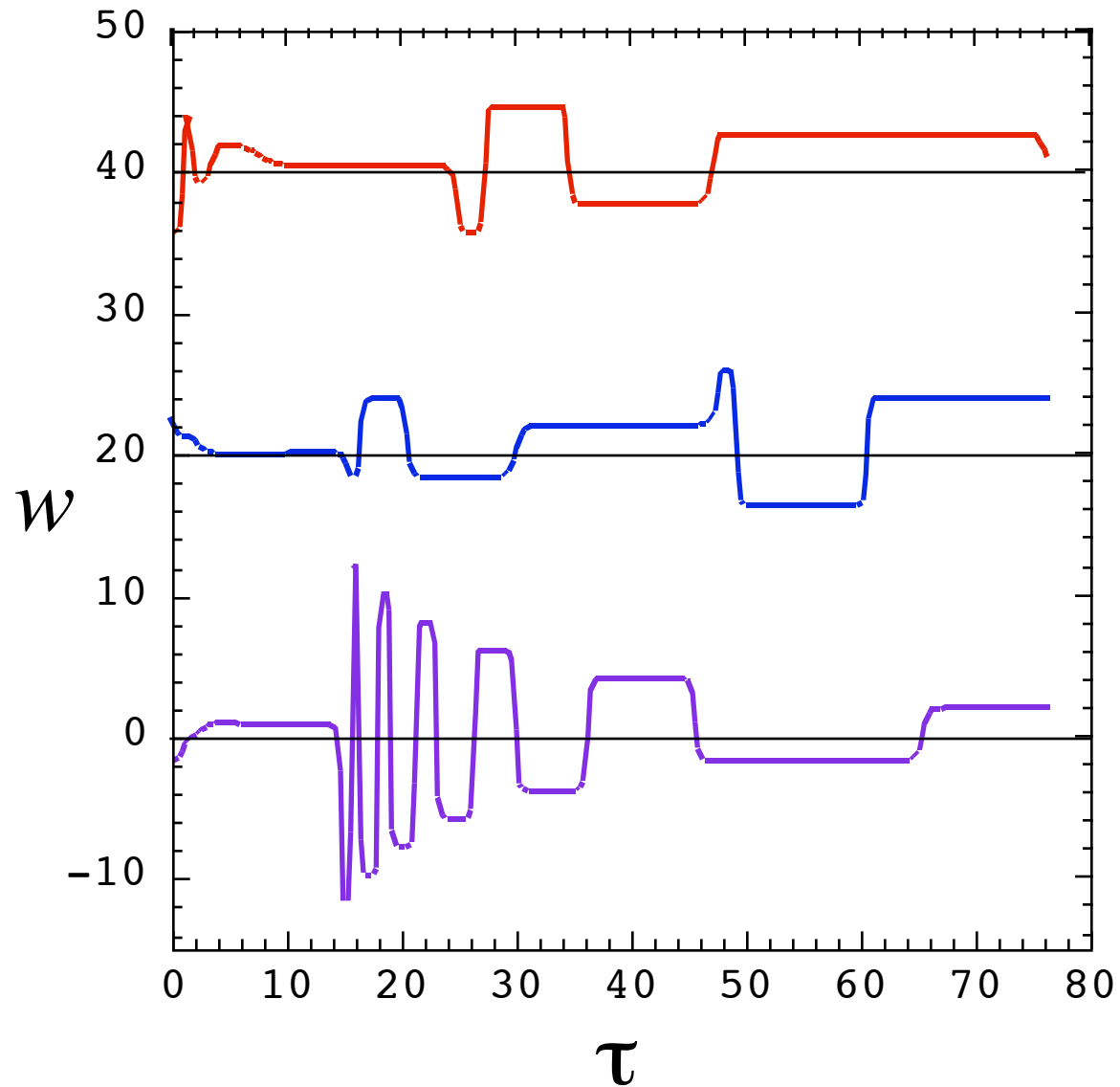
Behavior at 3 typical points (offset for clarity): $w = \dfrac{\pi_P}{2\pi_\lambda}$

Predictions for the next value of *w* after a bounce using conservation of momentum with exponential potentials:

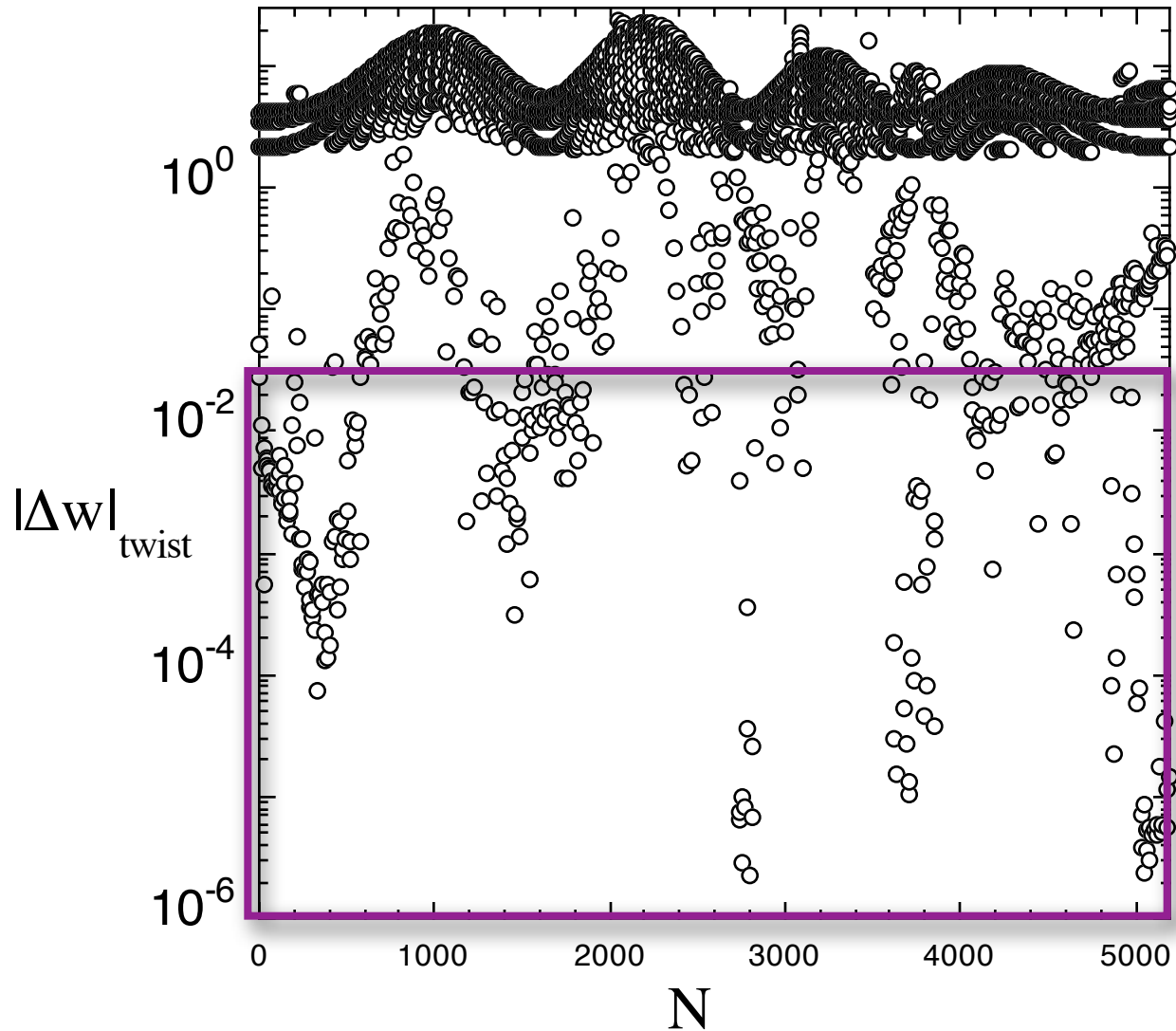| Bounce type	Bounce rule
Kinetic	$w' = -w$
Curvature	$w' = 2 - w$
Twist	$w' = \dfrac{w+3}{w-1}$
Curvature-Twist	$w' = \dfrac{w-5}{w-1}$
Kinetic-Twist	$w' = \dfrac{3-w}{w+1}$

Compare the sequences
of w's with those
predicted
by all the bounce rules.

The actual bounce is
of the type with the
smallest error.

3 spatial points are
shown.
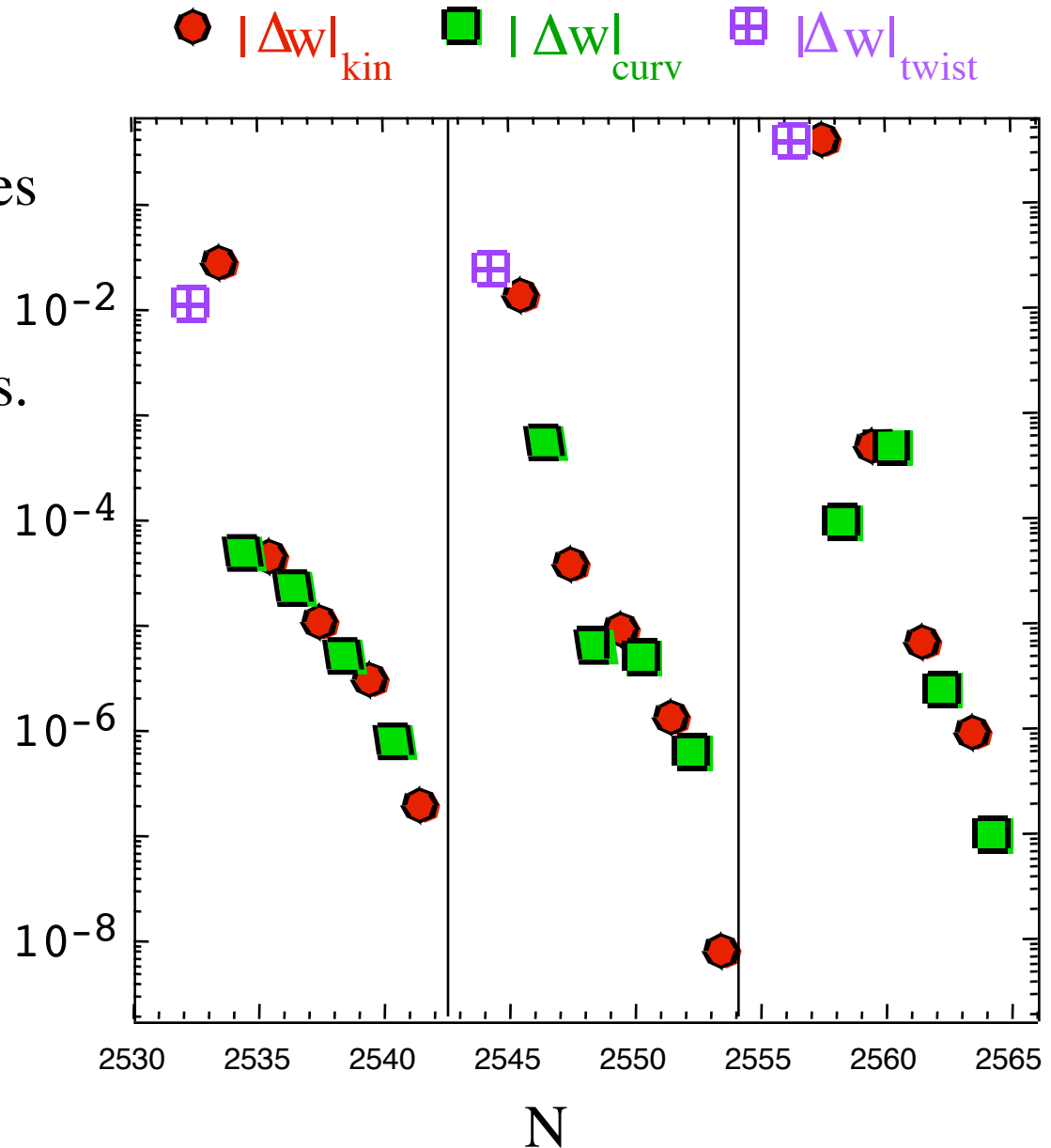
Second twist bounces indicated by small errors for twist bounce rule:

Open question

- Twisted Gowdy models are a subclass of U(1) symmetric models.

- Twist bounces are understood in terms of non-diagonal mixmaster models.

- All features of U(1) simulations are understood in terms of diagonal mixmaster.

- Where are the twist bounces in U(1) models?

To identify the U(1) variables, we must use the gauge $N = e^{-\mu/4}\, e^{\Lambda}$ rather than $N = e^{\Lambda}$ which was used in the U(1) simulations where $e^{\mu/4} = 2\pi_{\Lambda}$.
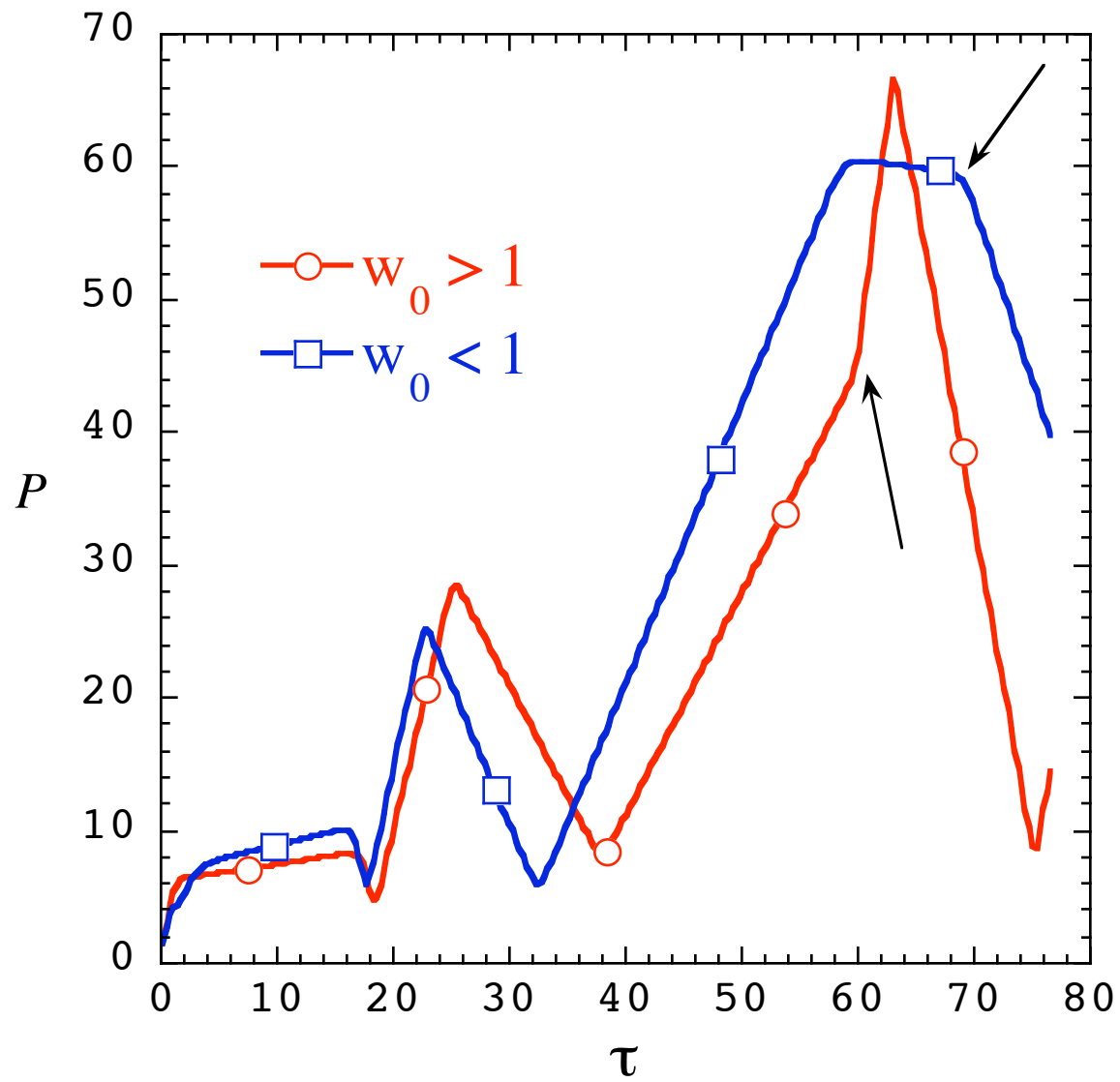
We find

$$2\varphi = P - \tau$$

$$\Lambda = \frac{\lambda}{4} + \frac{P}{2} + \frac{3\tau}{4} + \frac{\mu}{4}$$

$$e^{-2z} = e^{\Lambda} + e^{-\Lambda}\left(1 + \int^{\tau}\Theta\right)^2$$
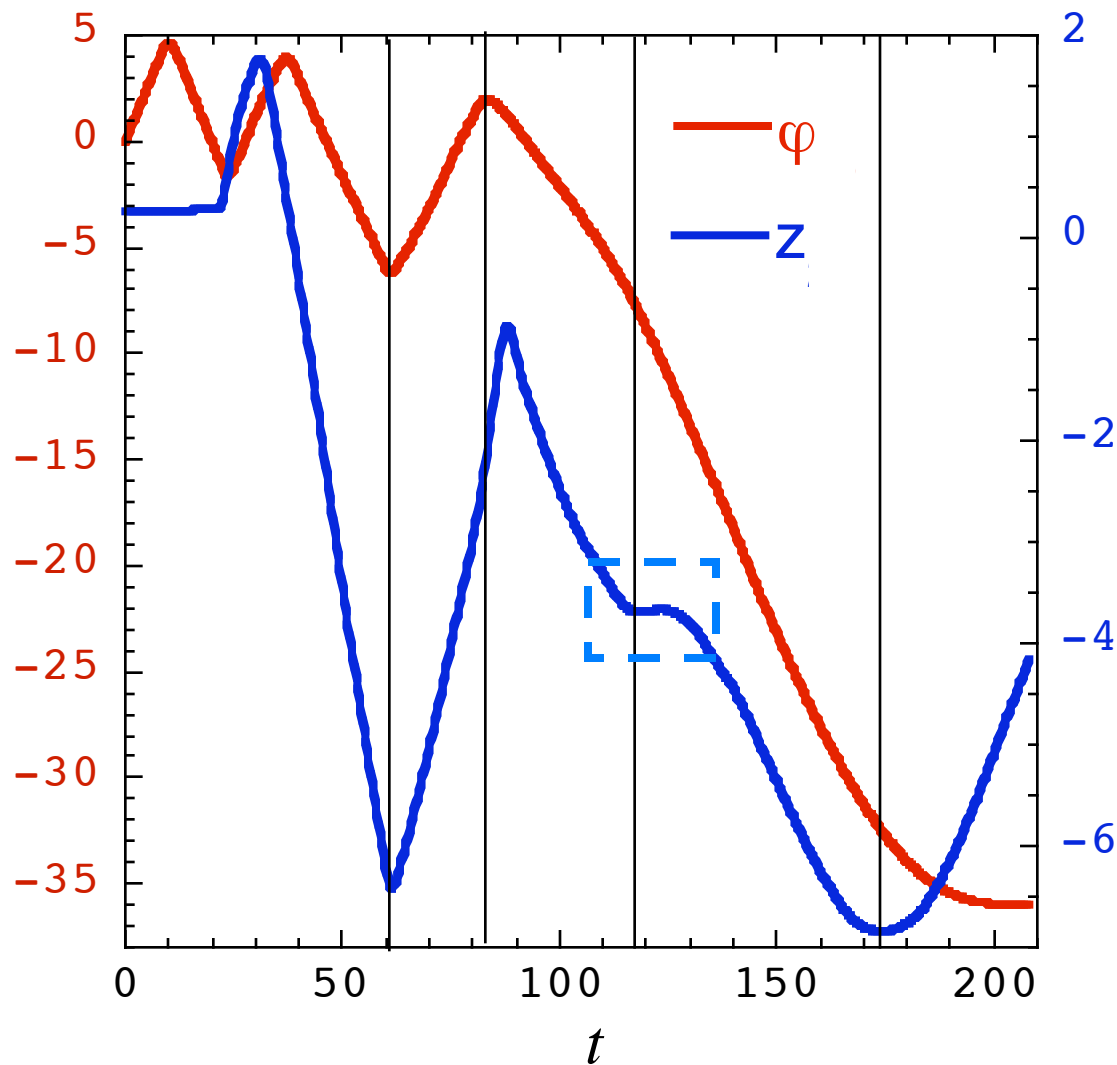
The twist potential is

$$V_3 = \frac{1}{8}p_z^2 + \frac{1}{2}p_x^2 e^{4z}.$$

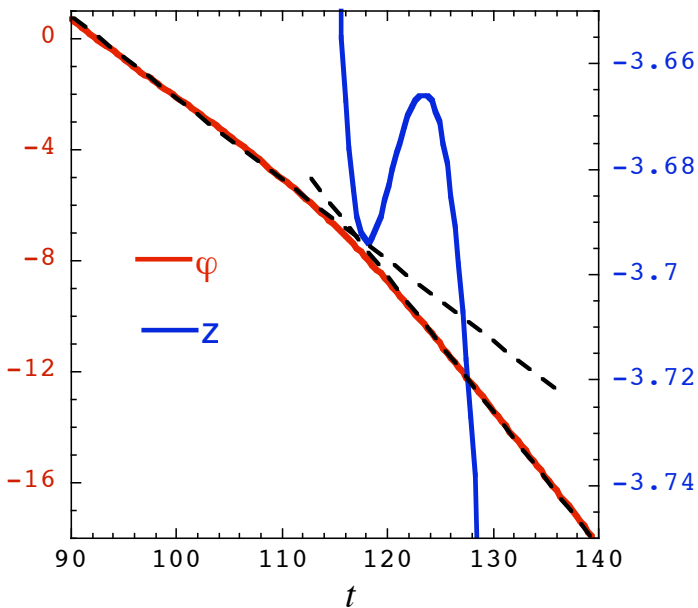# Twist bounce signatures in P.

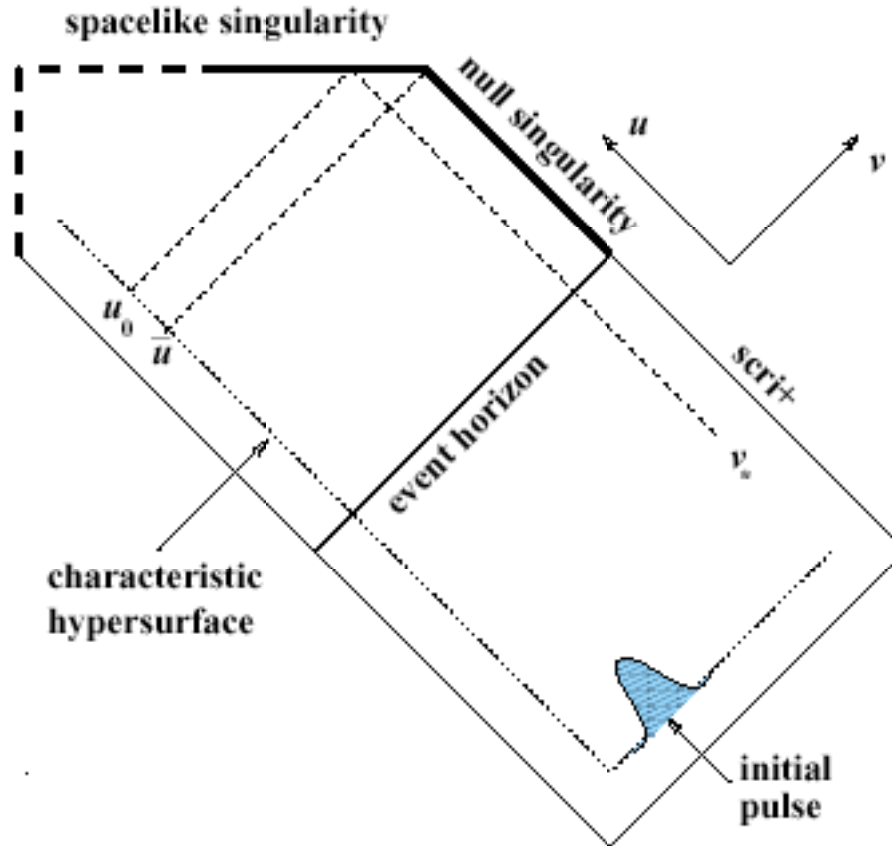# Use a more general gauge condition:

φ and z for a U(1) simulation in the new gauge

Twist bounce in a U(1) simulation

# Comment on singularities inside black holes:

Analytic and numerical evidence indicates that the singularity inside a Riessner-Nordstrom or Kerr black hole is first seen to be null and weak. (An infalling observer would not experience infinite tidal forces.) Many infalling observer world lines end at the null singularity. It eventually becomes spacelike and strong.

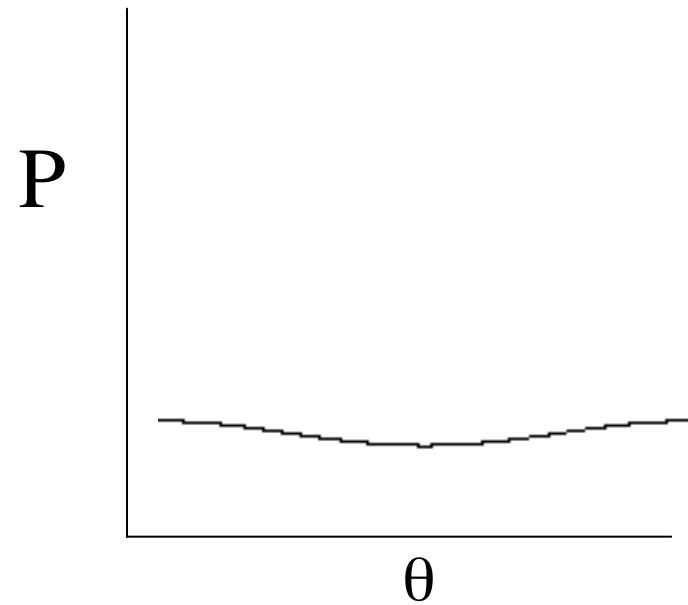P.R. Brady, J.D. Smith, Phys. Rev. Lett. **75**, 1256 (1995).
L.M. Burko, Phys. Rev. D **59**, 024011 (1999).

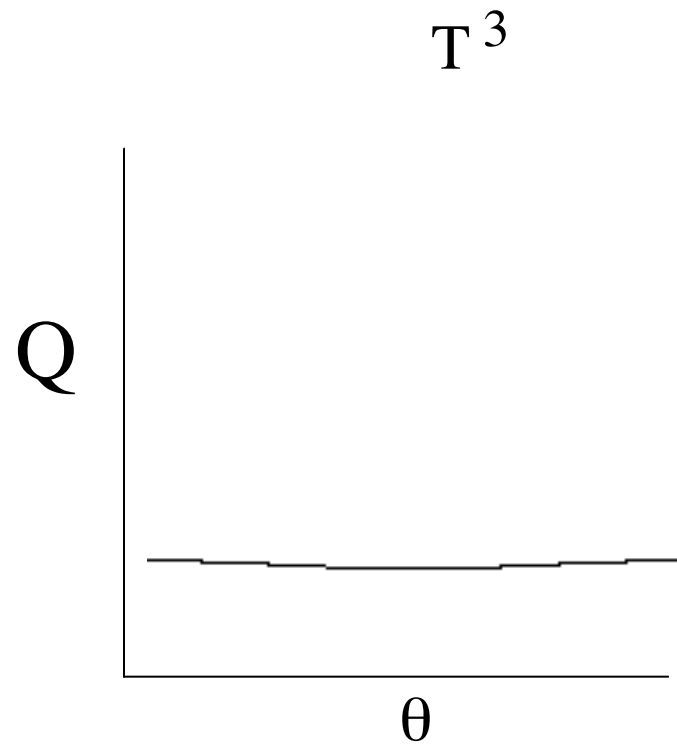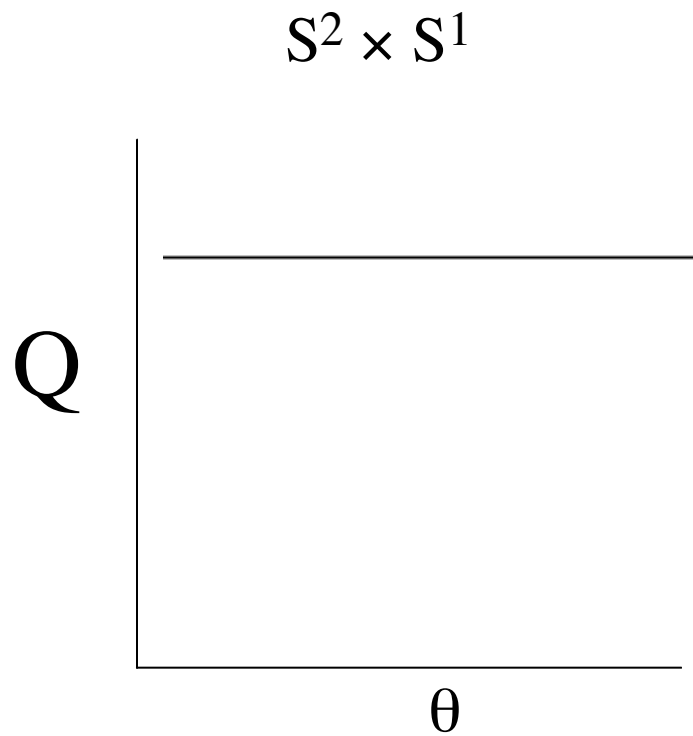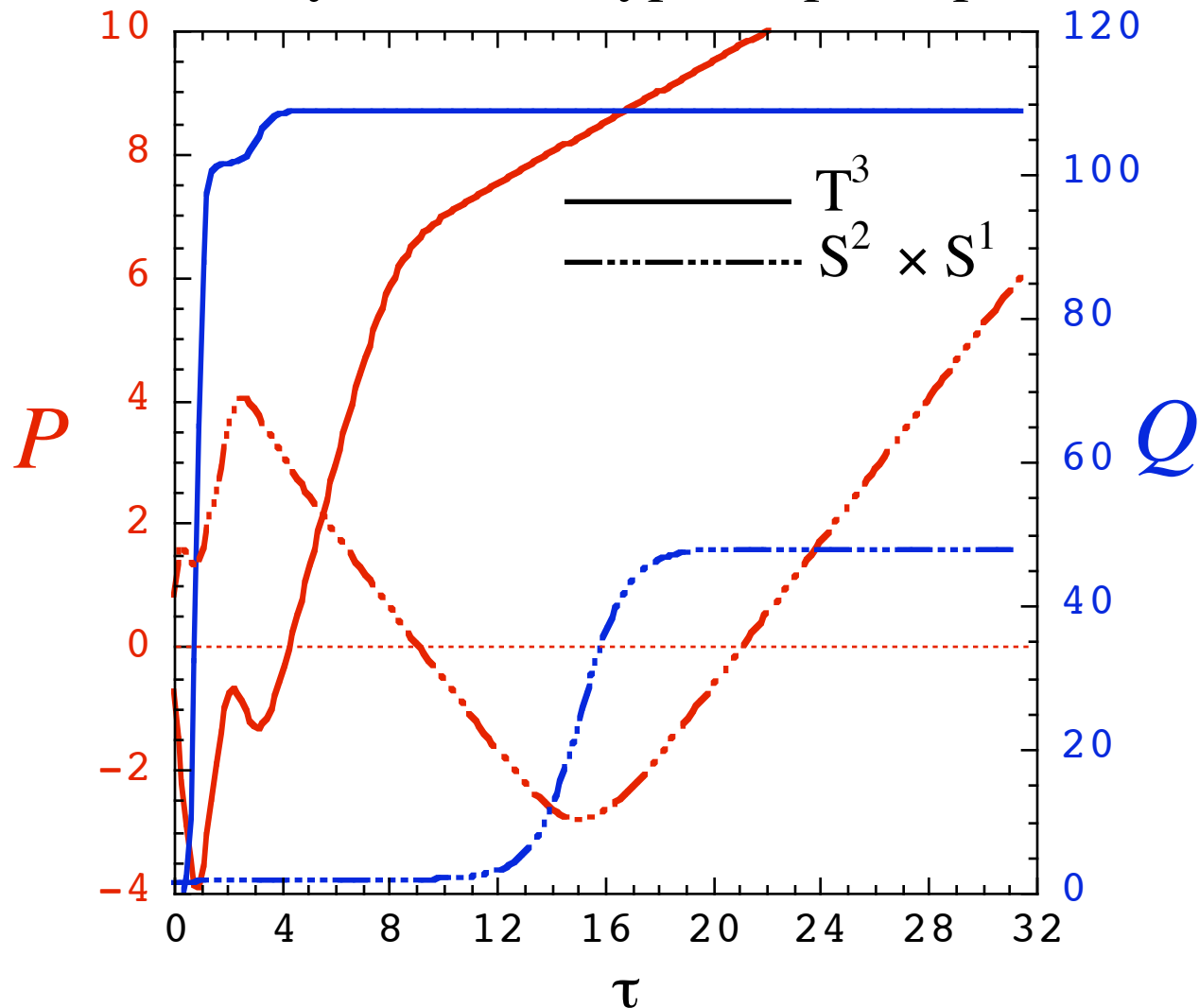# Gowdy models on $S^2 \times S^1$ and T^3 from similar initial data;

$S^2 \times S^1$

T^3

D. Garfinkle, Phys. Rev. D **60**, 104010 (1999)

Gowdy models on $S^2 \times S^1$ and T^3 from similar initial data;

$S^2 \times S^1$

T^3
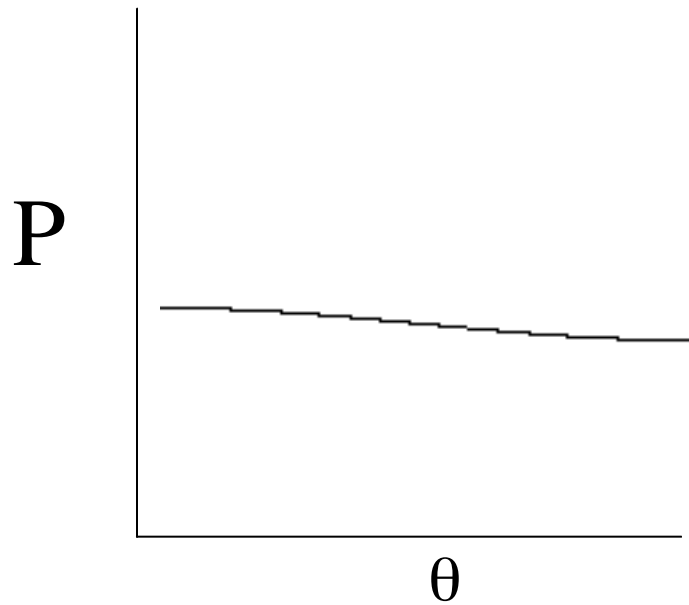
Gowdy models at typical spatial points

P Q

Legend:
- T^3 (solid)
- $S^2 \times S^1$ (dash-dot)

Horizontal axis: τ

Is the black hole horizon analogous to boundary conditions, i.e. a global effect, whose influence dies out as the singularity is approached?

Conclusions

- There is strong evidence from numerical simulation (and also mathematical theorems) that generic collapse leads to spacelike singularities dominated by local dynamics.

- Asymptotically, the local behavior closely follows the Kasner or Mixmaster solution.