Feynman propagators from the worldsheet

Yvonne Geyer

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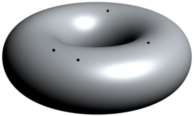
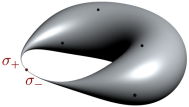




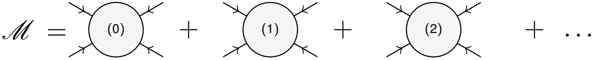
Scattering Amplitudes and Beyond

KITP

arXiv:2007.00623 with J Farrow, A. Lipstein, R. Monteiro and R. Stark-Muchão

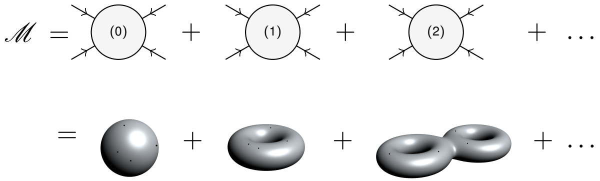
arXiv:1507.00321, 1511.06315, 1607.08887
with L. Mason, R. Monteiro, P. Tourkine

Representations of QFT amplitudes



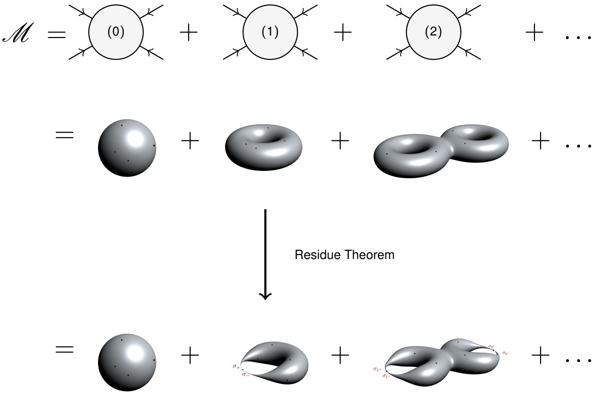
- Feynman diagrams
- Amplitudes program: On-shell methods, ...

Representations of QFT amplitudes



- string theory at $\alpha' \to 0$
- Witten's twistor string
- CHY formulae & ambitwistor strings

Representations of QFT amplitudes



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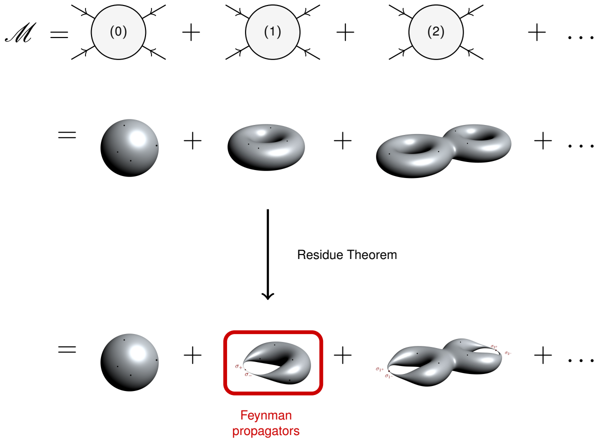

- CHY formulae & ambitwistor strings

CHY amplitudes [Cachazo-He-Yuan]

S-matrix for massless QFTs

$$\mathscr{M}_n = \int_{\mathfrak{M}_{0,n}} \frac{\mathrm{d}^n\sigma}{\mathrm{vol}\,\mathrm{SL}(2,\mathbb{C})} \prod_{i=1}^{n}{}' \,\bar{\delta}(\mathcal{E}_i)\;\; \mathscr{I}_n(\sigma_i, k_i, q_i)$$

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S-matrix for massless QFTs

D-dim momenta k_i

$k_i^2 = 0$

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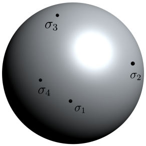
moduli space $\mathfrak{M}_{0,n}$

$\sigma_i \in \mathbb{CP}^1$



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holom. δ-fns
$\bar\delta(x) \equiv \bar\partial\left(\frac{1}{2i\pi\, x}\right)$

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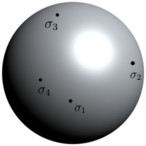

scattering equations $\mathcal{E}_i$

- Construction: $P_\mu = \sum_{i=1}^n \frac{k_{i\,\mu}}{\sigma-\sigma_i} d\sigma$

$$\mathcal{E}_i = \text{Res}_{\sigma_i} P^2(\sigma) = 2k_i \cdot P(\sigma_i)$$

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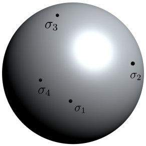

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- geometric interpretation: $P^2 = 0$
- fully localized

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S-matrix for massless QFTs

Integrand $\mathscr{I}_n$

- 'data' $q_i : T_i^{a_i}$, ϵ_i,
- theory-specific
  $\mathscr{I}_n = \mathscr{I}_n^{1/2} \tilde{\mathscr{I}}_n^{1/2}$

$$\mathscr{M}_n = \int_{\mathfrak{M}_{0,n}} \frac{d^n\sigma}{\text{vol}\, SL(2,\mathbb{C})} \prod_{i=1}^{n}{}' \,\bar{\delta}(\mathcal{E}_i)\; \mathscr{I}_n(\sigma_i, k_i, q_i)$$

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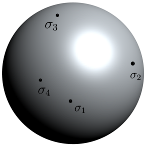

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A closer look at the integrand

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Building blocks $\mathscr{I}_n^{1/2}$

- Parke-Taylor factor: $C_n = \frac{\mathrm{tr}(T^{a_1}T^{a_2}\ldots T^{a_n})}{\sigma_{1\,2}\ldots\sigma_{n-1\,n}\sigma_{n\,1}} + \text{non-cyclic}$
- Reduced Pfaffian: $\mathrm{Pf}'(M) = \frac{(-1)^{i+j}}{\sigma_{ij}}\mathrm{Pf}(M_{[ij]})$

$$M = \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix} \qquad A_{ij} = \frac{k_i \cdot k_j}{\sigma_{ij}}\,, \qquad C_{ij} = \frac{\epsilon_i \cdot k_j}{\sigma_{ij}}\,, \qquad B_{ij} = \frac{\epsilon_i \cdot \epsilon_j}{\sigma_{ij}}$$

$$A_{ii} = 0\,, \qquad C_{ii} = -\sum_{j\neq i} C_{ij}\,, \qquad B_{ii} = 0$$

Theories

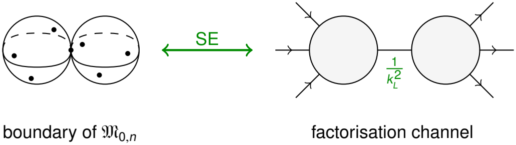
$$\mathscr{I}_n^{\text{grav}} = \mathrm{Pf}'(M)\,\mathrm{Pf}'(\tilde{M})\,, \qquad \mathscr{I}_n^{\text{YM}} = C_n\,\mathrm{Pf}'(M)\,, \qquad \mathscr{I}_n^{\text{BS}} = C_n\,\tilde{C}_n$$

Why Scattering Equations?

Scattering Equations

$$\mathcal{E}_i = \text{Res}_{\sigma_i} P^2(\sigma) = 2k_i \cdot P(\sigma_i) = \sum_{j \neq i} \frac{2k_i \cdot k_j}{\sigma_i - \sigma_j} = 0$$

- Factorisation: [Dolan-Goddard, YG-Mason-Monteiro-Tourkine, ...]



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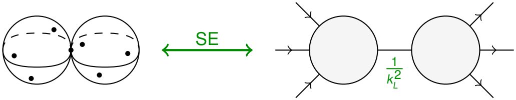

boundary of $\mathfrak{M}_{0,n}$ factorisation channel

- Parametrize $\partial\mathfrak{M}_{0,n} \supset \mathfrak{M}_{0,n_L+1} \otimes \mathfrak{M}_{0,n_R+1}$ by $\sigma_i = \sigma_L + \varepsilon x_i$ for $i \in L$:

$$0 = \sum_{i \in L} x_i \, \mathcal{E}_i^{(L)} = \sum_{i,j \in L} x_i \frac{2k_i \cdot k_j}{x_i - x_j} = \sum_{i,j \in L} k_i \cdot k_j = \frac{1}{2} \, k_L^2 \, .$$

Where did the Riemann Sphere come from?

CHY

$$\mathscr{M}_n^{(0)} = \int_{\mathfrak{M}_{0,n}} \frac{\mathrm{d}^n\sigma}{\mathrm{vol}\,\mathrm{SL}(2,\mathbb{C})} \prod_{i=1}^{n}{}' \bar{\delta}\left(\mathcal{E}_i\right) \mathscr{I}_n$$

'RNS' Ambitwistor String [Mason-Skinner, c.f. Berkovits]

- Chiral 2d CFT:

ambitwistor string

$$S = \frac{1}{2\pi}\int_{\Sigma} P\cdot\bar{\partial}X - e\,P\cdot\partial X - \frac{\tilde{e}}{2}P^2 + \frac{1}{2}\psi_r\cdot\bar{\partial}\psi_r - \frac{e}{2}\psi_r\cdot\partial\psi_r - \chi_r P\cdot\psi_r$$

NO α'

$$X^\mu \in \Omega^0(\Sigma),\ \ P_\mu \in \Omega^0(K_\Sigma),\ \ \psi^\mu_{r=1,2} \in \Pi\Omega^0(K_\Sigma^{1/2}).$$

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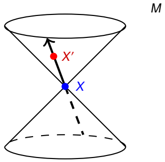
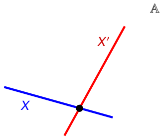
- c.f worldline formulations
- BRST quantisation: free, linear CFTs; $d_{\text{crit}} = 10$.
- target space: $\mathbb{A}$ = phase space of complexified null geodesics





Spectrum and correlators

- Spectrum: type II supergravity

NO STRINGY MODES

$$V_{\mathrm{NS}} = c\tilde{c}\,\delta(\gamma_1)\delta(\gamma_2)\,\epsilon_{\mu\nu}\psi_1^{\mu}\psi_2^{\nu}\,e^{ik\cdot X}$$

with $k^2 = \epsilon_{\mu\nu}k^{\nu} = \epsilon_{\mu\nu}k^{\mu} = 0$.

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$\Rightarrow$ Worldsheet theory for QFT amplitudes

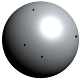
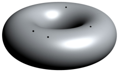
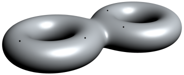
- correlator = CHY amplitude [Cachazo-He-Yuan]

$$\mathscr{M}_n^{(0)} \sim \left\langle \prod_{i=1}^{n} V(\sigma_i) \right\rangle = \int_{\mathfrak{M}_{0,n}} \frac{\mathrm{d}^n\sigma}{\text{vol}\,\mathrm{SL}(2,\mathbb{C})} \prod_i{}' \bar{\delta}\left(\mathcal{E}_i\right) \mathscr{I}_n$$

Main idea:

- P localizes onto EoM: $\bar{\partial}P_\mu = \sum_i k_{i\,\mu}\bar{\delta}(\sigma-\sigma_i)\,\mathrm{d}\sigma$
- Tree-level: $P_\mu = \sum_i \frac{k_{i\,\mu}}{\sigma-\sigma_i}\mathrm{d}\sigma$

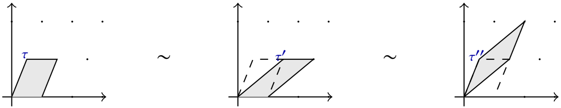
What about loops?

$\mathscr{M}_n =$  $+$  $+$  $+ \dots$

The one-loop integrand [Adamo-Casali-Skinner]

$$\mathscr{M}^{(1)} = \int d^{10}\ell \, \mathfrak{I}^{(1)} , \qquad \mathfrak{I}^{(1)} = \int_{\mathfrak{M}_{1,n}} d\tau \; \bar{\delta}(u) \prod_{i=2}^{n} \bar{\delta}\,(\mathcal{E}_i)\, \mathscr{I}^{(1)}$$

- $n-1$ marked points z_i (fix one)
- modular parameter τ



$$\tau \quad \sim \quad \tau+1 \quad \sim \quad \frac{\tau}{\tau+1}$$

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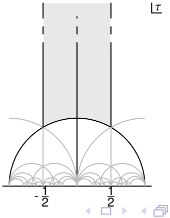
$\Rightarrow$ Fundamental domain
$\mathcal{F} \cong \mathcal{H}/\mathrm{PSL}(2,\mathbb{Z})$



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Localization for P:
$\bar\partial P_\mu = \sum_i k_{i\mu}\bar\delta(z - z_i)$

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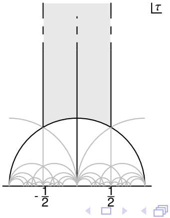

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Scattering equations

geometric interpret.: $P^2 = 0$

$$\mathcal{E}_i \equiv \mathrm{Res}_{z_i} P^2(z) = 0 \qquad i=2,\dots,n$$

$$\mathcal{E}_\tau \equiv u = 0 \qquad P^2 = u\,\mathrm{d}z^2$$

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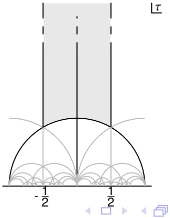

Properties

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Features:

- Localization
  Loop integrand $\mathfrak{I}$ is *fully localized* on $\mathcal{E}_A$.

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- Modular invariance

  This does NOT imply finiteness of the amplitude!

non-compact moduli space	$\Leftrightarrow$	integration over loop momentum ℓ

Question:

How does this relate to usual QFT integrands?

The residue theorem [YG-Mason-Monteiro-Tourkine]

Key features:

- localization
- modular invariance

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$\Longrightarrow$ Residue Theorem

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Key features:

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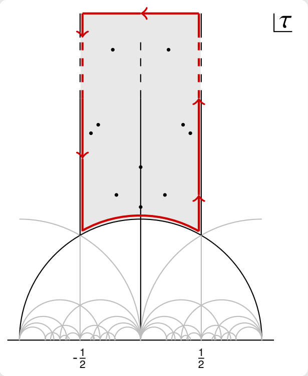
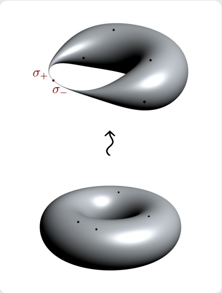
$\Rightarrow$ Residue Theorem

Cauchy residue theorem on $\mathcal{F}$

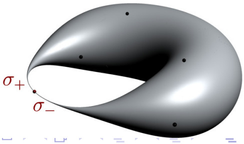
$u = 0$ on support of $\mathcal{E}_i = 0$ $\leftrightarrow$ $q \equiv e^{2i\pi\tau} = 0$





On the nodal Riemann Sphere

$$\mathscr{M}^{(1)} = \int d^{10}\ell \; \mathfrak{I}^{(1)}\,, \qquad \mathfrak{I}^{(1)} = \frac{1}{\ell^2} \int_{\mathfrak{M}_{0,n+2}} \frac{d^{n+2}\sigma}{\mathrm{vol}\,\mathrm{SL}(2,\mathbb{C})} \prod_{A=1}^{n+2}{}' \bar{\delta}\left(\mathcal{E}_A\right) \mathscr{I}_0^{(1)}$$



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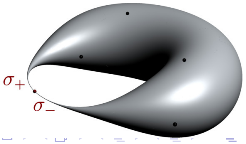

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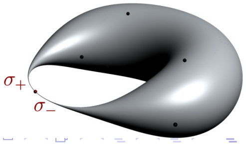
- $\mathcal{E}_A$ enforcing $P^2 - \ell^2 \omega_{+-}^2 = 0$:

$$\mathcal{E}_A = \mathrm{Res}_{\sigma_A}\left(P^2(\sigma) - \ell^2 \omega_{+-}^2\right)$$

- Möbius invariance

$$\mathscr{M}^{(1)} = \int d^{10}\ell\ \mathfrak{I}^{(1)}\,, \qquad \mathfrak{I}^{(1)} = \frac{1}{\ell^2} \int_{\mathfrak{M}_{0,n+2}} \frac{d^{n+2}\sigma}{\mathrm{vol}\,\mathrm{SL}(2,\mathbb{C})} \prod_{A=1}^{n+2}{}' \bar{\delta}\left(\mathcal{E}_A\right) \mathscr{I}_0^{(1)}$$



On the nodal Riemann Sphere

Localization for P:

$$P_\mu = \ell_\mu\, \omega_{+-} + \sum_i \frac{k_{i\,\mu}}{\sigma - \sigma_i}$$

- $\omega_{+-} = \frac{1}{\sigma-\sigma_+} - \frac{1}{\sigma-\sigma_-}$
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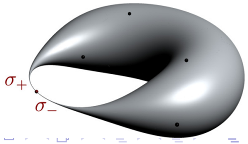

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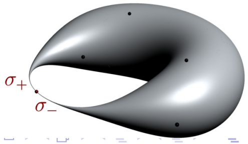
Comments:

- complexity $\sim$ tree-level
- ‘free lunch’: arbitrary dimension d
  different theories



One-loop integrand(s): Double Copy

integrand on nodal sphere

$$\mathfrak{I}_n^{(1)} = \frac{1}{\ell^2} \int_{\mathfrak{M}_{0,n+2}} \frac{d^{n+2}\sigma}{\mathrm{vol}\,\mathrm{SL}(2,\mathbb{C})} \prod_{A=1}^{n+2}{}' \bar{\delta}\left(\mathcal{E}_A\right) \mathscr{I}_0^{(1)}$$

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Supersymmetric:

- $\mathscr{I}_{\text{sugra}}^{(1)} = \mathcal{I}_{\text{kin}}^{(1)} \widetilde{\mathcal{I}}_{\text{kin}}^{(1)}$
- $\mathscr{I}_{\text{sYM}}^{(1)} = \mathcal{I}_{\text{kin}}^{(1)} \mathcal{C}^{(1)}$

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Building blocks

- Parke-Taylor: $\mathcal{C}^{(1)} = \sum_{i=1}^{n} \frac{tr(T^{a_1} T^{a_2} \ldots T^{a_n})}{\sigma_{\ell^+ i} \sigma_{i\, i+1} \ldots \sigma_{i+n\, \ell^-} \sigma_{\ell^- \ell^+}} + \text{non-cycl.}$
- Pfaffian: $\mathcal{I}^{(1)}_{\text{kin}} = \sum_r \text{Pf}\,'(M^r_{\text{NS}}) - \frac{c_d}{\sigma^2_{\ell^+ \ell^-}} \text{Pf}\,(M_2)$

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Non-supersymmetric

- $\mathscr{I}^{(1)}_{\text{YM}} = \left(\sum_r \text{Pf}\,'(M^r_{\text{NS}})\right) \mathcal{C}^{(1)}$
- $\mathscr{I}^{(1)}_{n\text{-gon}} = \left(\frac{1}{\sigma^2_{\ell^+\ell^-}} \prod_i \frac{\sigma_{\ell^+\ell^-}}{\sigma_{i\ell^+}\sigma_{i\ell^-}}\right) \mathcal{C}^{(1)}$

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- Puzzle: Only single factor of ℓ^{-2}, remainder $(2\ell \cdot k + k^2)^{-1}$

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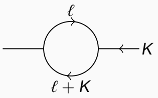
Example



$$\frac{1}{\ell^2(\ell+K)^2} = \frac{1}{\ell^2(2\ell \cdot K + K^2)} + \frac{1}{(\ell+K)^2(-2\ell \cdot K - K^2)}$$

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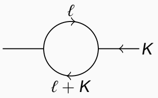

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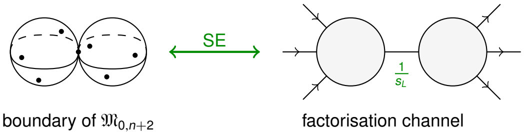
Integrand Representation Take II

nodal SE

$$\mathcal{E}_A = \mathrm{Res}_{\sigma_A}\left(P^2(\sigma) - \ell^2\omega_{+-}^2\right)$$

- Poles still determined by SE



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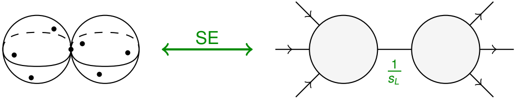
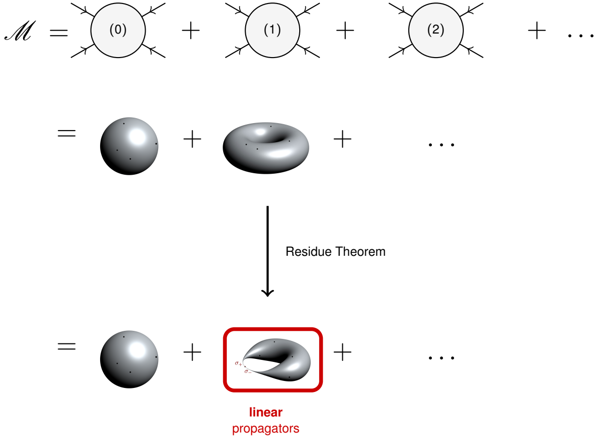


boundary of $\mathfrak{M}_{0,n+2}$ — factorisation channel

- Parametrize $\partial\mathfrak{M}_{0,n} \supset \mathfrak{M}_{0,n_L+1} \otimes \mathfrak{M}_{0,n_R+1}$ by $\sigma_i = \sigma_L + \varepsilon x_i$ for $i \in L$:

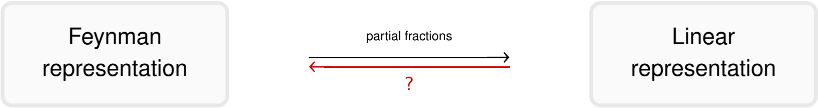
$$0 = \sum_{i \in L} x_i \, \mathcal{E}_i^{(L)} = \frac{1}{2}\, s_L \qquad s_L = \begin{cases} k_L^2 & L = L^{\text{ext}} \\ +2\ell \cdot k_L + k_L^2 & L = \{\ell^+\} \cup L^{\text{ext}} \\ -2\ell \cdot k_L + k_L^2 & L = \{\ell^-\} \cup L^{\text{ext}} \end{cases}$$

So far



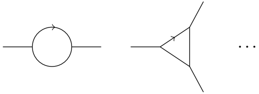
Question:

Is there a worldsheet representation
of integrands with Feynman propagators?



Toy example: the n-gon

n-gon integrand in linear representation:



$$\mathfrak{I}^{\text{lin}}_{n\text{-gon}} = \frac{1}{\ell^2} \sum_{\rho \in \text{cyc}(1\ldots n)} \frac{1}{\prod_{i=1}^{n} \left(2\ell \cdot k_{\rho_{1\ldots i}} + k^2_{\rho_{1\ldots i}}\right)}$$

$$= \frac{1}{\ell^2} \cdot \frac{1}{\left(2\ell \cdot k_1\right)\left(2\ell \cdot (k_1 + k_2) + (k_1 + k_2)^2\right) \cdots \left(-2\ell \cdot k_n\right)} + \cdots$$

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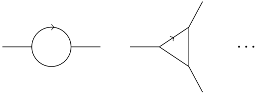

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(i) pick out first term

(ii) substitute $2\ell \cdot k_1 \to (\ell + k_1)^2$ and $2\ell \cdot k_n \to -(\ell - k_n)^2$

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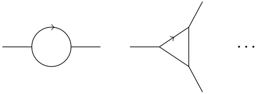

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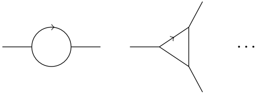

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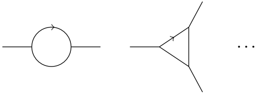

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Result: usual *n*-gon integrand

$$\mathfrak{I}^{\text{lin}}_{n\text{-gon}} \to \mathfrak{I}^{\text{Fey}}_{n\text{-gon}} = \frac{1}{\ell^2 \big(\ell + k_1\big)^2 \big(\ell + k_1 + k_2\big)^2 \cdots \big(\ell - k_n\big)^2}$$

Problems with the naive integrand-level approach

- n-gon integrand for different cyclic ordering $(213\ldots n)$:

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Naive algorithm only works for planar integrands

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- $(n-1)$-gon integrand with massive corner $(12\ldots[n-1,n])$:

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Lesson 2

No ‘BCFW-like’ shift without changing the (WS) integrand

$$k_1 \to k_1 + \alpha q\,, \qquad k_n \to k_n - \alpha q\,, \qquad \ell = \ell_0 + \alpha\, q \to \ell_0$$

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What went wrong: BCFW Recursion for the integrand

Deformation:

$$\begin{aligned} \hat{k}_1 &= k_1 + z\,q \\ \hat{k}_n &= k_n - z\,q \\ \hat{\ell} &= \ell - z\,q \end{aligned}$$

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$$\mathfrak{I} = \mathfrak{I}(z=0) = \oint_{|z|=\varepsilon} \frac{1}{z}\mathfrak{I}(z) = -\oint_{|z|=\varepsilon} \frac{1}{z}\mathfrak{I}(z)$$

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Poles: $0 = \hat{k}_L^2 = k_L^2 + 2z\,q\cdot k_L$

$0 = \hat{\ell}^2 = \ell^2 - 2z\,q\cdot\ell$

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$\longleftarrow$ Note: $\hat{\ell} + \hat{k}_1 = \ell + k_1$, $\quad \hat{\ell} - \hat{k}_n = \ell - k_n$

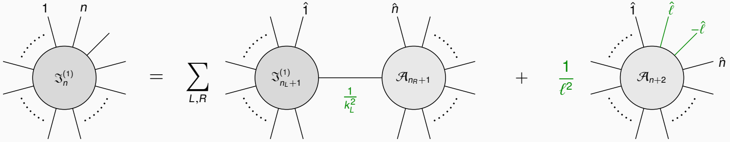
Cauchy:

$$\mathfrak{I} = \mathfrak{I}(z=0) = \oint_{|z|=\varepsilon} \frac{1}{z}\mathfrak{I}(z) = -\oint_{|z|=\varepsilon} \frac{1}{z}\mathfrak{I}(z)$$

Poles:

$$0 = \hat{k}_L^2 = k_L^2 + 2z\,q\cdot k_L$$
$$0 = \hat{\ell}^2 = \ell^2 - 2z\,q\cdot\ell$$

BCFW recursion



What went wrong: BCFW Recursion for the integrand

Deformation:

$$\hat{k}_1 = k_1 + z\,q$$
$$\hat{k}_n = k_n - z\,q$$
$$\hat{\ell} = \ell - z\,q$$

$\longleftarrow$ Note: $\hat{\ell} + \hat{k}_1 = \ell + k_1$, $\hat{\ell} - \hat{k}_n = \ell - k_n$

Cauchy:

$$\mathfrak{I} = \mathfrak{I}(z=0) = \oint_{|z|=\varepsilon} \frac{1}{z}\mathfrak{I}(z) = -\oint_{|z|=\varepsilon} \frac{1}{z}\mathfrak{I}(z)$$

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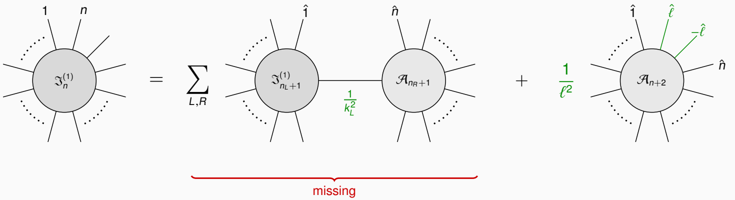

Main idea: Modification on WS

naive algorithm

(i) pick out first term

(ii) substitute $2\ell \cdot k_1 \to (\ell + k_1)^2$ and $2\ell \cdot k_n \to -(\ell - k_n)^2$

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algorithm

(i) for 'colour' half- integrand

$$C^{(1)} = \sum_{\rho \in \mathbb{Z}_n} \frac{tr(T^{a_{\rho_1}} T^{a_{\rho_2}} \dots T^{a_{\rho_n}})}{\sigma_{\ell+\,\rho_1} \sigma_{\rho_1 \rho_2} \dots \sigma_{\rho_n \ell-} \sigma_{\ell-\,\ell+}} \quad \to \quad C^{(0)}_{n+2} = \frac{tr(T^{a_1} T^{a_2} \dots T^{a_n})}{\sigma_{\ell+\,1} \sigma_{1\,2} \dots \sigma_{n\,\ell-} \sigma_{\ell-\,\ell+}}$$

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(ii) directly substitute

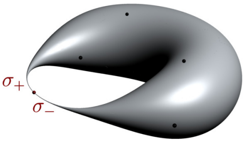
$$\mathcal{E}_A \quad \to \quad \mathcal{E}_A^{\ell^2\text{-def}} = \mathcal{E}_A \Big|_{\substack{2\ell \cdot k_1 \mapsto +(\ell + k_1)^2 \\ 2\ell \cdot k_n \mapsto -(\ell - k_n)^2}}$$

Recall: Scattering Equations on nodal sphere

SE on nodal sphere

$$\mathcal{E}_A = \mathrm{Res}_{\sigma_A}\left(P^2(\sigma) - \ell^2\omega_{+-}^2\right)$$

with $P = \left(\frac{\ell}{\sigma - \sigma_{\ell+}} - \frac{\ell}{\sigma - \sigma_{\ell-}} + \sum_{i=1}^{n} \frac{k_i}{\sigma - \sigma_i}\right) d\sigma$



- Explicit form:

$$\mathcal{E}_i = \frac{2k_i \cdot \ell}{\sigma_i - \sigma_{\ell+}} - \frac{2k_i \cdot \ell}{\sigma_i - \sigma_{\ell-}} + \sum_{j \neq i} \frac{2k_i \cdot k_j}{\sigma_i - \sigma_j}$$

$$\mathcal{E}_1 = \frac{2\ell \cdot k_1}{\sigma_1 - \sigma_{\ell+}} - \frac{2\ell \cdot k_1}{\sigma_1 - \sigma_{\ell-}} + \sum_{j \neq 1} \frac{2k_1 \cdot k_j}{\sigma_1 - \sigma_j}$$

$$\mathcal{E}_n = \frac{2\ell \cdot k_n}{\sigma_n - \sigma_{\ell+}} - \frac{2\ell \cdot k_n}{\sigma_n - \sigma_{\ell-}} + \sum_{j \neq n} \frac{2k_n \cdot k_j}{\sigma_n - \sigma_j}$$

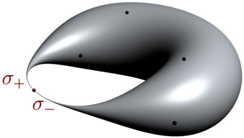
$$\mathcal{E}_{\ell\pm} = \pm \sum_{j \neq 1,n} \frac{2\ell \cdot k_j}{\sigma_{\ell\pm} - \sigma_j} \pm \frac{2\ell \cdot k_1}{\sigma_{\ell\pm} - \sigma_1} \mp \frac{2\ell \cdot k_n}{\sigma_{\ell\pm} - \sigma_n}$$

- Möbius invariance: $\sum_{j \neq i} k_j \cdot k_i = \sum_j \ell \cdot k_j = 0$

ℓ^2-deformed scattering equations

ℓ^2-deformed SE

$$\mathcal{E}_A^{\ell^2\text{-def}} := \mathcal{E}_A \Big|_{\substack{2\ell\cdot k_1 \mapsto +(\ell+k_1)^2 \\ 2\ell\cdot k_n \mapsto -(\ell-k_n)^2}}$$



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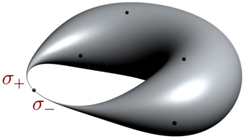

- Explicit form:

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$$\mathcal{E}_1 = +\frac{(\ell+k_1)^2}{\sigma_1-\sigma_{\ell+}} - \frac{(\ell+k_1)^2}{\sigma_1-\sigma_{\ell-}} + \sum_{j\neq 1}\frac{2k_1\cdot k_j}{\sigma_1-\sigma_j}$$

$$\mathcal{E}_n = -\frac{(\ell-k_n)^2}{\sigma_n-\sigma_{\ell+}} + \frac{(\ell-k_n)^2}{\sigma_n-\sigma_{\ell-}} + \sum_{j\neq n}\frac{2k_n\cdot k_j}{\sigma_n-\sigma_j}$$

$$\mathcal{E}_{\ell\pm} = \pm\sum_{j\neq 1,n}\frac{2\ell\cdot k_j}{\sigma_{\ell\pm}-\sigma_j} \pm \frac{(\ell+k_1)^2}{\sigma_{\ell\pm}-\sigma_1} \mp \frac{(\ell-k_n)^2}{\sigma_{\ell\pm}-\sigma_n}$$

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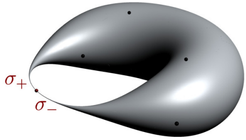

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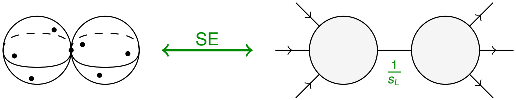
Why this deformation?

nodal SE

$$\mathcal{E}_A = \mathrm{Res}_{\sigma_A}\left(P^2(\sigma) - \ell^2\omega_{+-}^2\right)$$

- Poles still determined by SE



boundary of $\mathfrak{M}_{0,n+2}$ factorisation channel

- Parametrize $\partial\mathfrak{M}_{0,n} \supset \mathfrak{M}_{0,n_L+1} \otimes \mathfrak{M}_{0,n_R+1}$ by $\sigma_i = \sigma_L + \varepsilon x_i$ for $i \in L$:

$$0 = \sum_{i\in L} x_i\,\mathcal{E}_i^{(L)} = \frac{1}{2}\,s_L \qquad\qquad s_L = \begin{cases} k_L^2 & L = L^{\mathrm{ext}} \\ +2\ell\cdot k_L + k_L^2 & L = \{\ell^+\}\cup L^{\mathrm{ext}} \\ -2\ell\cdot k_L + k_L^2 & L = \{\ell^-\}\cup L^{\mathrm{ext}} \end{cases}$$

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ℓ^2-deformed SE

$$\mathcal{E}_A^{\ell^2\text{-def}} := \mathcal{E}_A \Big|_{\substack{2\ell\cdot k_1 \mapsto +(\ell+k_1)^2 \\ 2\ell\cdot k_n \mapsto -(\ell-k_n)^2}}$$

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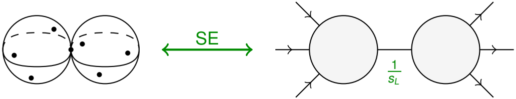


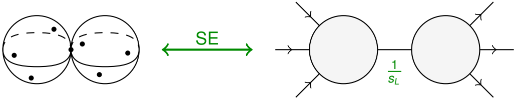
boundary of $\mathfrak{M}_{0,n+2}$ — factorisation channel

Why this deformation?

ℓ^2-deformed SE

$$\mathcal{E}_A^{\ell^2\text{-def}} := \mathcal{E}_A \Big|_{\substack{2\ell\cdot k_1 \mapsto +(\ell+k_1)^2 \\ 2\ell\cdot k_n \mapsto -(\ell-k_n)^2}}$$

- Poles still determined by SE



boundary of $\mathfrak{M}_{0,n+2}$ factorisation channel

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algorithm

(i) for 'colour' half- integrand

$$C^{(1)} = \sum_{\mathbb{Z}_n} \frac{tr(T^{a_{\rho_1}} T^{a_{\rho_2}} \ldots T^{a_{\rho_n}})}{\sigma_{\ell+\,\rho_1}\sigma_{\rho_1\rho_2}\ldots\sigma_{\rho_n\ell-}\sigma_{\ell-\ell+}} \quad \rightarrow \quad C^{(0)}_{n+2} = \frac{tr(T^{a_1} T^{a_2} \ldots T^{a_n})}{\sigma_{\ell+1}\sigma_{1\,2}\ldots\sigma_{n\,\ell-}\sigma_{\ell-\ell+}}$$

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Integrands: Super Yang-Mills

integrand with Feynman propagators

$$\mathfrak{I}^{(1)}_{\text{sYM}} = \frac{1}{\ell^2} \int_{\mathfrak{M}_{0,n+2}} \frac{d^{n+2}\sigma}{\text{vol}\,\text{SL}(2,\mathbb{C})} \prod_{A=1}^{n+2}{}' \bar{\delta}\left(\mathcal{E}_A^{\ell^2\text{-def}}\right) \mathcal{I}^{(1)}_{\text{kin}}\, \mathcal{C}^{(0)}_{n+2}$$

Integrands: Super Yang-Mills

integrand with Feynman propagators

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Building blocks

- Parke-Taylor: $C^{(0)}_{n+2} = \frac{tr(T^{a_1} T^{a_2} \ldots T^{a_n})}{\sigma_{\ell^+1}\sigma_{12}\ldots\sigma_{n\,\ell^-}\sigma_{\ell^-\ell^+}}$

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- Kinematics: $\mathcal{I}^{(1)}_{\text{MHV}} = \sum_{\rho \in S_n} \frac{N^{(1)}_{\text{MHV}}(\rho)}{\sigma_{\ell^+ \rho_1}\sigma_{\rho_1 \rho_2} \ldots \sigma_{\rho_n \ell^-}\sigma_{\ell^- \ell^+}}$

Checks:

- 4- and 5-particle integrands
- factorization

Integrands: Super Yang-Mills

integrand with Feynman propagators

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Checks:

- 4- and 5-particle integrands
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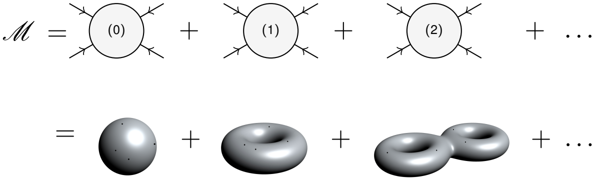
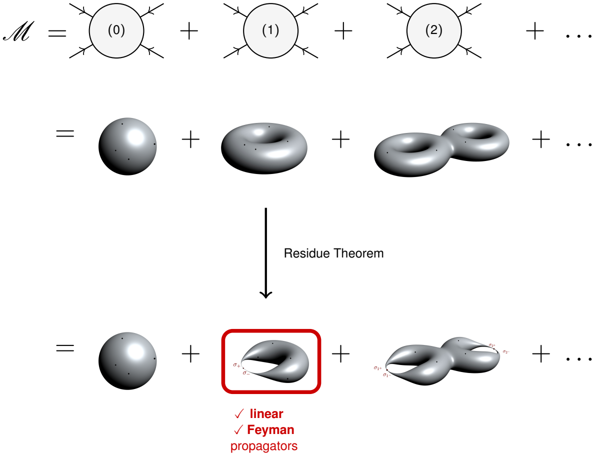
Proposal: $\mathcal{I}^{(1)}_{\text{kin}} = \sum_r \text{Pf}\,'(M^r_{\text{NS}}) - \frac{c_d}{\sigma^2_{\ell^+\,\ell^-}}\text{Pf}\,(M_2)$

Summary



Summary

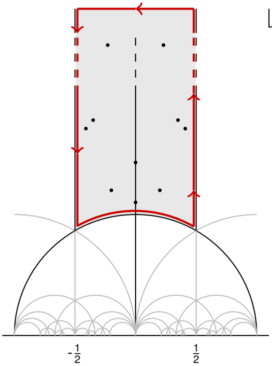


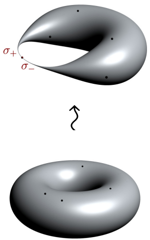
Outlook

- Super Yang-Mills general case: proof
- Origin of the deformation and supergravity case

Cauchy residue theorem on $\mathcal{F}$

$u = 0$ on support of $\mathcal{E}_i^{\text{def}} = 0 \quad \leftrightarrow \quad q \equiv e^{2i\pi\tau} = 0$





Thank you!