# SINGULARITIES, SYMMETRIES & HIGHER ORDER CORRECTIONS

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## SUMMARY


- Generalize BKL approach to arbitrary matter coupled gravity systems, and interpret it in terms of 'small tension expansion' in spatial gradients.
- Exhibit link between BKL dynamics and infinite dimensional hidden symmetries of (super)gravity.

...a new twist to the BKL story...

- What can we learn about the possible symmetries of a unified theory from an analysis of Einstein's field equations and their solutions in the vicinity of a space-like (cosmological) singularity?


Literature (I):


1. M.P. Ryan and L.C. Shepley: *Homogeneous Relativistic Cosmologies*, Princeton University Press (1975)
2. C.W. Misner: *Deterministic chaos in general relativity*, gr-qc/9405068
3. R.T. Jantzen: *Spatially homogeneous dynamics: a unified picture*, gr-qc/0102035
4. A. Rendall, *Fuchsian methods and spacetime singularities*, gr-qc/0303071


Literature (II):


1. T. Damour, M. Henneaux and H. Nicolai: *Cosmological Billiards*, Class. Quant. Grav. **20**, R1 (2003), hep-th/0212256
2. T. Damour, M. Henneaux und H. Nicolai: *$E_{10}$ and a small tension expansion of M theory*, Phys. Rev. Lett. **28**, 221601(2002), hep-th/0207267

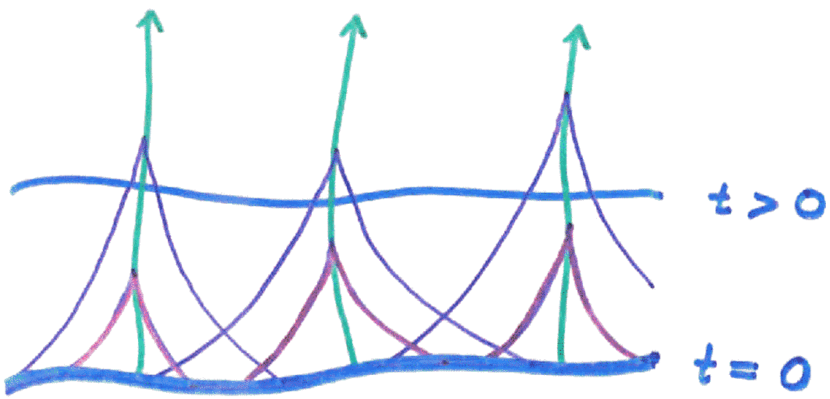
## BKL: THE BASIC PICTURE

Singularity Theorems (S.W. Hawking, R. Penrose) predict the emergence of space time singularities under certain generic conditions, but make no specific prediction about their precise nature.

In its most general form this problem is probably too difficult, but:

> can we characterize and understand the behavior of the gravitational field at least in the vicinity of a spacelike singularity ?

BKL = V.A. Belinskii, I.M. Khalatnikov und E.M. Lifschitz (1970)




*spacelike* singularity ⟷ causal decoupling
spatial gradients $(\partial_i) \ll$ time derivatives $(\partial_t)$

If correct: in the limit $t \to 0^+$ Einstein's equations reduce to a continuous superposition of ordinary differential equations, one for each spatial point $x \in \Sigma$ → drastic simplification via effective reduction to one (time) dimension!

(cf. 'horizon problem' of inflationary cosmology)

## KASNER SOLUTION (1926)

Einstein's equations for the gravitational field:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G_N T_{\mu\nu}$$

Kasner solution = simplest solution depending only on one coordinate (in a suitable gauge):

$$ds^2 = -dt^2 + t^{2p_1} dx^2 + t^{2p_2} dy^2 + t^{2p_3} dz^2$$

where the Kasner exponents $p_i$ must satisfy

$$\begin{aligned} p_1^2 + p_2^2 + p_3^2 &= 1 \\ p_1 + p_2 + p_3 &= 1 \end{aligned}$$

⇒ solution is *homogeneous* (= invariant under spatial translations), but *not isotropic* (= not invariant under spatial rotations).

Kasner solution is singular on the spacelike hypersurface $t = 0$

⟶ models Big Bang Singularity

Because one of the Kasner exponents is negative, e.g.

$$p_1 < 0$$

the observer is simultaneously squashed and stretched, but such that spatial volume $V \propto t^2 \longrightarrow 0^+$.

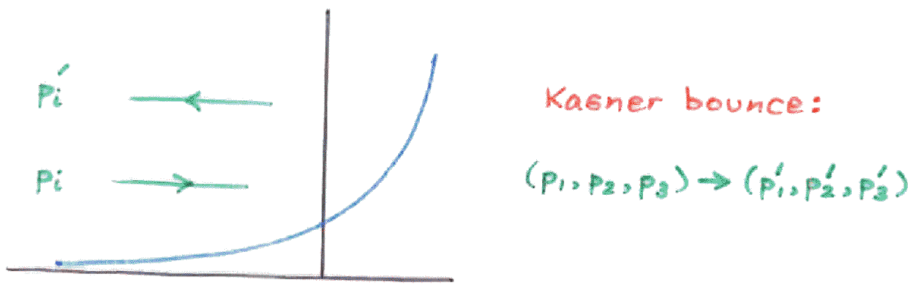
> **Idea:** Causal decoupling ⟶ near $t = 0$ solution behaves like Kasner solution at each spatial point $x \in \Sigma$, but with $x$-dependent Kasner exponents $p_i = p_i(x)$ and controllable corrections.

## STABILITY VS. CHAOS

BKL: Test stability of ansatz with homogeneous curved spatial geometry

$\longrightarrow$ substitute Kasner-like ansatz into Bianchi-type solution and check whether spatial gradients remain small or not.

Because $p_i < 0$ for some $i$, approximation breaks down eventually for pure gravity (and spacetime dimension $D \leq 10$).




Dynamics = sequence of Kasner flights and Kasner bounces

**Two kinds of behavior are possible as $t \to 0^+$:**

1. infinite number of bounces $\longleftrightarrow$ chaos
2. finite number of bounces $\longrightarrow$ monotonic Kasner-like behavior for sufficiently small $\varepsilon$ and $0 < t < \varepsilon$ (i.e. AVD= asymptotically velocity dominated)

## GENERAL CASE

**But:** Einstein's equations in their general form are far more complicated $\longrightarrow$ must take into account all other degrees of freedom:

- spatial inhomogeneities ($x$ dependence)
- off-diagonal components of metric
- matter fields (dilatons, gauge fields, etc.)

General ansatz for metric:

$$ds^2 = -dt^2 + \sum_{a=1}^{d} e^{-2\beta^a} \theta^a \otimes \theta^a$$

Here $\theta^a$ parametrizes off-diagonal components of metric.

The diagonal components (scale factors) $\beta^a$ will now be re-interpreted as coordinates in a fictitious $d$-dimensional space, such that the time evolution of the metric can be thought of as a motion in $\beta$-space.

**Aim:** construct *effective low energy* theory for diagonal degrees of freedom $(\beta^1, ..., \beta^d)$ by 'integrating out' off-diagonal metric components, matter fields, spatial inhomogeneities,...!

In this description, the Kasner solution corresponds to a free relativistic motion in $\beta$-space:

$$\beta^a(\tau) = p^a \tau + \beta_0^a$$

with the new 'Zeno-like' time coordinate:

$$\tau \equiv -\log t \qquad (\text{thus} \quad \tau \to +\infty \;\Leftrightarrow\; t \to 0^+)$$

The previous conditions on the Kasner exponents are now interpreted as a *relativistic dispersion relation* in $\beta$-space:

$$G_{ab}\dot\beta^a\dot\beta^b \equiv \sum_a (\dot\beta^a)^2 - \Big(\sum_a \dot\beta^a\Big)^2 = 0$$

NB: the $\beta$-space metric $G_{ab}$ has Lorentzian signature $(- + \cdots +)$!

**Result (I):** modulo some technical assumptions the limit $t \to 0^+$ yields a radical simplification: the dynamics takes place only in the diagonal metric degrees of freedom, whereas all other degrees of freedom get asymptotically "frozen" and manifest themselves only via certain *effective potentials* which modify the free Kasner motion.

**NB:** For proper derivation, use *hyperbolic coordinates* $\gamma^a$

$$\beta^a = \rho\gamma^a \quad , \quad \gamma^a\gamma_a = -1$$

and consider limit $\rho^2 \to \infty$.

*E.g.*, for pure gravity in $D = 4$, hyperbolic coordinates parametrize the Poincaré disk $\mathcal{D} \equiv \{z \in \mathbb{C} | z\bar{z} \leq 1\} \subset \mathbb{C}$ with metric

$$d\sigma^2 = \frac{dzd\bar{z}}{1 - |z|^2}$$

**Conclusion:** for small $t$ the equation of motion in $\beta$-space becomes

$$G_{ab}\dot{\beta}^a\dot{\beta}^b + V(\beta) = 0$$

with the 'effective potential'

$$V(\beta) \sim \sum_A c_A(P,Q)e^{-w_A(\beta)} \qquad (*)$$

where the coefficients $c_A$ depend on the remaining degrees of freedom (off-diagonal metric components, matter fields, spatial curvature,...)

## Crucial properties of effective potential $(*)$:

1. Non-negativity of the leading terms (indexed with $A'$):

$$c_{A'}(P,Q) \geq 0$$

2. Exponentials depend linearly on the $\beta$'s, therefore we can think of the $w_A$ as *linear forms*:

$$w_A(\beta) = G_{ab}w_A^a\beta^b$$

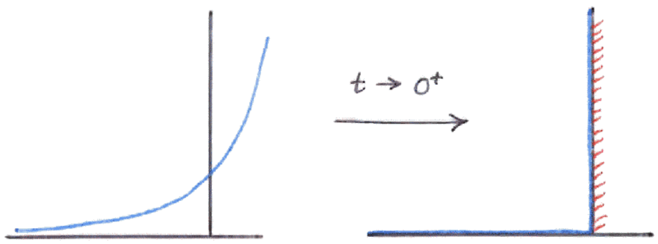
In the limit $t \to 0^+ \Leftrightarrow \beta^a \to \pm\infty$ exponential potentials become infinite potential barriers (dominant contributions indexed by $A'$):

$$\lim_{\beta\to\infty} \sum_A c_A(P,Q)e^{-w_A(\beta)} \sim \sum_{A'} \Theta_\infty\big(-w_{A'}(\beta)\big)$$

with

$$\Theta_\infty(x) := \begin{cases} 0 & \text{if } x < 0 \\ +\infty & \text{if } x > 0 \end{cases}$$

Thus, for $t \to 0^+$ (C.W. Misner, D.M. Chitre):




That is, soft walls (=exponential potential barriers) become sharp walls in the limit $t \to 0^+$. The walls associated with $w_A$ are the hyperplanes in $\beta$-space defined by

$$w_A(\beta) = 0$$

and partition $\beta$-space into wedge-like regions.

## WALLS

1. Symmetry (or centrifugal) walls (from eliminating off-shell metric components):
$$w^{(S)}_{(ab)}(\beta) := \beta^a - \beta^b \qquad \text{for} \quad a > b$$
2. Curvature (or gravitational) walls (from eliminating spatial inhomogeneities):
$$w^{(C)}_{abc}(\beta) := 2\beta^a + \sum_{m \neq a,b,c} \beta^m$$
3. $p$-form walls (from eliminating electric or magnetic $p$-form degrees of freedom):
$$w^{(electric)}_{a_1 \dots a_p}(\beta) := \beta^{a_1} + \dots \beta^{a_p}$$
$$w^{(magnetic)}_{a_1 \dots a_p}(\beta) := \sum_{m \neq a_1, \dots, a_p} \beta^m$$

All these walls are *timelike*, i.e. the associated hyperplanes have *spacelike normal vector.*

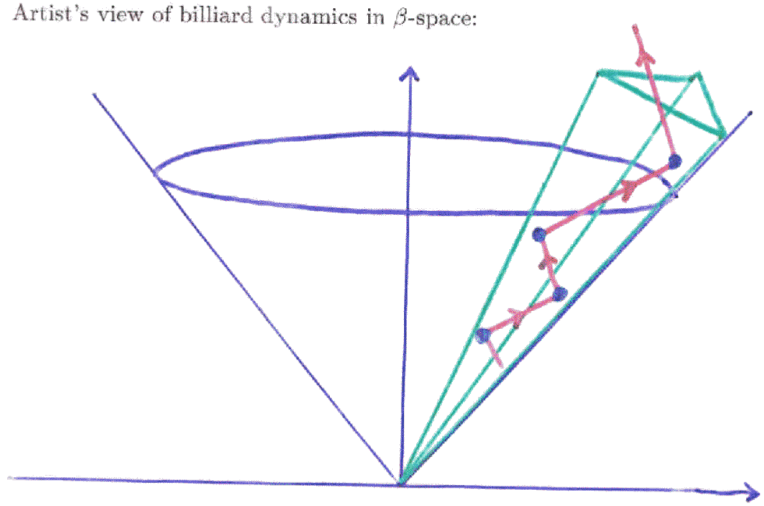
$\longrightarrow$ Reflections against these walls belong to orthochronous Lorentz group $SO^+(1, d-1)$ in $\beta$-space, and sometimes (but not always) to a discrete subgroup thereof.

**NB:** Only known *spacelike walls* from cosmological constant $\Rightarrow$ bounce reverses time direction in $\beta$-space.

(and walls from higher order curvature terms)

*In summa* the whole dynamics of Einstein's theory and its generalizations in the vicinity of a spacelike singularity reduces to a **relativistic billiard in $\beta$-space**, whose "cushions" are defined by the sharp wall potentials $\Theta_\infty$.

Artist's view of billiard dynamics in $\beta$-space:



- If billiard wedge is contained in forward light cone: infinitely many reflections against the walls (cushions) of the billiard $\longleftrightarrow$ chaotic oscillations of the metric $\longleftrightarrow$ **BKL chaos**
- If wedge extends beyond light cone: only finitely many reflections $\longleftrightarrow$ Kasner-like behavior near the singularity
- Equivalent description in terms of hyperbolic billiard on manifold of constant negative curvature (finite volume $\leftrightarrow$ chaos).

Very convenient method to check whether a given model of matter coupled gravity exhibits chaos or not!

**Examples:**

- Chaotic behavior for Einstein gravity in $D=4$ (BKL!)
- Kasner-like (regular) behavior for gravity in $D > 10$ dimensions or gravity with a dilaton for all $D$.
- Chaos can be reinstated by electric or magnetic $p$-form walls

**BUT...**

**Result (II):** All maximally supersymmetric candidate models for a mathematically consistent unfied theory (type IIA and type IIB superstrings, heterotic string, $D=11$ supergravity) exhibit BKL chaos in the vicinity of the initial singularity.

## Symmetry from Chaos......

**Result (III):** For many theories of interest, the $\beta$-space of scale factors can be identified with the Cartan subalgebra of some indefinite Kac Moody algebra. The wedge, in which the billiard takes place, is identified with the Weyl-chamber of this algebra.

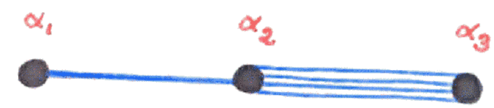
- Timelike walls $\longleftrightarrow$ normal vectors $\equiv$ real roots
  Dominant walls $\longleftrightarrow$ normal vectors $\equiv$ *simple* roots
- Kasner bounces = Weyl reflections ($\rightarrow$ Weyl group)
- Chaotic oscillations if Kac-Moody algebra is hyperbolic

## Example: pure (Einstein) gravity in $D = 4$

The relevant Lie algebra is the rank-3 hyperbolic Kac-Moody algebra $AE_3$ with Cartan matrix

$$A_{ij} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix}$$

and Dynkin diagram




with two centrifugal walls (or symmetry walls) $\alpha_1$ and $\alpha_2$, and one gravitational wall (or curvature wall) $\alpha_3$.

Regular subalgebras of $AE_3$ with known action on restricted classes of solutions of Einstein's equations:

- $\alpha_2$: Matzner-Misner $SL(2,\mathbb{R})$
- $\alpha_3$: Ehlers $SL(2,\mathbb{R})$
- $\{\alpha_1, \alpha_2\}$: spatial $SL(3,\mathbb{R})$ rotations
- $\{\alpha_2, \alpha_3\}$: Geroch group (with Lie algebra $A_1^{(1)}$)

The $AE_3$ Weyl group is $PGL(2,\mathbb{Z})$ (Feingold, Frenkel) $\Longrightarrow$ Billiard takes place in a fundamental region

$$\mathcal{F} = \mathcal{D}/PGL(2,\mathbb{Z})$$

Walls behind walls behind walls... = Weyl images of dominant walls.

## But what is an indefinite Kac Moody algebra?

True answer: nobody knows...

V. G. Kac: *"The theory of Kac-Moody algebras is a disaster..."*

**Recursive definition:** Lie Algebra $\mathfrak{g}(A)$ defined by generators and relations (Chevalley Serre presentation) with Cartan matrix $A_{ij}$:

$$[e_i, f_j] = \delta_{ij} h_j \quad , \quad [h_i, e_j] = A_{ij} e_j \quad , \quad [h_i, f_j] = -A_{ij} f_j$$

$$(\mathrm{ad}\, e_i)^{1-A_{ij}}(e_j) = (\mathrm{ad}\, f_i)^{1-A_{ij}}(f_j) = 0$$

where $\{h_i\}$ span Cartan subalgebra $\mathfrak{h}$ (CSA): $[h_i, h_j] = 0$.

Thus, a Kac Moody algebra generalizes the algebraic structures known from the analysis of the rotation group and its representations in quantum mechanics ($h \sim J^3$, $e \sim J^+$, $f \sim J^-$). Always:

$$\mathfrak{g}(A) = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$$

For $A$ **finite** (= positive definite), recover Cartan's classification of finite dimensional simple Lie algebras.

For $A$ **affine** (= positive semi-definite), $\mathfrak{g}(A)$ is infinite dimensional and can be realized via a current algebra in one dimension.

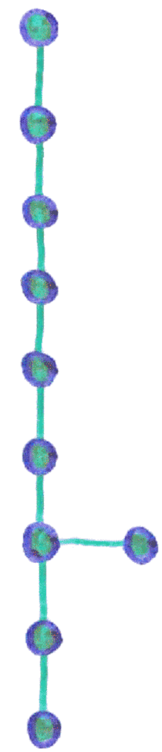
For $A$ **indefinite**, Cartan matrix has both positive and negative eigenvalues $\Longleftrightarrow$ Lorentzian metric in the CSA $\Longleftrightarrow$ Lorentzian metric in Wheeler-DeWitt superspace of $\beta$'s!

$A$ **hyperbolic** $\Leftrightarrow$ $A$ indefinite *and* regular subalgebras affine or finite.

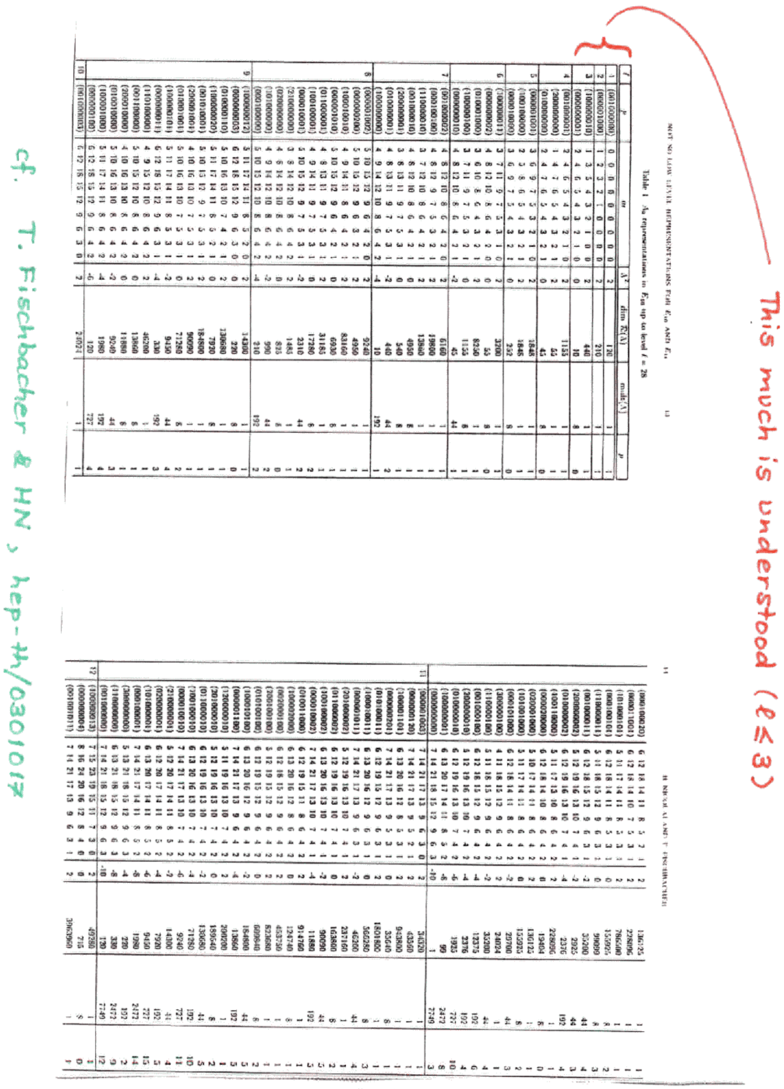
In this case the number of linearly independent raising and lowering operators 'explodes', and even more than 35 years after their discovery we still don't know much about these algebras..... it is not even possible to write them down explicitly!

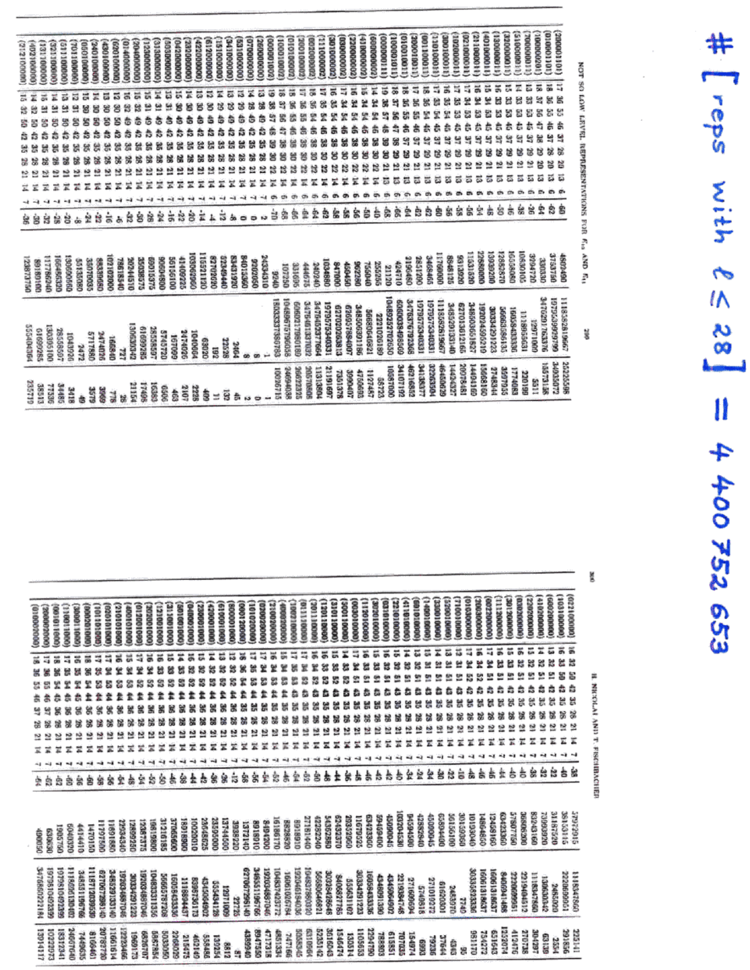
D = 11, IIA & IIB SUPERGRAVITY :

$E_{10}$



Thus: maximal supergravity $\longleftrightarrow$ "maximally extended" hyperbolic algebra


This much is understood (ℓ ≤ 3)
cf. T. Fischbacher & HN, hep-th/0301017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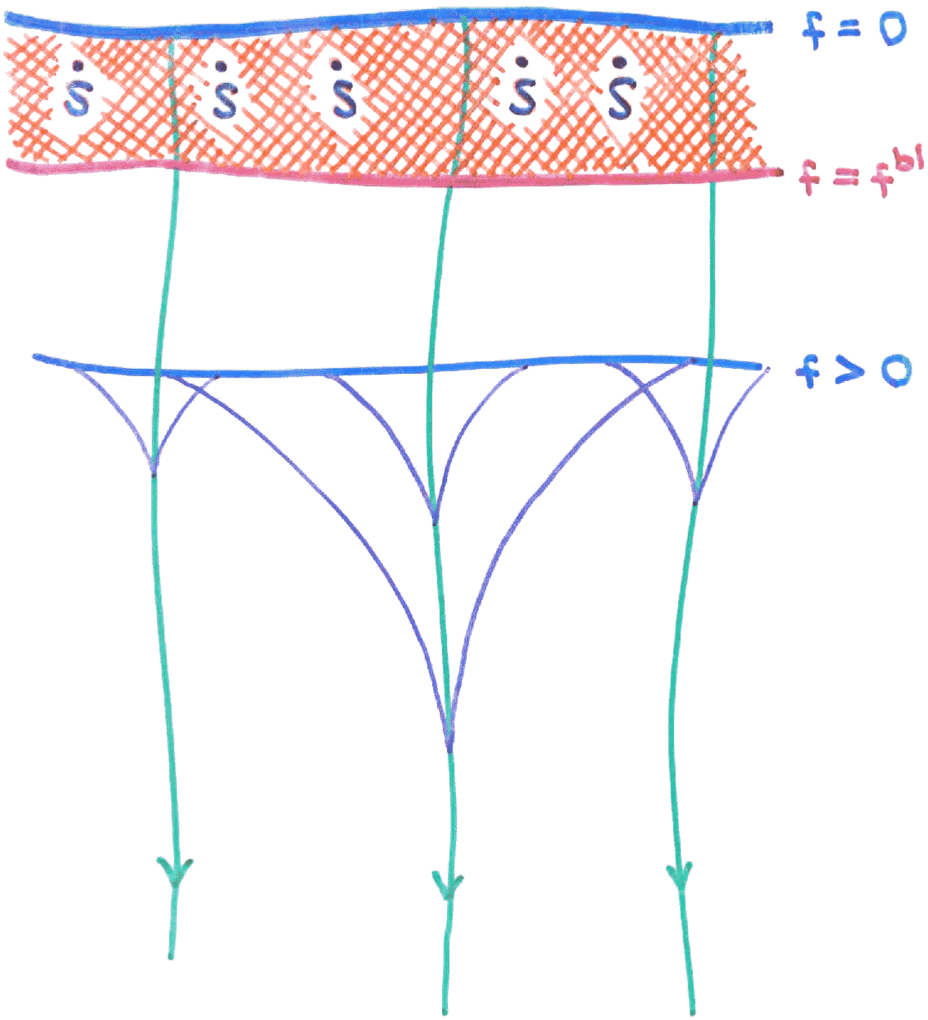


## M THEORY PERSPECTIVES?

An alternative view: space-time as an elastic medium

$$S = \frac{c^2}{16\pi G_N} \int d^D x \sqrt{-g} R = \int dt \int d^d x \left[ \rho_b (\partial_t h)^2 - T_b (\nabla h)^2 \right] + \dots$$

for small perturbations of the background metric

$$g_{\mu\nu} = \eta_{\mu\nu} + c h_{\mu\nu}$$

$\Longrightarrow$ BKL limit as a 'zero tension limit' $T_b \to 0$??

A similar limit has been considered in string theory (though without much success so far .....)

**Proposal:** String theory, as known today, is only the spontaneously broken version of some other and more fundamental pre-geometrical (and background independent) theory exhibiting a much larger symmetry. The 'symmetric phase' of M theory could well be related to the zero tension limit of string theory.

A first indication (D. Gross, 1988): infinitely many linear relations between string scattering amplitudes for tensionless bosonic string (i.e. for $\alpha' \to \infty$, where all string excitations become massless).