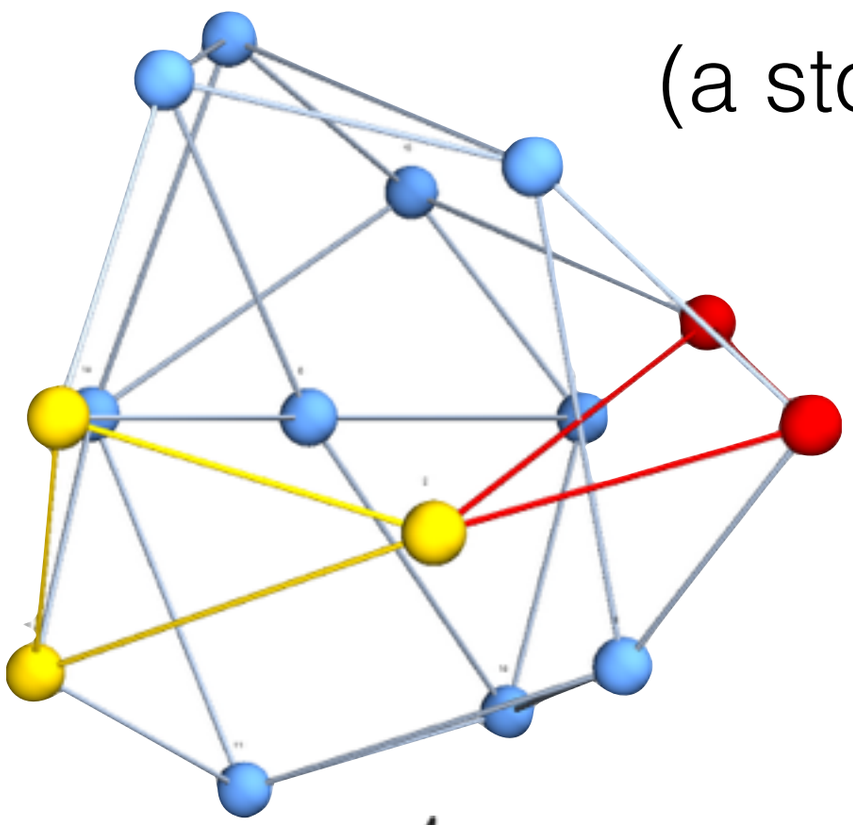
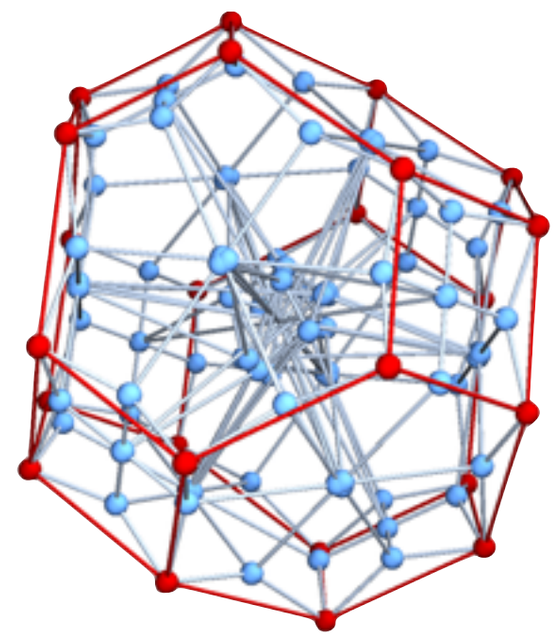
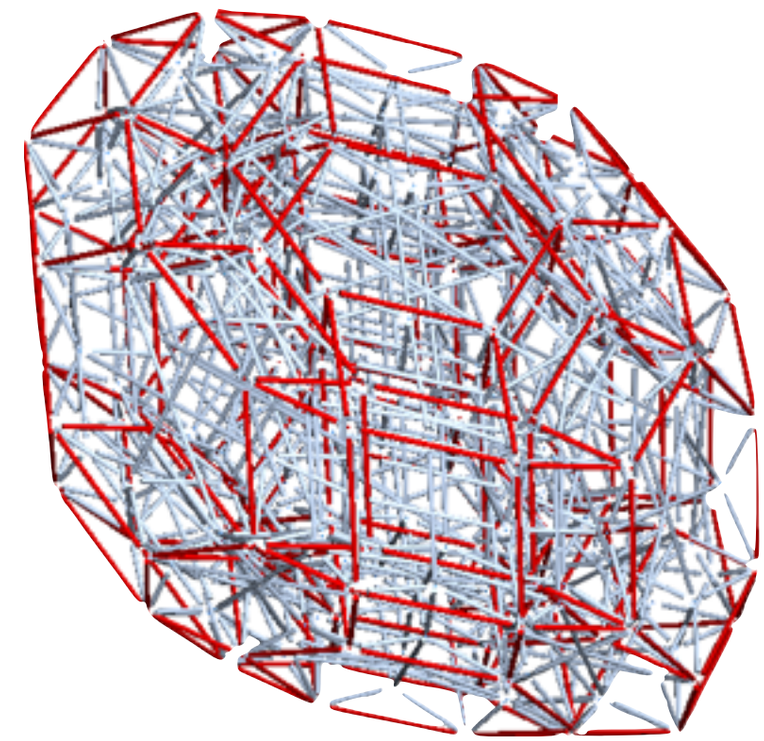
Playful Constructions in Double Copy Predictions

(a story about stories)



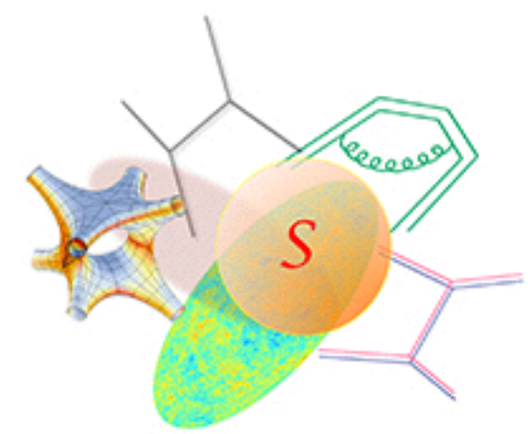




IPhT cea saclay

John Joseph M Carrasco

17 April 2017



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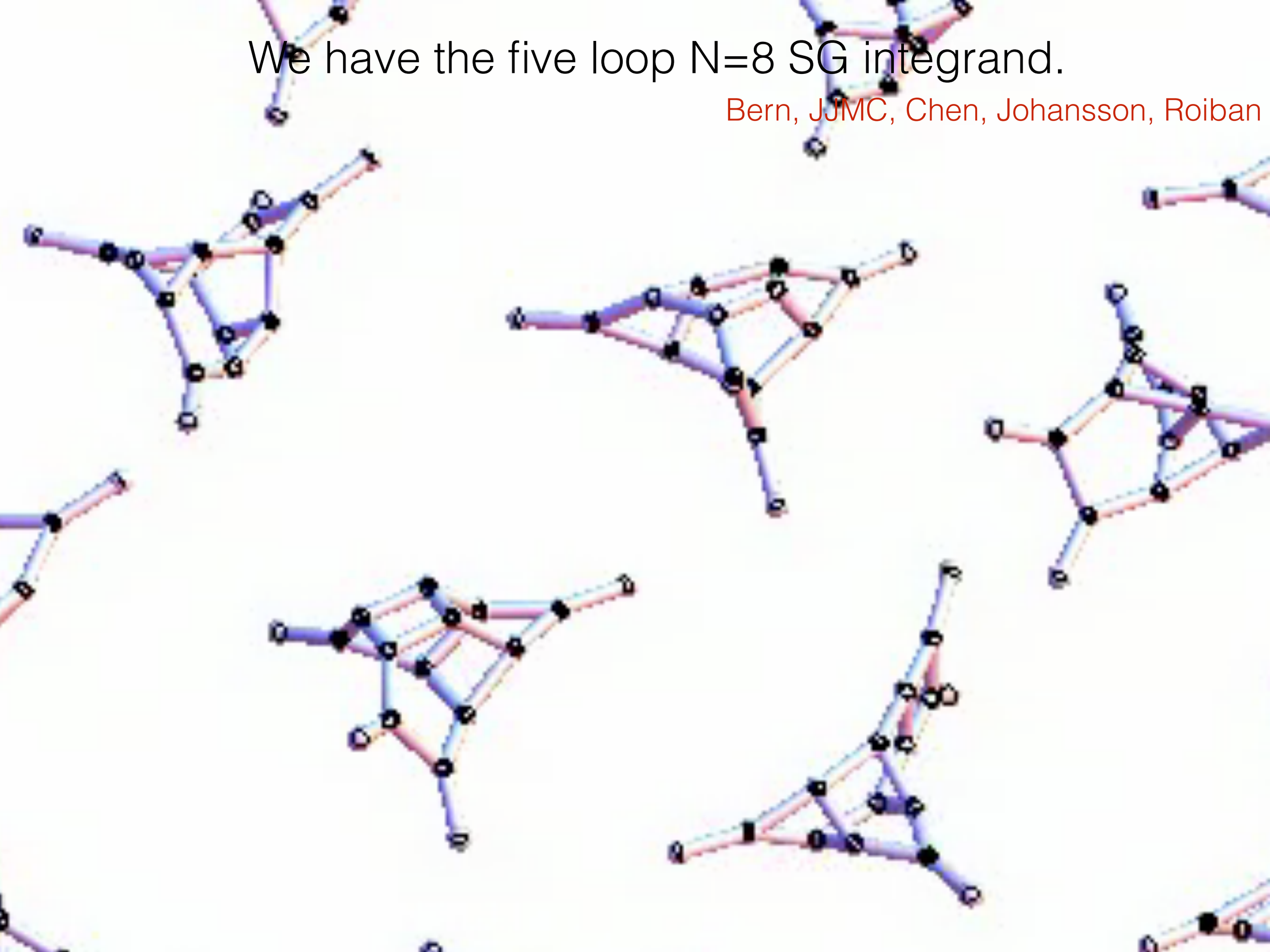
preQFT: Strategic Predictions for Quantum Field Theories

Scattering Amplitudes: from Gauge Theory to Gravity

We have the five loop N=8 SG integrand.

Bern, JJMC, Chen, Johansson, Roiban



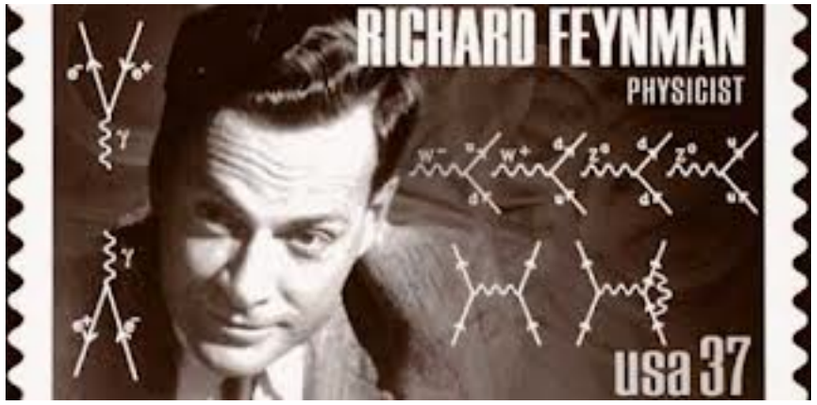
For more details see last talk of the conference on Friday:

- 11:30AM - Zvi Bern (UCLA) - *Generalized Double Copies and the Five-Loop Integrand of N = 8 Supergravity*

Stories

Quantum stories from Actions?

Use Feynman rules.



trees: semi-classical

loops: increasing quantum corrections

Why don’t we (typically) do this?

Off-shell three-graviton vertex:

$$
\begin{aligned}
\frac{\delta S^3}{\delta\varphi_{\mu\nu}\delta\varphi_{\sigma\tau}\delta\varphi_{\rho\lambda}} \quad \to \quad & 2\eta^{\mu\tau}\eta^{\nu\sigma}k_1{}^{\lambda}k_1{}^{\rho} + 2\eta^{\mu\sigma}\eta^{\nu\tau}k_1{}^{\lambda}k_1{}^{\rho} - 2\eta^{\mu\nu}\eta^{\sigma\tau}k_1{}^{\lambda}k_1{}^{\rho} + \\
& 2\eta^{\lambda\tau}\eta^{\mu\nu}k_1{}^{\sigma}k_1{}^{\rho} + 2\eta^{\lambda\sigma}\eta^{\mu\nu}k_1{}^{\tau}k_1{}^{\rho} + \eta^{\mu\tau}\eta^{\nu\sigma}k_2{}^{\lambda}k_1{}^{\rho} + \eta^{\mu\sigma}\eta^{\nu\tau}k_2{}^{\lambda}k_1{}^{\rho} + \eta^{\lambda\tau}\eta^{\nu\sigma}k_2{}^{\mu}k_1{}^{\rho} + \\
& \eta^{\lambda\sigma}\eta^{\nu\tau}k_2{}^{\mu}k_1{}^{\rho} + \eta^{\lambda\tau}\eta^{\mu\sigma}k_2{}^{\nu}k_1{}^{\rho} + \eta^{\lambda\sigma}\eta^{\mu\tau}k_2{}^{\nu}k_1{}^{\rho} + \eta^{\lambda\tau}\eta^{\nu\sigma}k_3{}^{\mu}k_1{}^{\rho} + \eta^{\lambda\sigma}\eta^{\nu\tau}k_3{}^{\mu}k_1{}^{\rho} - \\
& \eta^{\lambda\nu}\eta^{\sigma\tau}k_3{}^{\mu}k_1{}^{\rho} + \eta^{\lambda\tau}\eta^{\mu\sigma}k_3{}^{\nu}k_1{}^{\rho} + \eta^{\lambda\sigma}\eta^{\mu\tau}k_3{}^{\nu}k_1{}^{\rho} - \eta^{\lambda\mu}\eta^{\sigma\tau}k_3{}^{\nu}k_1{}^{\rho} + \eta^{\lambda\nu}\eta^{\mu\tau}k_3{}^{\sigma}k_1{}^{\rho} + \\
& \eta^{\lambda\mu}\eta^{\nu\tau}k_3{}^{\sigma}k_1{}^{\rho} + \eta^{\lambda\nu}\eta^{\mu\sigma}k_3{}^{\tau}k_1{}^{\rho} + \eta^{\lambda\mu}\eta^{\nu\sigma}k_3{}^{\tau}k_1{}^{\rho} + 2\eta^{\mu\nu}\eta^{\rho\tau}k_1{}^{\lambda}k_1{}^{\sigma} + 2\eta^{\mu\nu}\eta^{\rho\sigma}k_1{}^{\lambda}k_1{}^{\tau} - \\
& 2\eta^{\lambda\rho}\eta^{\mu\nu}k_1{}^{\sigma}k_1{}^{\tau} + 2\eta^{\lambda\nu}\eta^{\mu\rho}k_1{}^{\sigma}k_1{}^{\tau} + 2\eta^{\lambda\mu}\eta^{\nu\rho}k_1{}^{\sigma}k_1{}^{\tau} + \eta^{\mu\tau}\eta^{\nu\rho}k_1{}^{\sigma}k_2{}^{\lambda} + \eta^{\mu\rho}\eta^{\nu\tau}k_1{}^{\sigma}k_2{}^{\lambda} + \\
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& \eta^{\lambda\rho}\eta^{\nu\tau}k_1{}^{\sigma}k_2{}^{\mu} + \eta^{\lambda\nu}\eta^{\rho\tau}k_1{}^{\sigma}k_2{}^{\mu} + \eta^{\lambda\sigma}\eta^{\nu\rho}k_1{}^{\tau}k_2{}^{\mu} - \eta^{\lambda\rho}\eta^{\nu\sigma}k_1{}^{\tau}k_2{}^{\mu} + \eta^{\lambda\nu}\eta^{\rho\sigma}k_1{}^{\tau}k_2{}^{\mu} + \\
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& k_2 - \eta^{\lambda\tau}\eta^{\mu\rho}\eta^{\nu\sigma}k_1\cdot k_2 + \eta^{\lambda\rho}\eta^{\mu\tau}\eta^{\nu\sigma}k_1\cdot k_2 - \eta^{\lambda\sigma}\eta^{\mu\rho}\eta^{\nu\tau}k_1\cdot k_2 + \eta^{\lambda\rho}\eta^{\mu\sigma}\eta^{\nu\tau}k_1\cdot k_2 + \\
& 2\eta^{\lambda\tau}\eta^{\mu\nu}\eta^{\rho\sigma}k_1\cdot k_2 - \eta^{\lambda\nu}\eta^{\mu\tau}\eta^{\rho\sigma}k_1\cdot k_2 - \eta^{\lambda\mu}\eta^{\nu\tau}\eta^{\rho\sigma}k_1\cdot k_2 + 2\eta^{\lambda\sigma}\eta^{\mu\nu}\eta^{\rho\tau}k_1\cdot k_2 - \\
& \eta^{\lambda\nu}\eta^{\mu\sigma}\eta^{\rho\tau}k_1\cdot k_2 - \eta^{\lambda\mu}\eta^{\nu\sigma}\eta^{\rho\tau}k_1\cdot k_2 - 2\eta^{\lambda\rho}\eta^{\mu\nu}\eta^{\sigma\tau}k_1\cdot k_2 + 2\eta^{\lambda\nu}\eta^{\mu\rho}\eta^{\sigma\tau}k_1\cdot k_2 + \\
& 2\eta^{\lambda\mu}\eta^{\nu\rho}\eta^{\sigma\tau}k_1\cdot k_2 - \eta^{\lambda\tau}\eta^{\mu\sigma}\eta^{\nu\rho}k_1\cdot k_3 - \eta^{\lambda\sigma}\eta^{\mu\tau}\eta^{\nu\rho}k_1\cdot k_3 - \eta^{\lambda\tau}\eta^{\mu\rho}\eta^{\nu\sigma}k_1\cdot k_3 + \\
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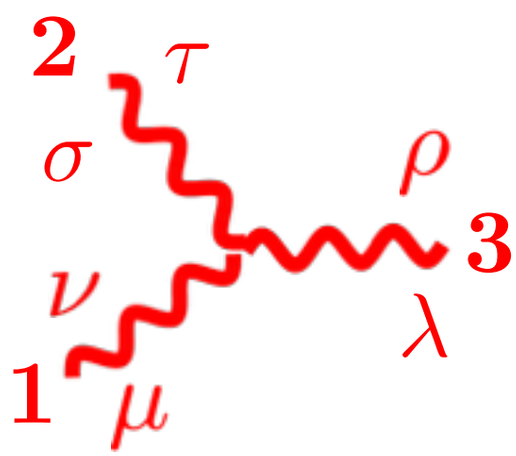
& \eta^{\lambda\nu}\eta^{\mu\tau}\eta^{\rho\sigma}k_1\cdot k_3 - \eta^{\lambda\mu}\eta^{\nu\tau}\eta^{\rho\sigma}k_1\cdot k_3 + 2\eta^{\lambda\sigma}\eta^{\mu\nu}\eta^{\rho\tau}k_1\cdot k_3 - \eta^{\lambda\nu}\eta^{\mu\sigma}\eta^{\rho\tau}k_1\cdot k_3 - \\
& \eta^{\lambda\mu}\eta^{\nu\sigma}\eta^{\rho\tau}k_1\cdot k_3 - 2\eta^{\lambda\rho}\eta^{\mu\nu}\eta^{\sigma\tau}k_1\cdot k_3 + \eta^{\lambda\nu}\eta^{\mu\rho}\eta^{\sigma\tau}k_1\cdot k_3 + \eta^{\lambda\mu}\eta^{\nu\rho}\eta^{\sigma\tau}k_1\cdot k_3 - \\
& \eta^{\lambda\tau}\eta^{\mu\sigma}\eta^{\nu\rho}k_2\cdot k_3 - \eta^{\lambda\sigma}\eta^{\mu\tau}\eta^{\nu\rho}k_2\cdot k_3 - \eta^{\lambda\tau}\eta^{\mu\rho}\eta^{\nu\sigma}k_2\cdot k_3 + 2\eta^{\lambda\rho}\eta^{\mu\tau}\eta^{\nu\sigma}k_2\cdot k_3 - \\
& \eta^{\lambda\sigma}\eta^{\mu\rho}\eta^{\nu\tau}k_2\cdot k_3 + 2\eta^{\lambda\rho}\eta^{\mu\sigma}\eta^{\nu\tau}k_2\cdot k_3 + \eta^{\lambda\tau}\eta^{\mu\nu}\eta^{\rho\sigma}k_2\cdot k_3 - \eta^{\lambda\nu}\eta^{\mu\tau}\eta^{\rho\sigma}k_2\cdot k_3 - \\
& \eta^{\lambda\mu}\eta^{\nu\tau}\eta^{\rho\sigma}k_2\cdot k_3 + \eta^{\lambda\sigma}\eta^{\mu\nu}\eta^{\rho\tau}k_2\cdot k_3 - \eta^{\lambda\nu}\eta^{\mu\sigma}\eta^{\rho\tau}k_2\cdot k_3 - \eta^{\lambda\mu}\eta^{\nu\sigma}\eta^{\rho\tau}k_2\cdot k_3 - \\
& 2\eta^{\lambda\rho}\eta^{\mu\nu}\eta^{\sigma\tau}k_2\cdot k_3 + 2\eta^{\lambda\nu}\eta^{\mu\rho}\eta^{\sigma\tau}k_2\cdot k_3 + 2\eta^{\lambda\mu}\eta^{\nu\rho}\eta^{\sigma\tau}k_2\cdot k_3
\end{aligned}
$$

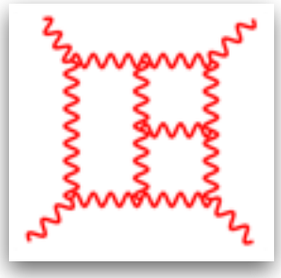
171 terms

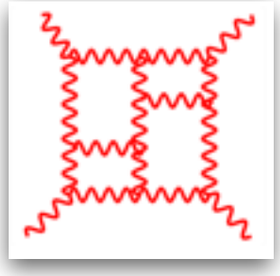


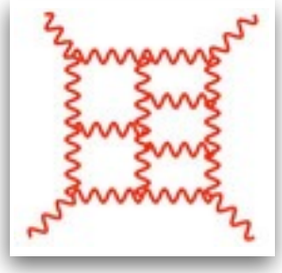
[DeWitt, 1967]

Textbook approach crumbles:

Feynman rules for a graviton: 171 terms per vertex
3 terms per edge

A single 3 loop diagram:  **~10^{20}** TERMS

4 loop diagram:  **~10^{26}** TERMS

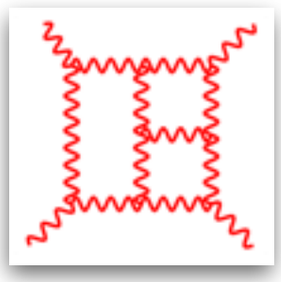
5 loop diagram:  **~10^{31}** TERMS

BUT FINAL EXPRESSIONS ARE TRACTABLE

Vast majority of terms: unphysical freedom that must cancel

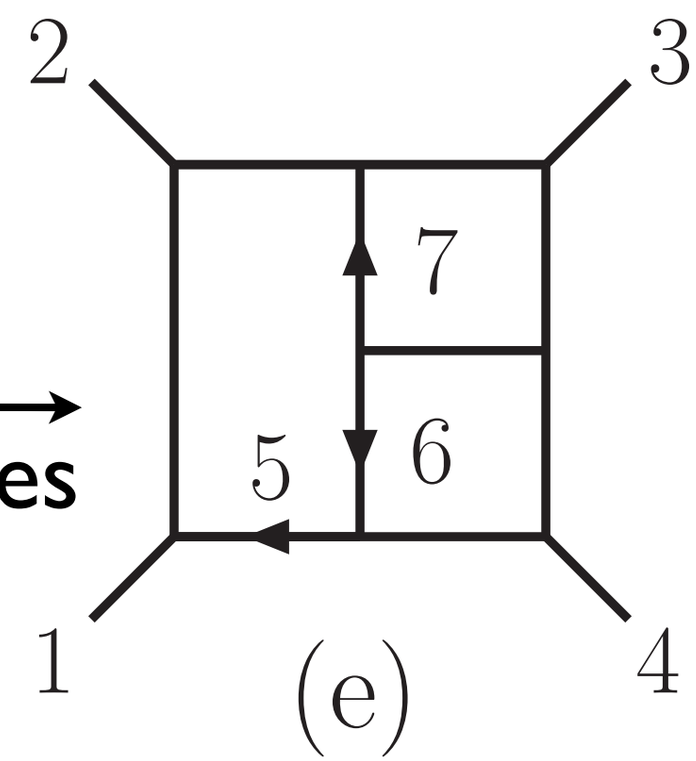
Bern, JJMC, Dixon, Kosower, Roiban '07

MOST SYMMETRIC 4D THEORY, N=8 SUGRA



~10^{20} TERMS

add all other particles



$$\propto \int stu\mathcal{M}_4^{(0)} \frac{\left(s\left(k_4+l_5\right)^2\right)^2}{d\circ(e) \equiv (l_5^2 l_6^2 l_7^2 (k_1-l_5)^2 \cdots)}$$

Some secrets obscured by actions:

Calculate with physical (on-shell) quantities: $k_i^2 = 0$

Physical (on-shell) **tree-level amplitudes contain all the information necessary to verify and build *all* loop-level amplitudes**

Bern, Dixon, Dunbar, and Kosower ('94,'95)

Bern, Dixon, and Kosower ('96)

Physical (on-shell) **three-vertices contain all the information necessary to build *all* tree-level amplitudes**

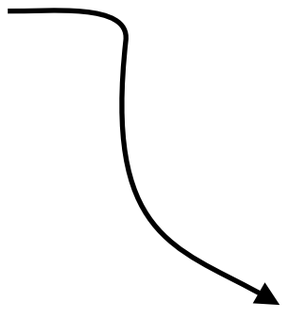
Britto, Cachazo, Feng, and Witten ('05)

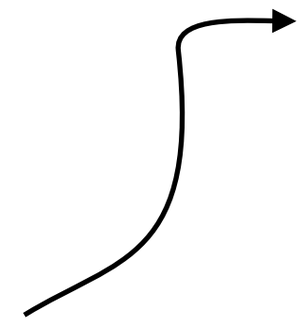
Easy verification => Natural construction. **Method of maximal cuts.**

Bern, JJMC, Johansson, Kosower ('07)


S-MATRIX
REVOLUTIONS

the game of **Scattering Amplitudes**





IN

free states

(no interactions)

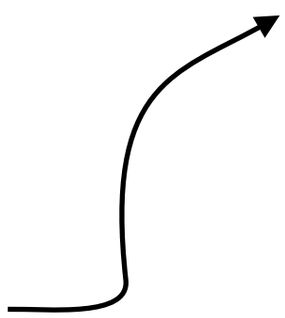
$$[\mathrm{S} - \mathrm{matrix}]$$

OUT

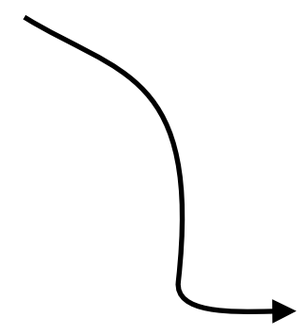
free states

(no interactions)

(all the interactions!)



Key Property: GAUGE INVARIANT



the game of **Scattering Amplitudes**

IN
free states
(no interactions)

$$[\mathrm{S-matrix}]$$

(all the interactions!)

OUT
free states
(no interactions)

the game of Scattering Amplitudes

QFT, NR-QM, String Theory, Classical Physics…

IN
free states
(no interactions)

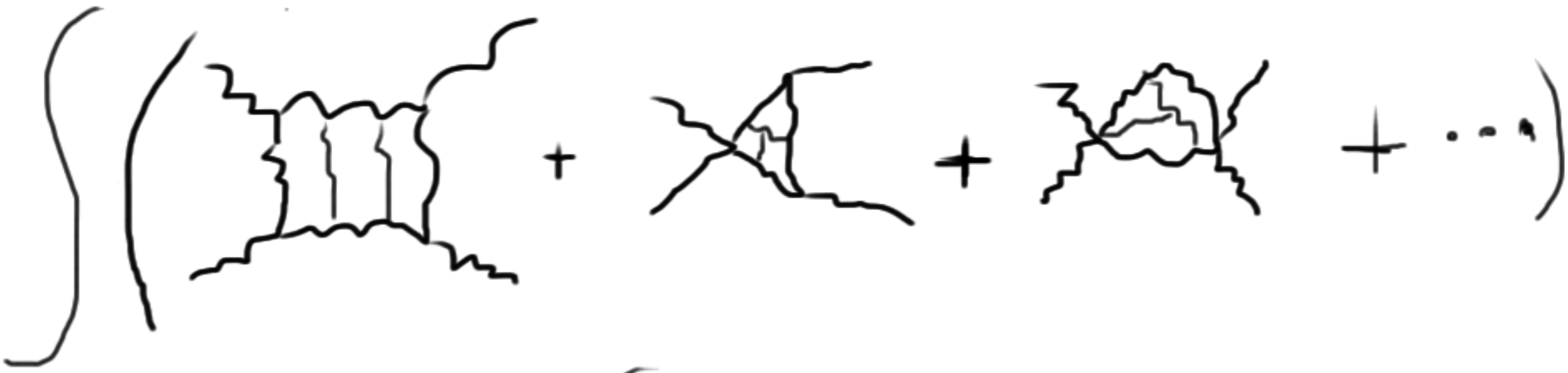
$$[\mathrm{S-matrix}]$$

(all the interactions!)

OUT
free states
(no interactions)

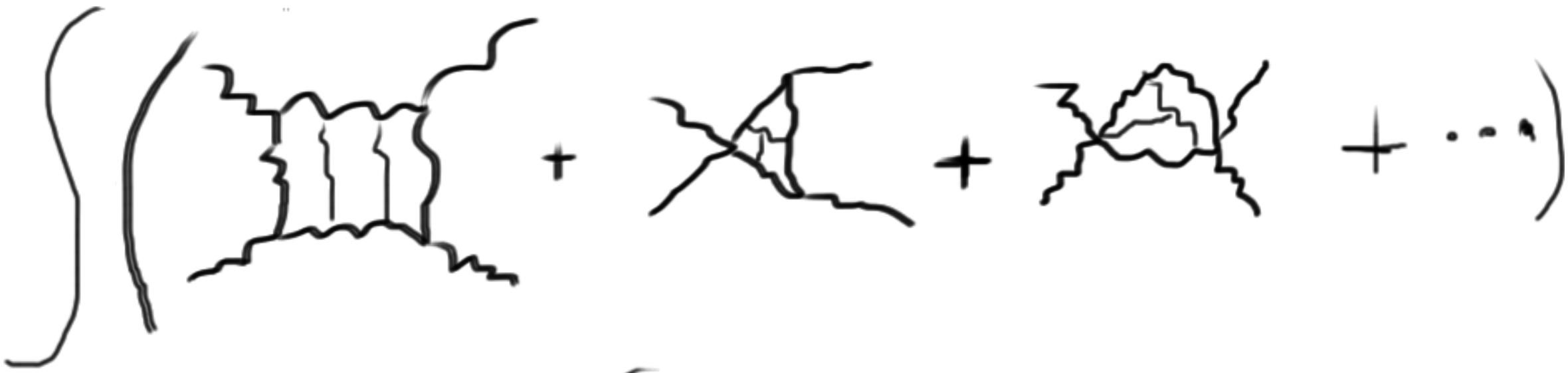
Same predictions, but definitely different stories

NECESSARY



expression

NECESSARY



$$= \int \sum_{g} \frac{n^{\circ} g}{d^{\circ} g}$$

SUFFICIENT

$$= \mathcal{U}_c \sum_g \frac{n_g}{d_g}$$

Bern, Dixon, Dunbar, and Kosower ('94,'95)

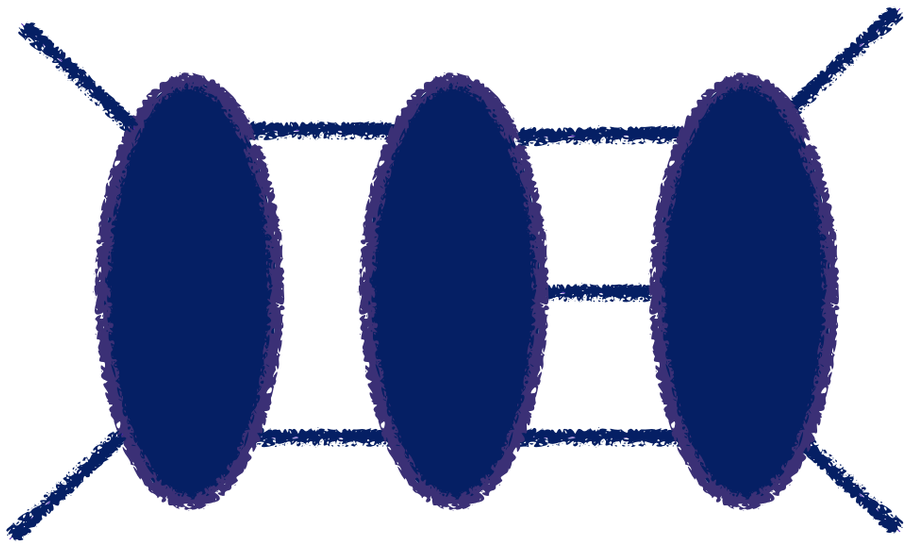
Bern, Dixon, and Kosower ('96)

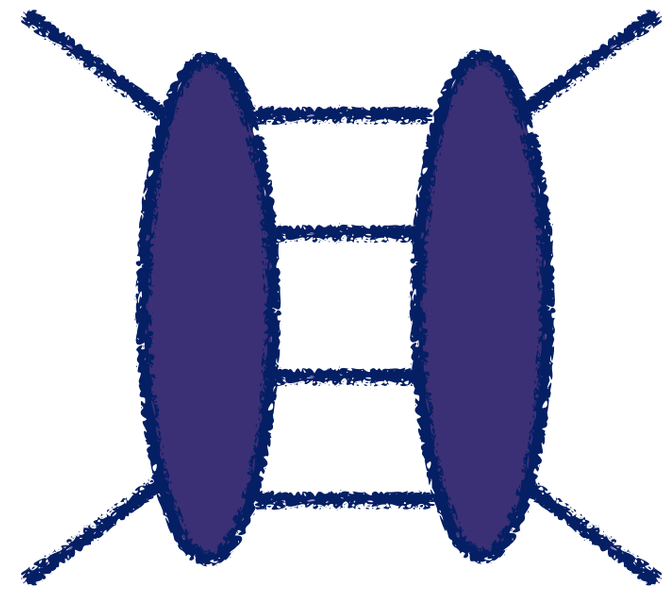
Britto, Cachazo, and Feng ('04)

$\forall\ \mathcal{U}_c \in$ unitarity cuts

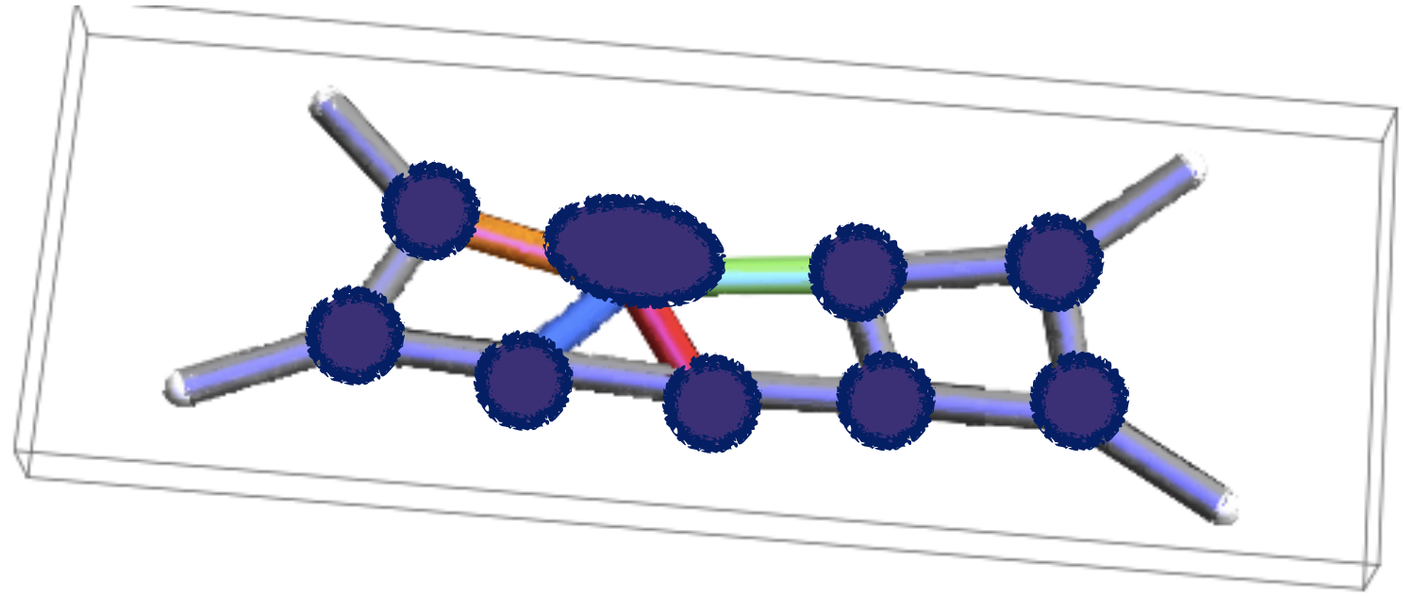
TREE-LEVEL CUT

$$\equiv \sum_{\text{int states}} \mathcal{M}^{(0)}_{U_c,1}\mathcal{M}^{(0)}_{U_c,2}\cdots\mathcal{M}^{(0)}_{U_c,m}$$

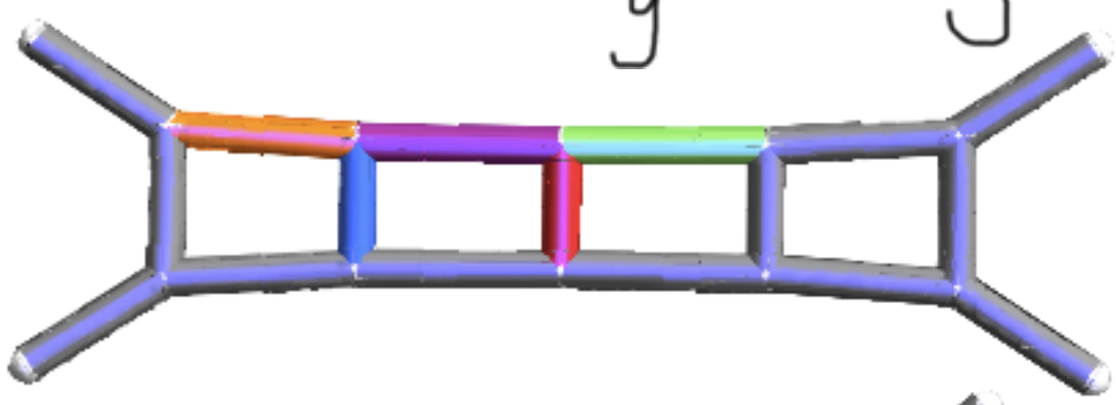


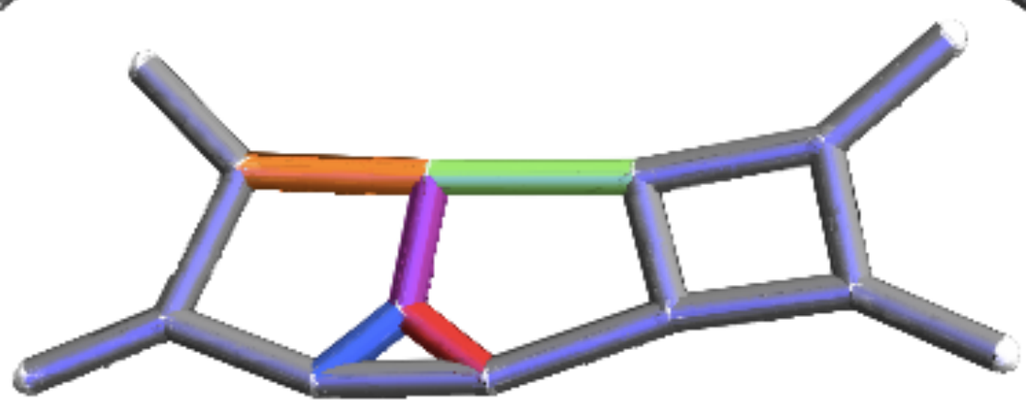


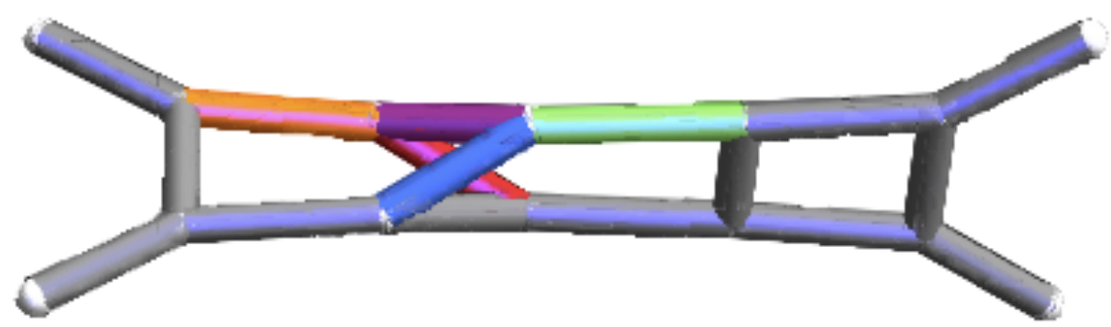
Simple Cut:



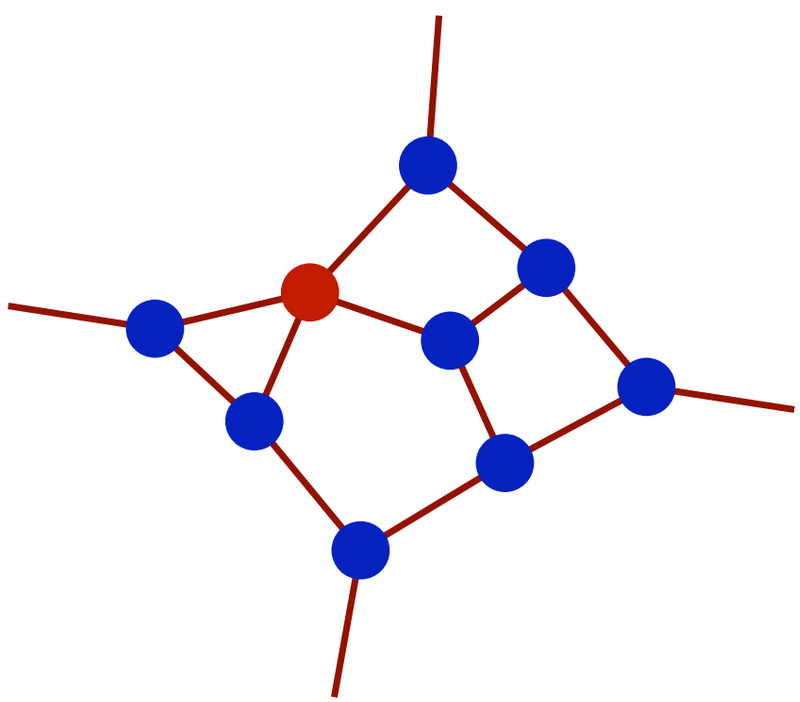
$$\mathcal{U}_c \sum_{g} \frac{n(g)}{d(g)} = \sum_{g_c} \frac{n(g_c)}{d(g_c)}$$







SPANNING CUTS



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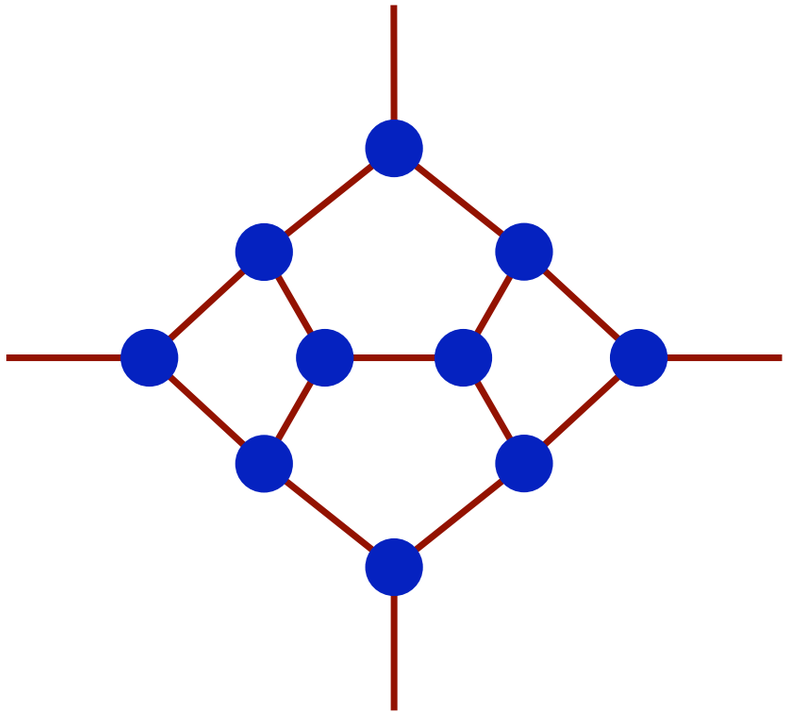


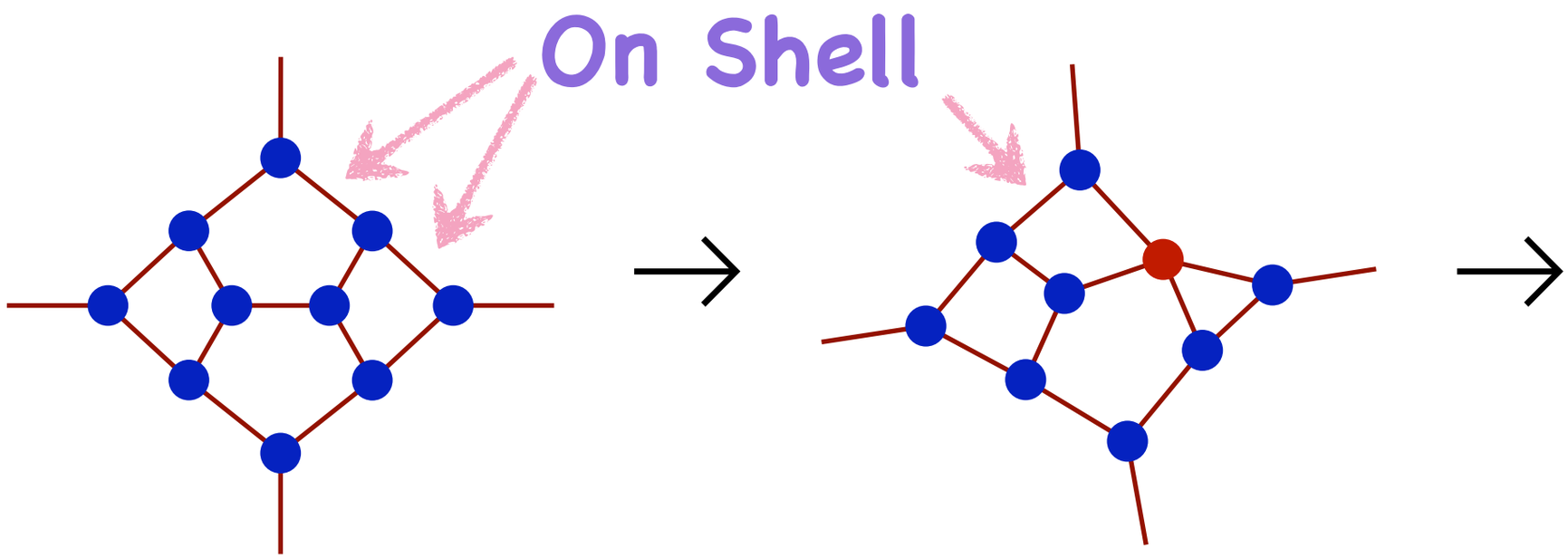
leads to notion of a **Minimal Spanning Set**

EASY VERIFICATION

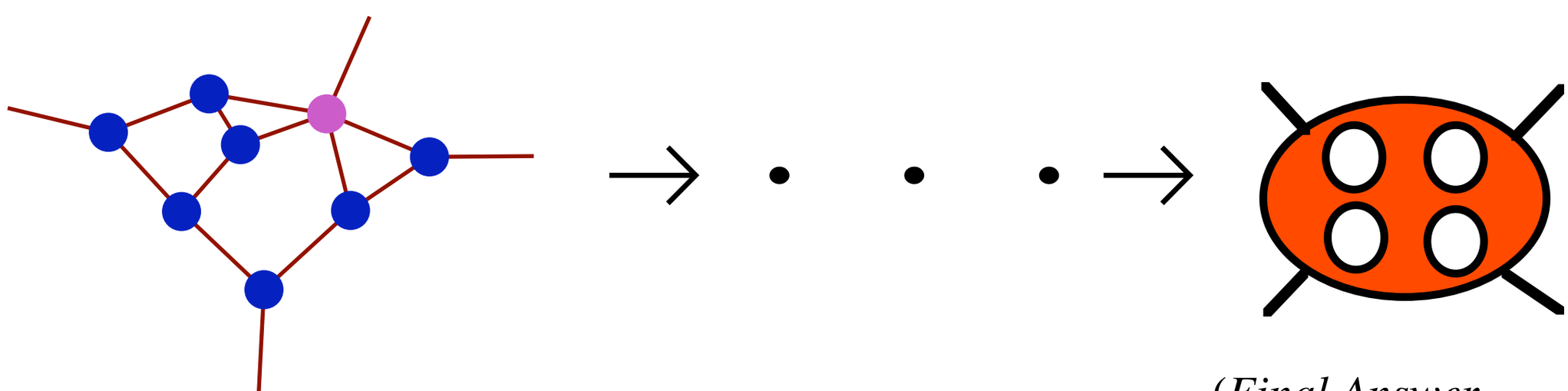
EASY VERIFICATION ⟶ NATURAL CONSTRUCTION

METHOD OF MAXIMAL CUTS

Bern, JJMC, Kosower, Johansson (`07)

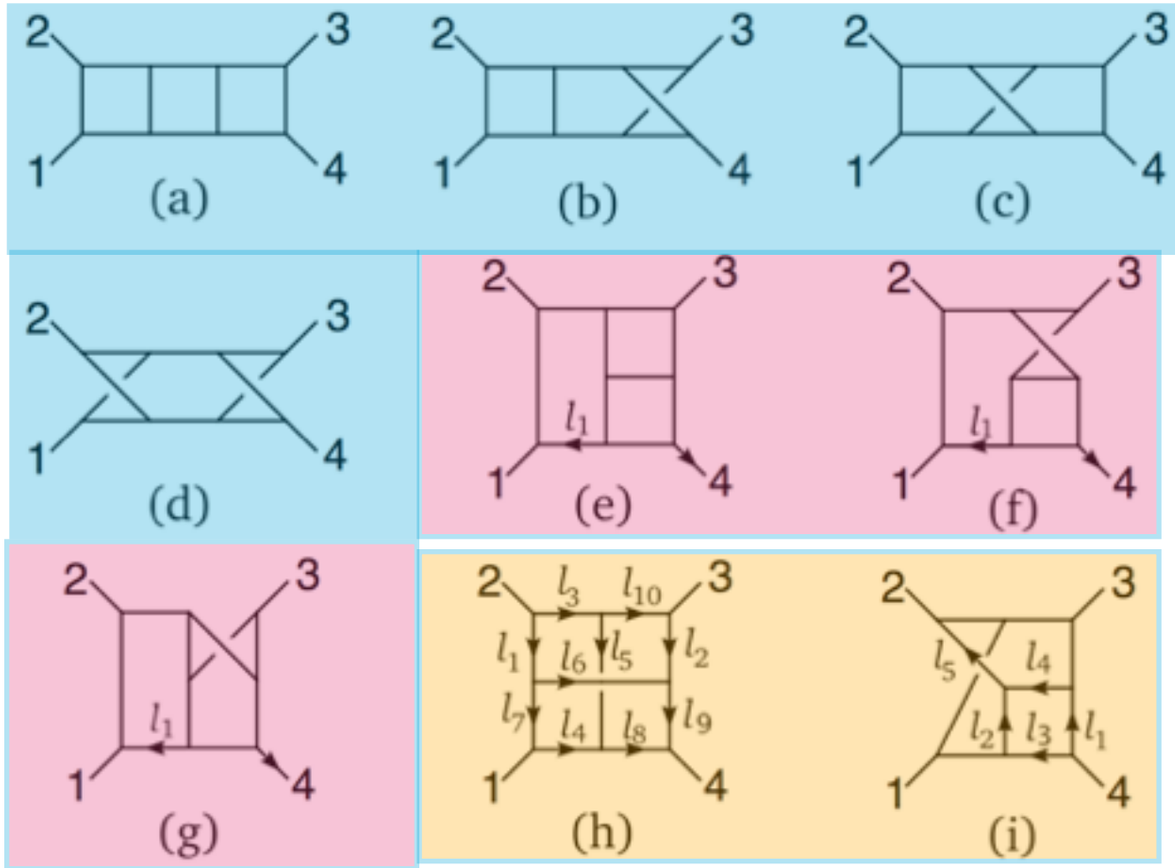


$(\forall$ exposed propagators $p^2 = 0\ )$



(*Final Answer,*
no cut conditions !)

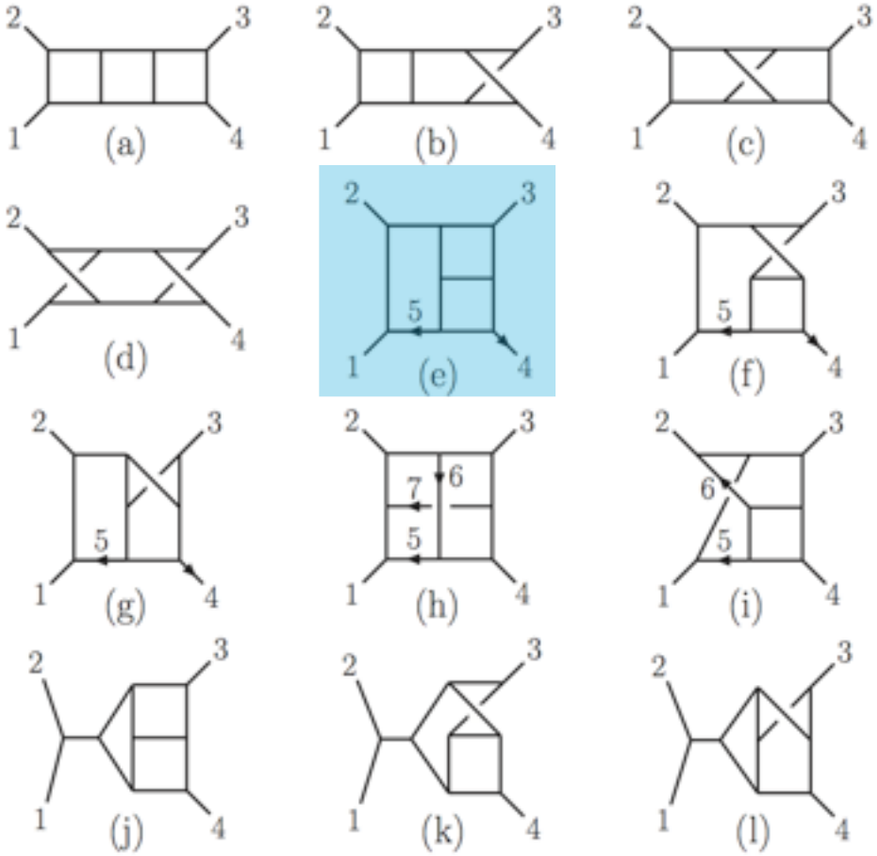
Bern, JJMC, Dixon, Kosower, Johansson, Roiban '07



Original solution of three-loop four-point N=4 sYM and N=8 sugra

Integral	$\mathcal{N}=4$ Yang-Mills	$\mathcal{N}=8$ Supergravity
(a)–(d)	s^2	$[s^2]^2$
(e)–(g)	$s(l_1+k_4)^2$	$[s\,(l_1+k_4)^2]^2$
(h)	$s(l_1+l_2)^2+t(l_3+l_4)^2$ $-sl_5^2-tl_6^2-st$	$(s(l_1+l_2)^2+t(l_3+l_4)^2-st)^2-s^2(2((l_1+l_2)^2-t)+l_5^2)l_5^2$ $-t^2(2((l_3+l_4)^2-s)+l_6^2)l_6^2-s^2(2l_7^2l_2^2+2l_1^2l_9^2+l_2^2l_9^2+l_1^2l_7^2)$ $-t^2(2l_3^2l_8^2+2l_{10}^2l_4^2+l_8^2l_4^2+l_3^2l_{10}^2)+2stl_5^2l_6^2$
(i)	$s(l_1+l_2)^2-t(l_3+l_4)^2$ $-\frac{1}{3}(s-t)l_5^2$	$(s(l_1+l_2)^2-t(l_3+l_4)^2)^2$ $-(s^2(l_1+l_2)^2+t^2(l_3+l_4)^2+\frac{1}{3}stu)l_5^2$

BCJ (2010)



Cubic Double-Copy Solution

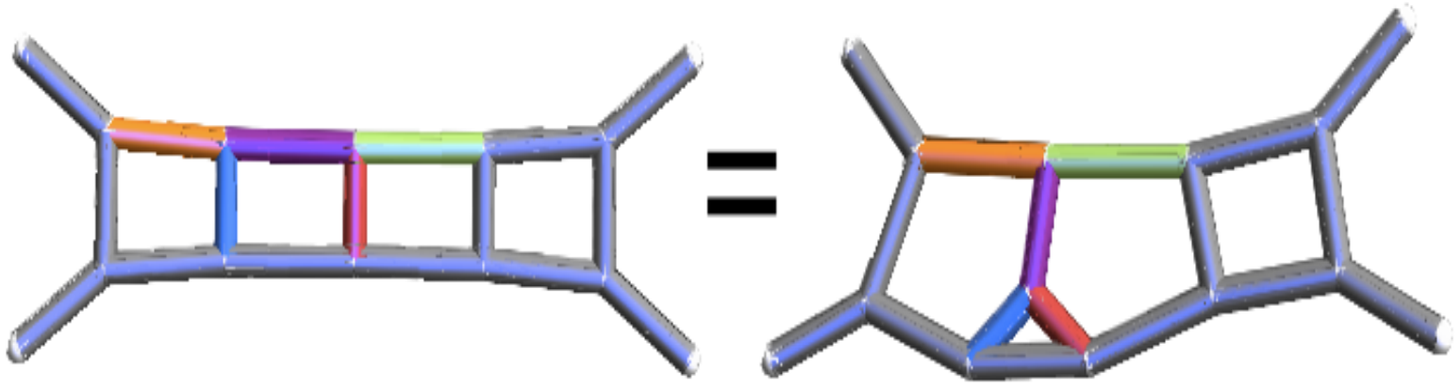
$$s = (k_1 + k_2)^2 \quad t = (k_1 + k_4)^2 \quad u = (k_1 + k_3)^2 \quad \tau_{i,j} = 2k_i \cdot l_j$$

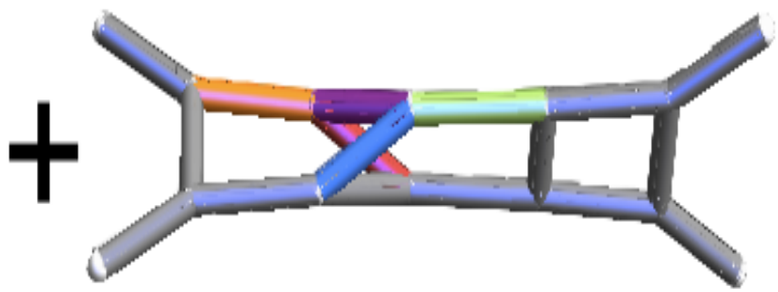
Integral $I^{(x)}$	$\mathcal{N} = 4$ Super-Yang-Mills ($\sqrt{\mathcal{N} = 8}$ supergravity) numerator
(a)–(d)	s^2
(e)–(g)	$\left(s\left(-\tau_{35} + \tau_{45} + t\right) - t\left(\tau_{25} + \tau_{45}\right) + u\left(\tau_{25} + \tau_{35}\right) - s^2 \right)/3$
(h)	$\left(s\left(2\tau_{15} - \tau_{16} + 2\tau_{26} - \tau_{27} + 2\tau_{35} + \tau_{36} + \tau_{37} - u\right) + t\left(\tau_{16} + \tau_{26} - \tau_{37} + 2\tau_{36} - 2\tau_{15} - 2\tau_{27} - 2\tau_{35} - 3\tau_{17}\right) + s^2 \right)/3$
(i)	$\left(s\left(-\tau_{25} - \tau_{26} - \tau_{35} + \tau_{36} + \tau_{45} + 2t\right) + t\left(\tau_{26} + \tau_{35} + 2\tau_{36} + 2\tau_{45} + 3\tau_{46}\right) + u\,\tau_{25} + s^2 \right)/3$
(j)-(l)	$s(t-u)/3$

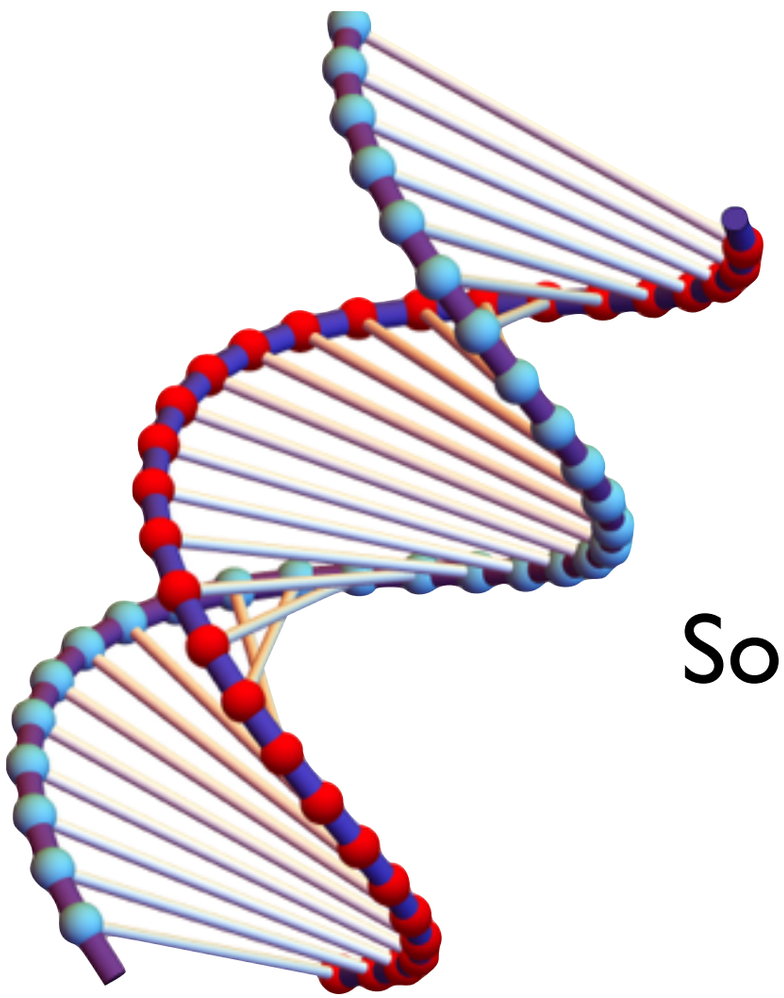
Color-Kinematics and Double Copy Construction

Bern, JJMC, Johansson ('08,'10)

Color and Kinematics dance together.



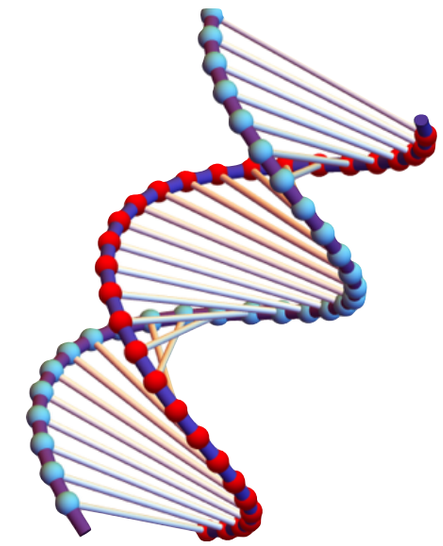




Solving Yang-Mills theories means solving Gravity theories.

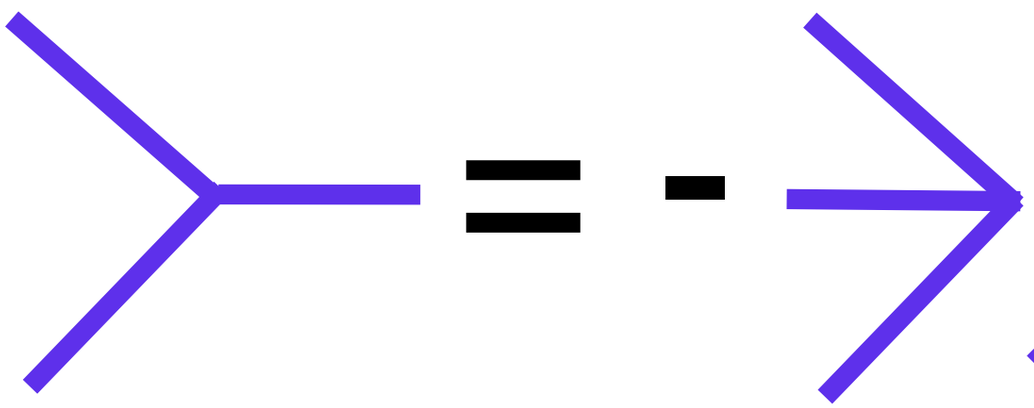
Bern, JJMC, Johansson ('08,'10)

Generic D-dimensional YM theories have a fascinating structure at tree-level

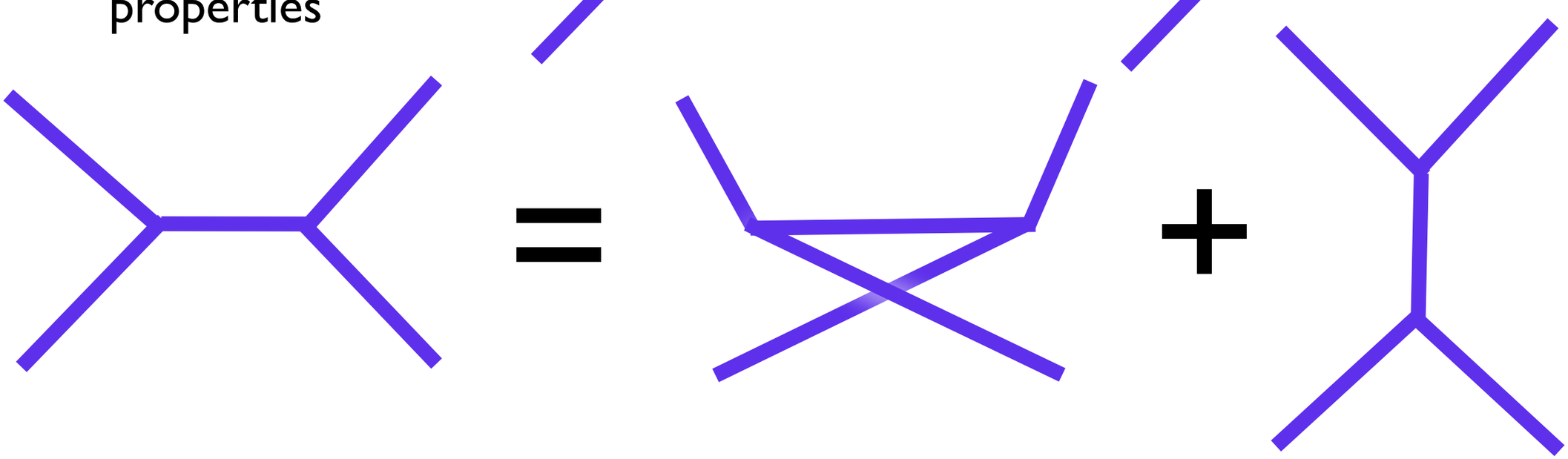


$$\mathcal{A}_m^{\text{tree}} = \sum_{\mathcal{G}\in\text{cubic}} \frac{c(\mathcal{G})n(\mathcal{G})}{D(\mathcal{G})}$$

Color factors and numerator factors satisfy similar lie algebra properties



Vertex Antisymmetry



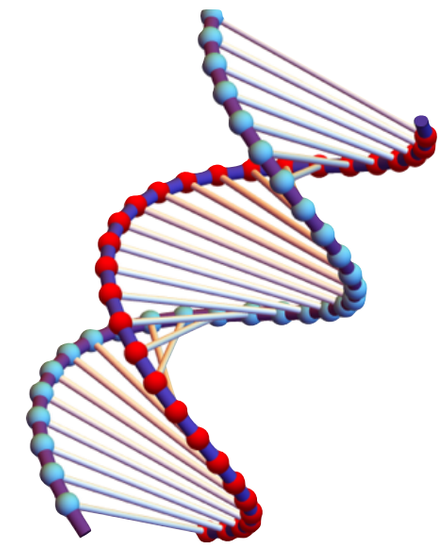
Jacobi

Color-Kinematic Duality!

Bern, JJMC, Johansson ('08,'10)

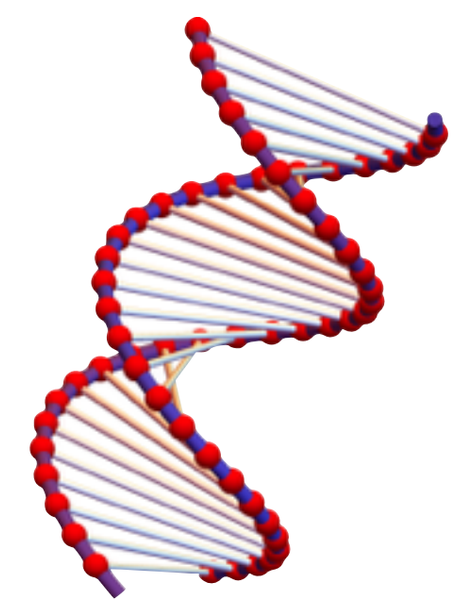
Generic D-dimensional YM theories have a fascinating structure at tree-level

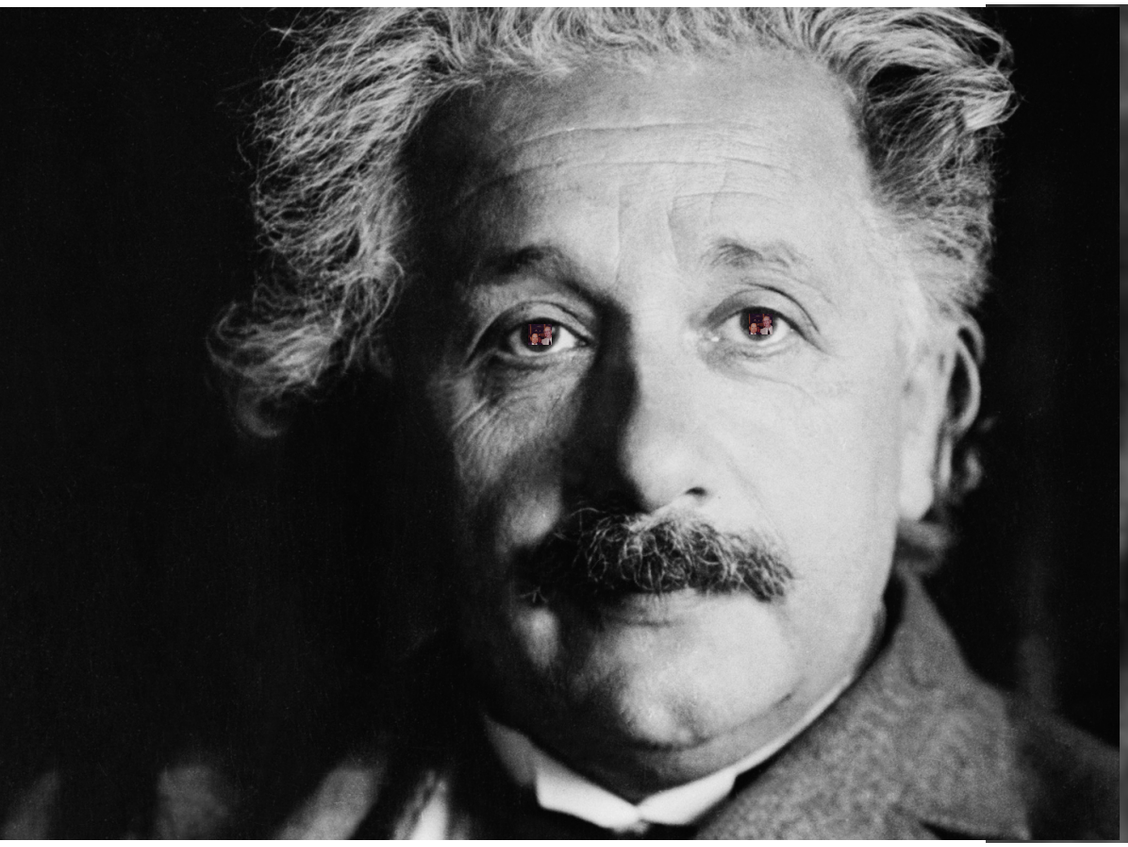
$$\mathcal{A}_m^{\text{tree}} = \sum_{\mathcal{G}\in\text{cubic}} \frac{c(\mathcal{G})n(\mathcal{G})}{D(\mathcal{G})}$$

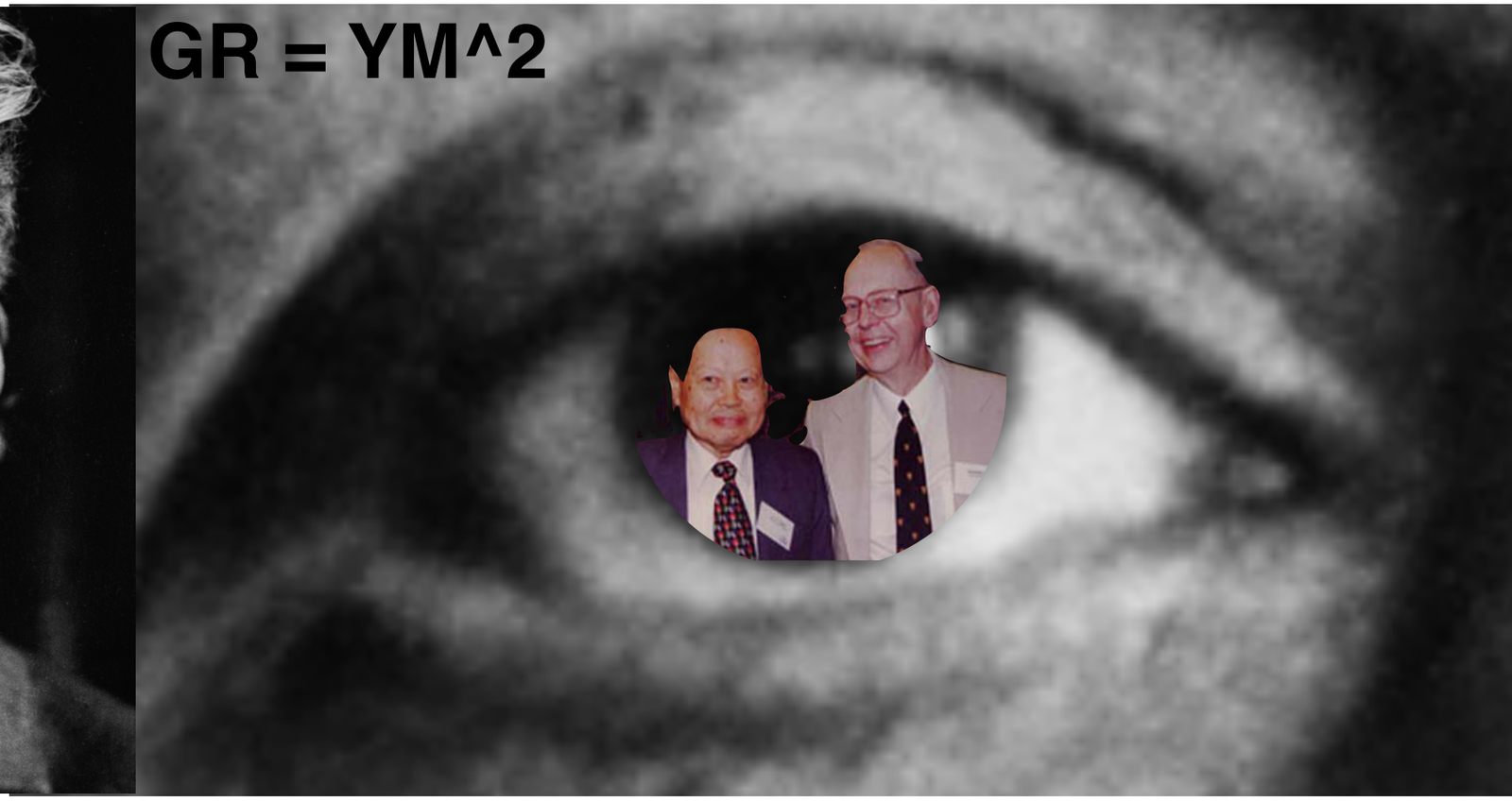


YM: Color-Kinematic Duality, makes manifest gravitational double copy structure:

$$-iM_n^{\text{tree}} = \sum_{\mathcal{G}\in\text{cubic}} \frac{n(\mathcal{G})\tilde{n}(\mathcal{G})}{D(\mathcal{G})}$$



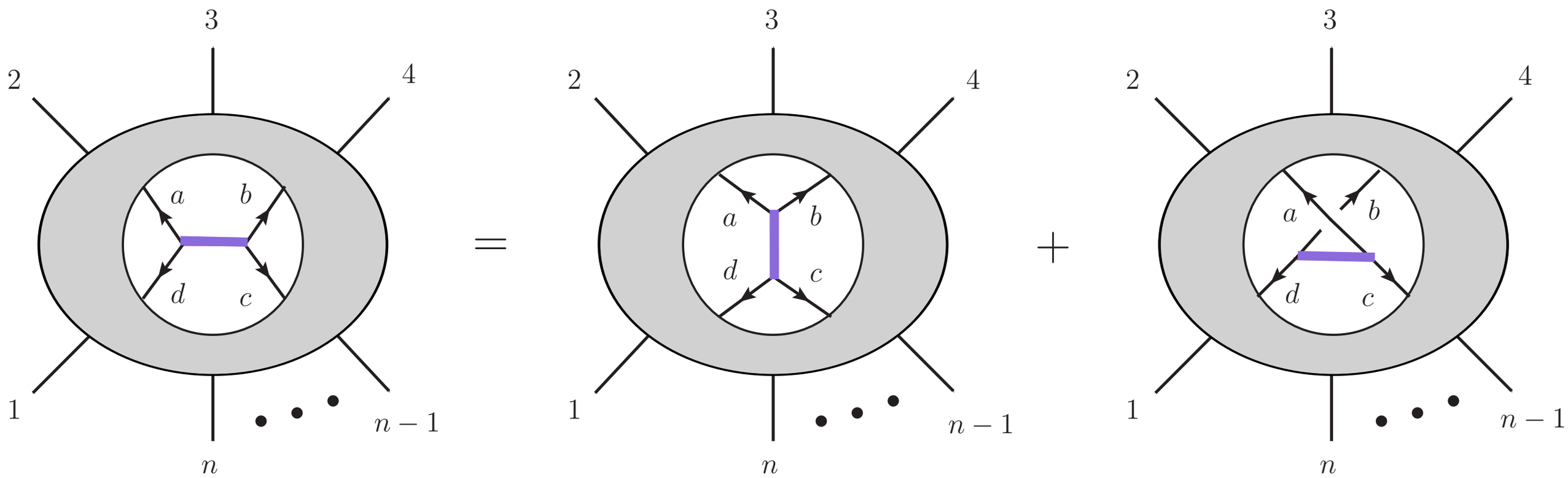



GR = YM^2

Valid multi-loop generalization?

$$\frac{(-i)^L}{g^{n-2+2L}}\mathcal{A}^{\text{loop}} = \sum_{\mathcal{G}\in\text{cubic}} \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S(\mathcal{G})} \frac{n(\mathcal{G})c(\mathcal{G})}{D(\mathcal{G})}$$

CONJECTURE: for all graphs, can impose CK on every edge:



Consequence of unitarity: double copy structure holds.

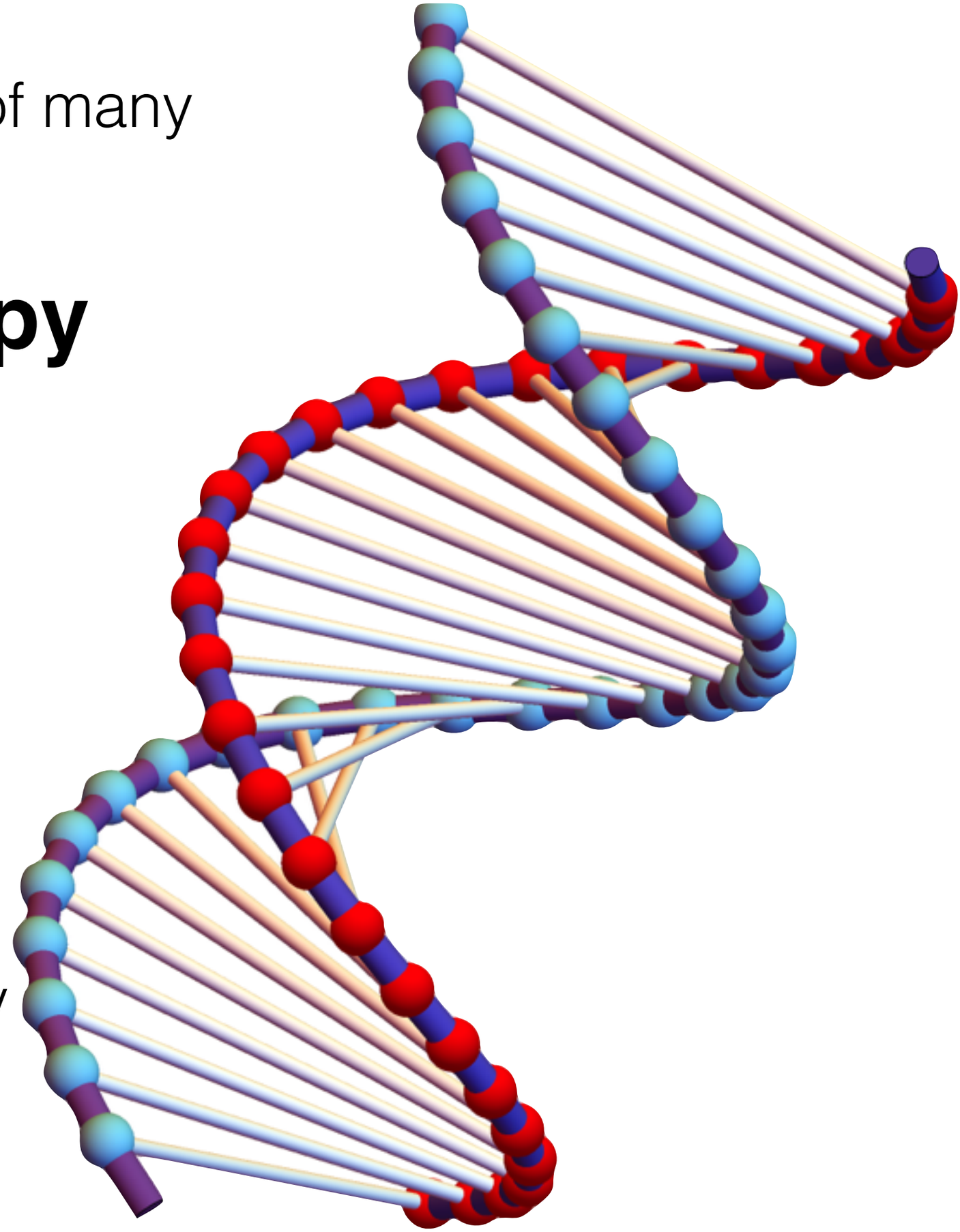
$$\frac{(-i)^{L+1}}{(\kappa/2)^{n-2+2L}}\mathcal{M}^{\text{loop}} = \sum_{\mathcal{G}\in\text{cubic}} \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S(\mathcal{G})} \frac{n(\mathcal{G})\tilde{n}(\mathcal{G})}{D(\mathcal{G})}$$

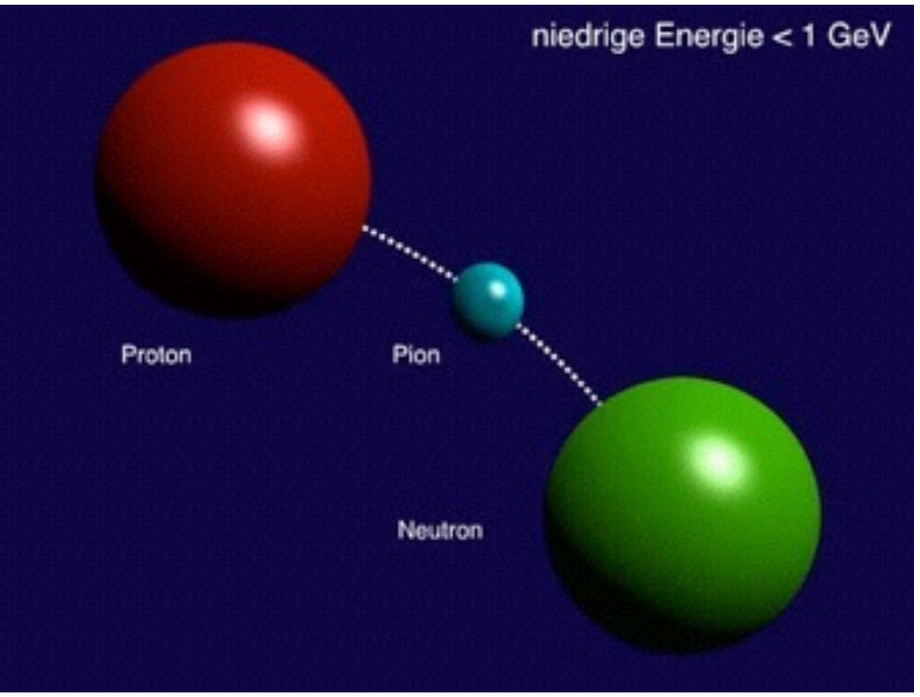
The scattering amplitudes of many relativistic theories admit a:

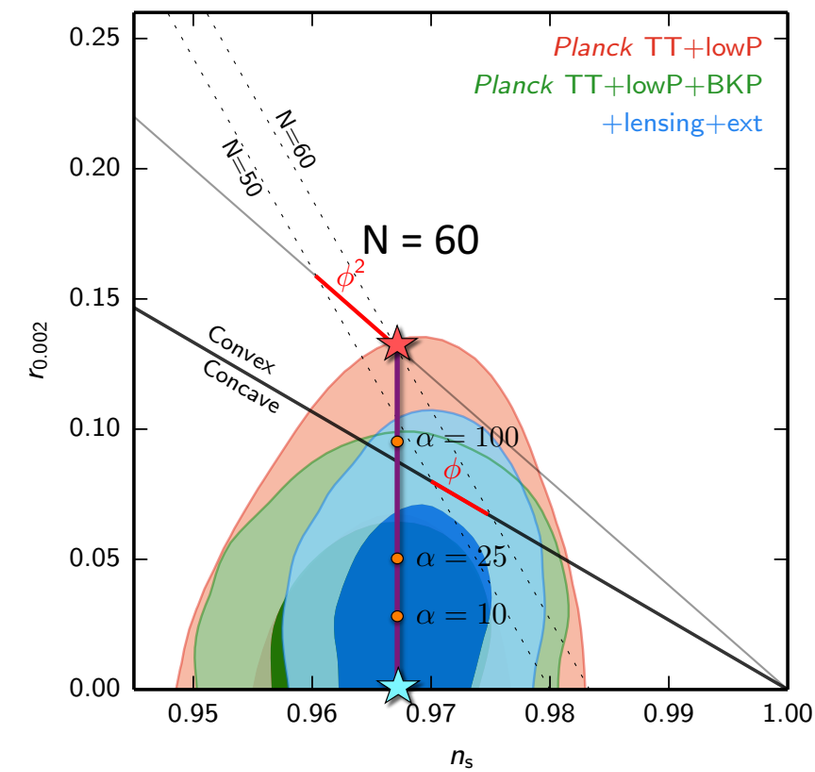
Double-copy
Numerator
Algebra

This points to previously hidden structure in many theories.

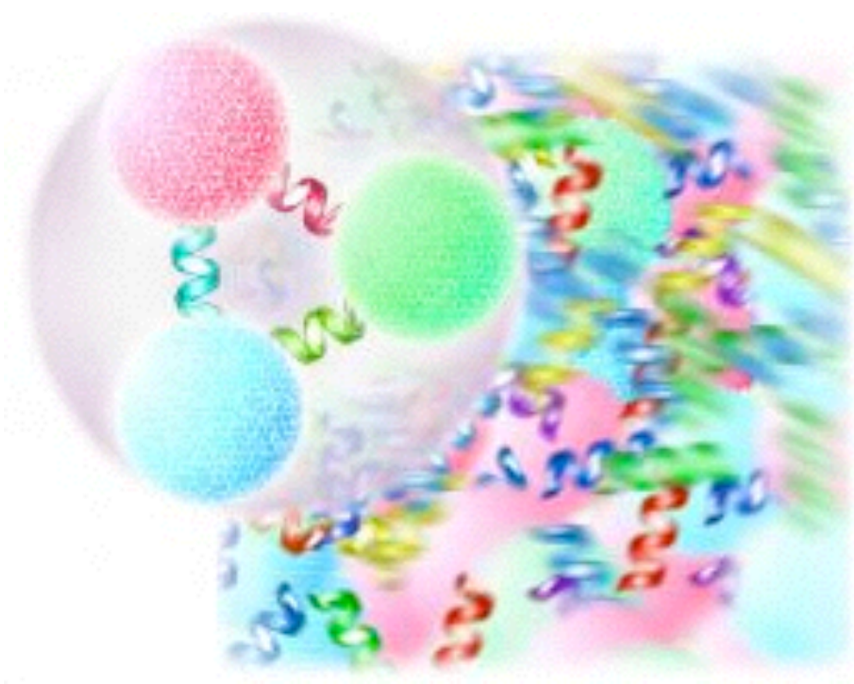
Structure yet to be generally understood at the level of the action.

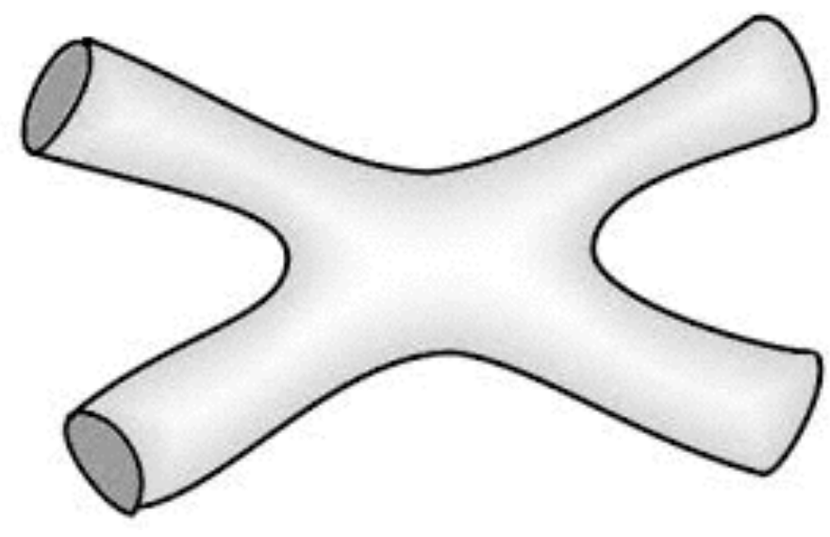






Many theories are double copy!





Key Point: **MANY Theories are Double Copies**

Bi-Adjoint Scalar: color $\otimes$ color

Bern, de Freitas, Wong ('99); Bern, Dennen, Huang; Du, Feng, Fu; Bjerrum-Bohr, Damgaard, Monteiro, O'Connell

(S)YM (…(S)QCD…): color $\otimes$ spin-1

BCJ ('08) Bjerrum-Bohr, Damgaard, Vanhove; Steiberger; Feng et al; Mafra, Schlotterer, ('08-'11); Johansson, Ochirov

(S)Gr (…(S)Einstein-YM…): spin-1 $\otimes$ spin-1

KLT('86); BCJ ('08); Chiodaroli, Gunaydin, Johansson, Roiban; Johansson,Ochirov

NLSM / Chiral Lagrangian: "color" $\otimes$ even-spin-0

Chen, Du '13 Cachazo, He, Yuan '14 Cheung, Shen '16

(S)Born-Infeld: spin-1 $\otimes$ even-spin-0

Cachazo, He, Yuan '14

Galileon: even-spin-0 $\otimes$ even-spin-0

Cachazo, He, Yuan '14 Cheung, Shen '16

Open String: α' $\otimes$ spin-1

Broedel, Schlotterer, Stieberger

Closed String: spin-1 $\otimes$ α' corrected spin-1

Broedel, Schlotterer, Stieberger

Z-theory: α' $\otimes$ "color"

Broedel, Schlotterer, Stieberger; JJMC, Mafra, Schlotterer

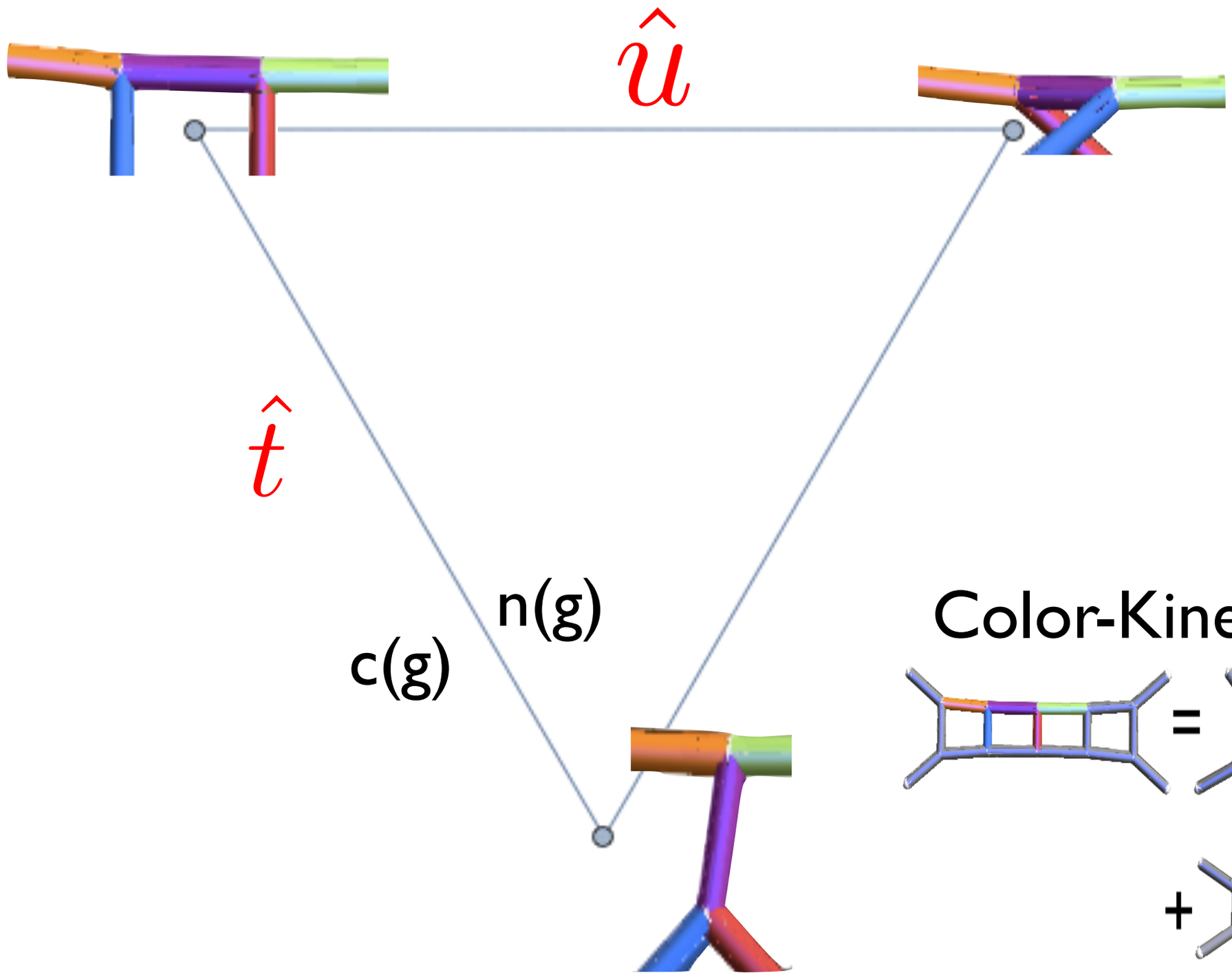
a geometric guide to color-kinematics

Physics = Geometry

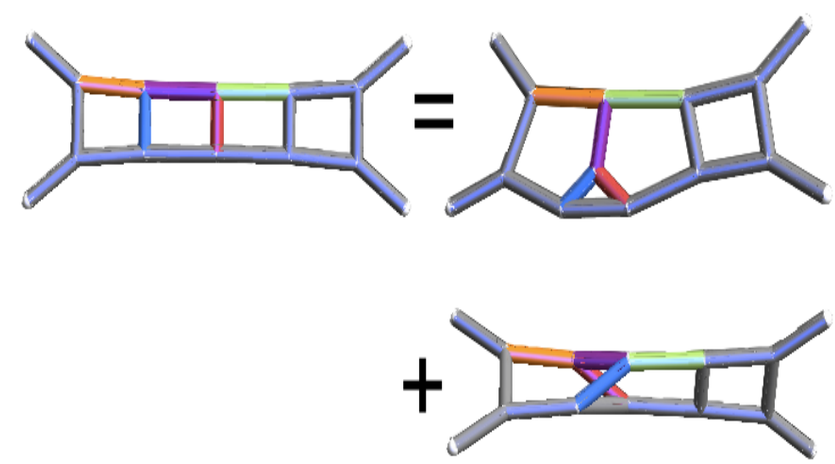
(the best polytopes are graphs of graphs!)

JJMC (to appear)

Convenient language: graphs of graphs

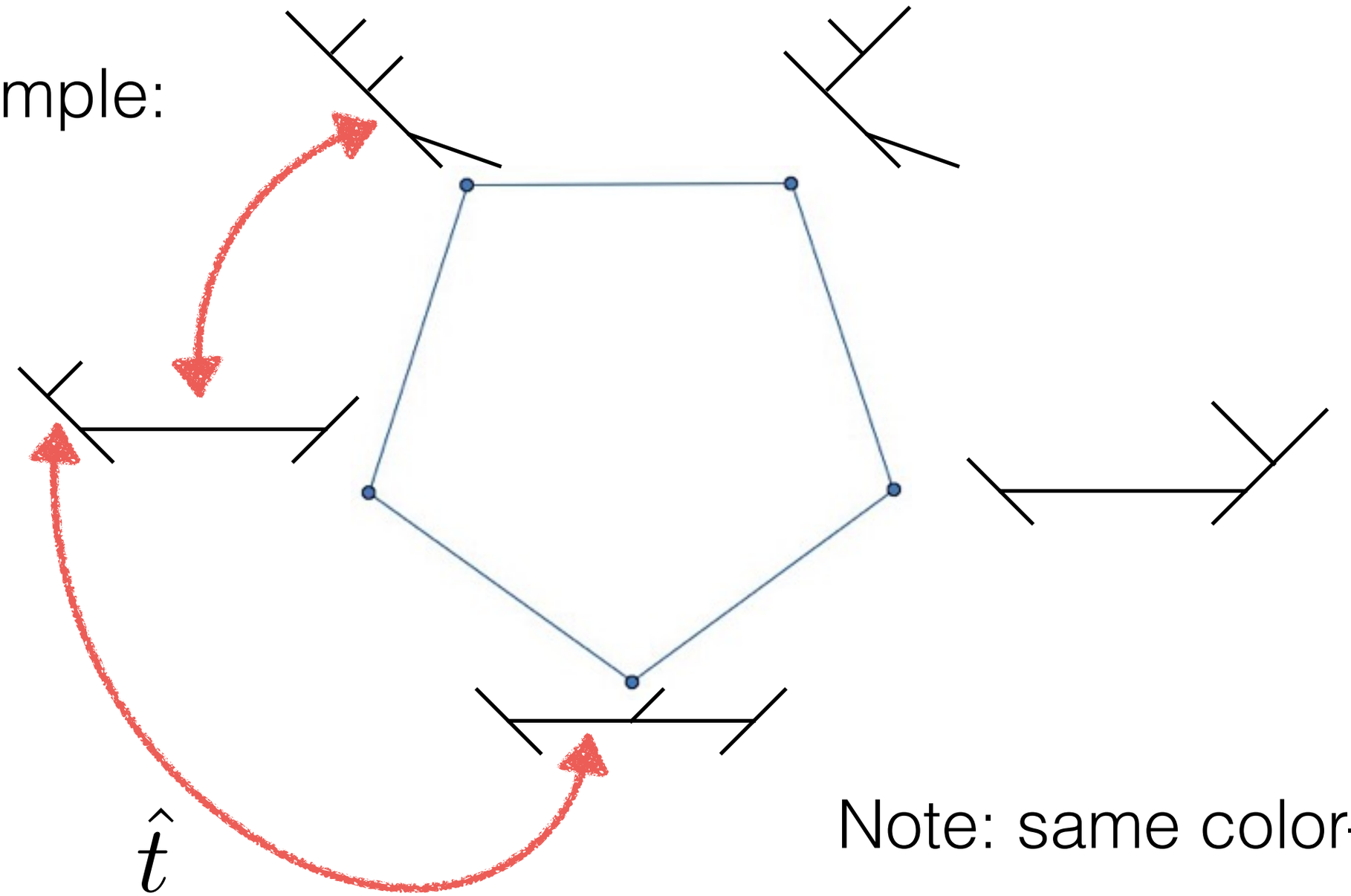


Color-Kinematics



Graphs contributing to a color-stripped tree, generate the 1-skeleton of **Stasheff polytopes** joined only by $\hat{t}$

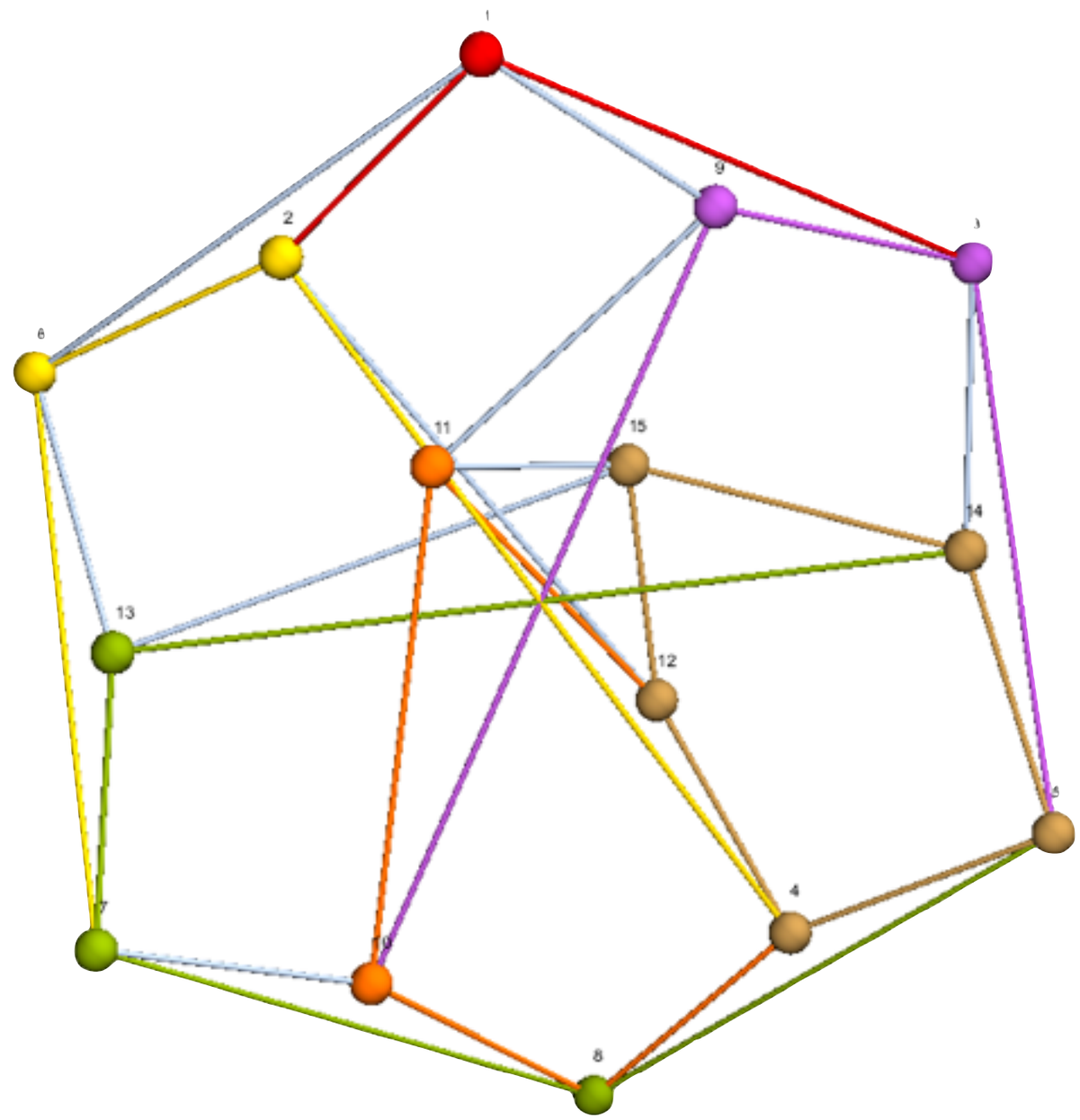
5pt example:



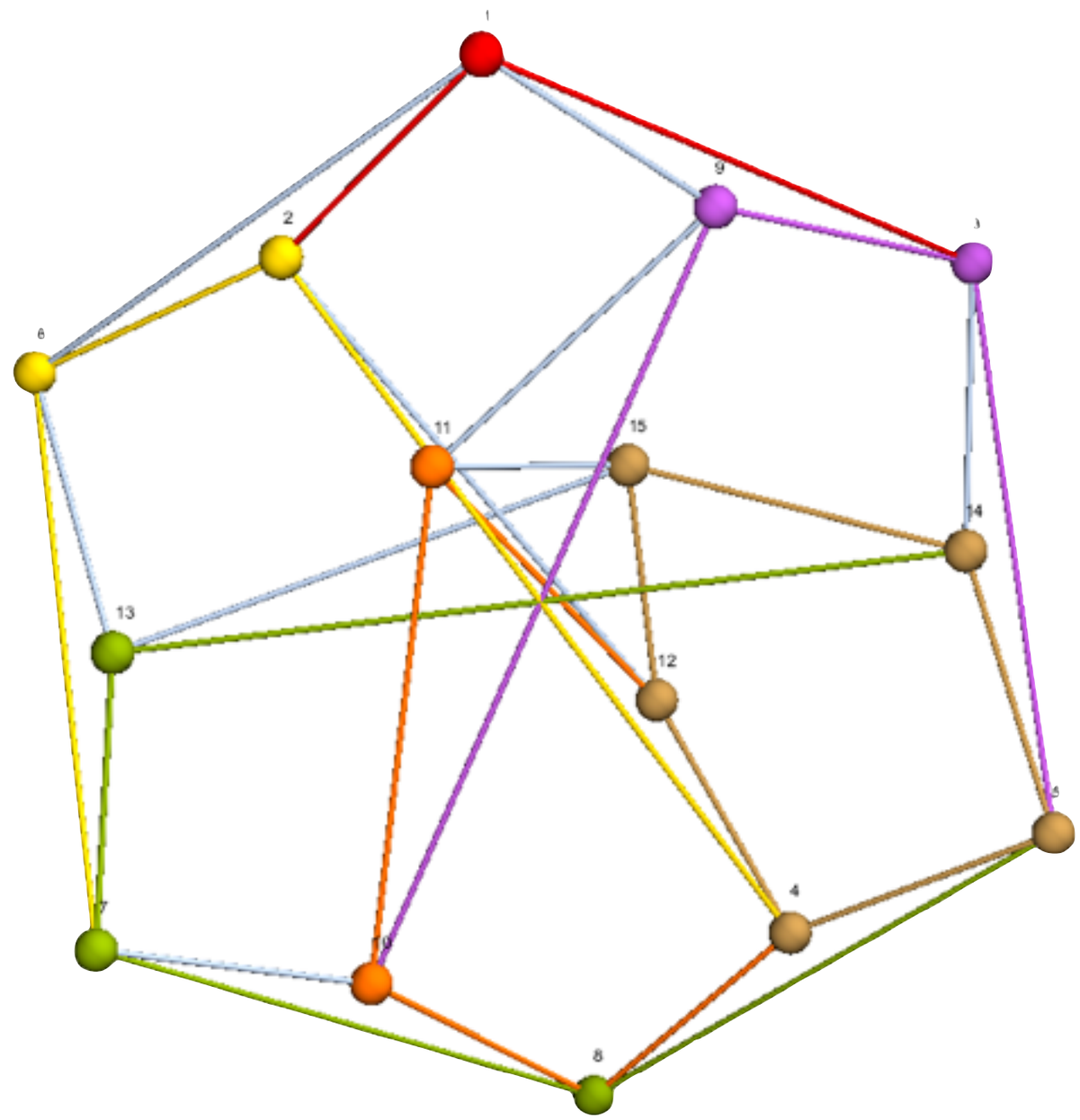
Note: same color-order!

(these polytopes are also called **associahedra**)

You might think you need (m-2)! of these color-ordered amplitudes to capture everything because this is what is required to touch every vertex at least once:



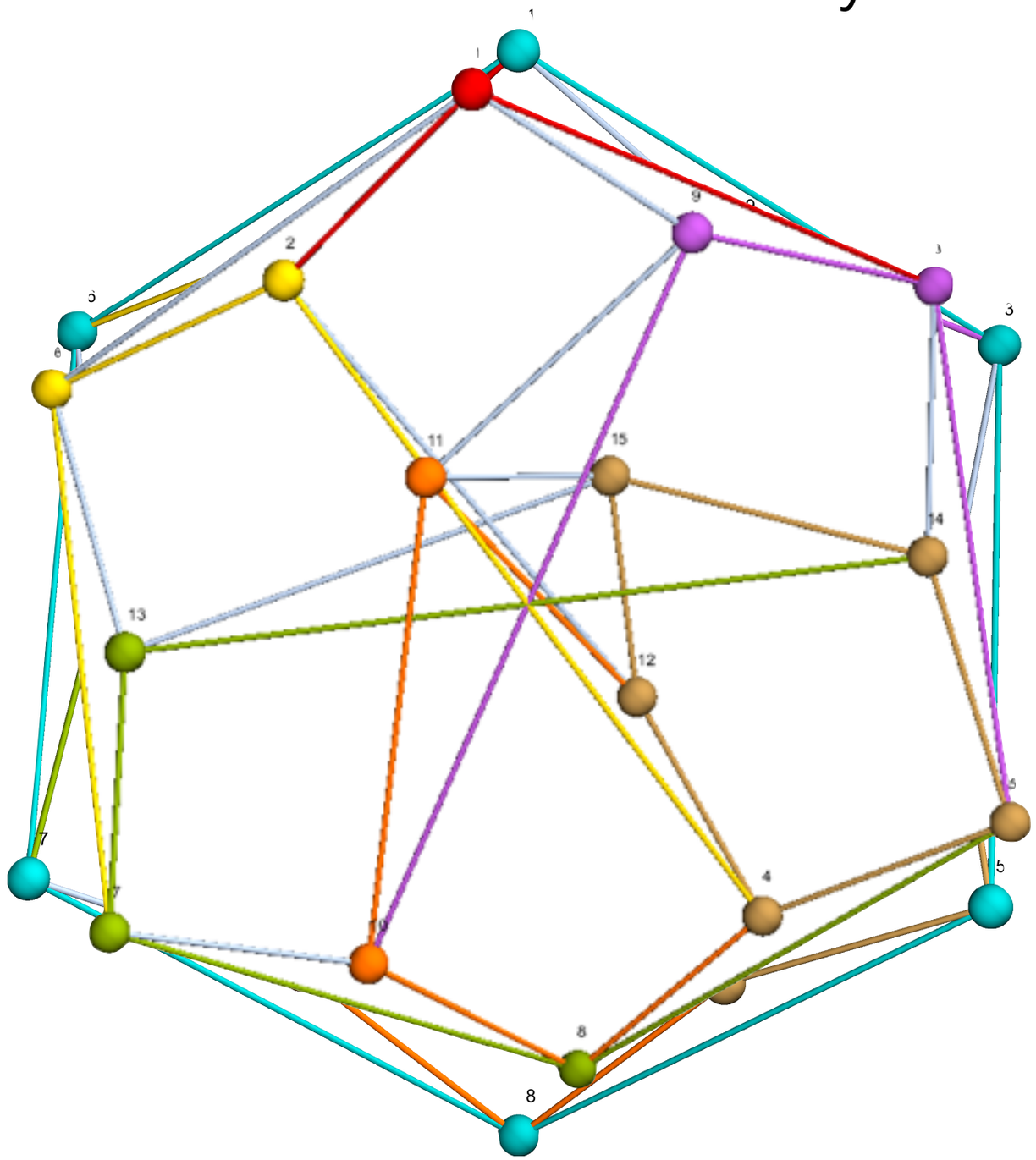
You might think you need (m-2)! of these color-ordered amplitudes to capture everything because this is what is required to touch every vertex at least once:



In fact, such a choice is the KK-basis, proven sufficient by Del Duca, Dixon, and Maltoni

JJMC

But notice, because of color-kinematics, only (m-2)! nodes are needed to specify both the color factors and numerator factors of everyone



JJMC

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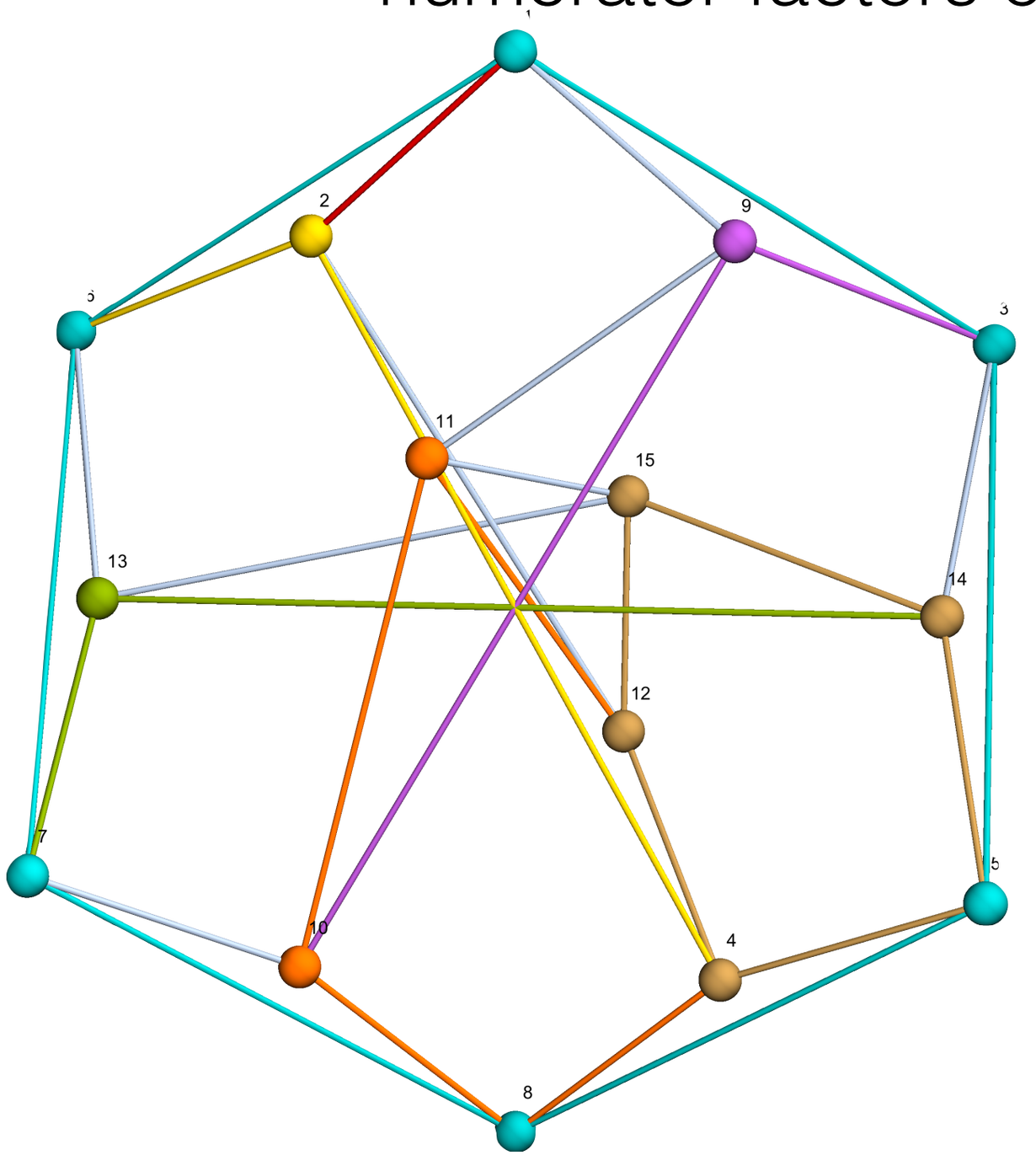
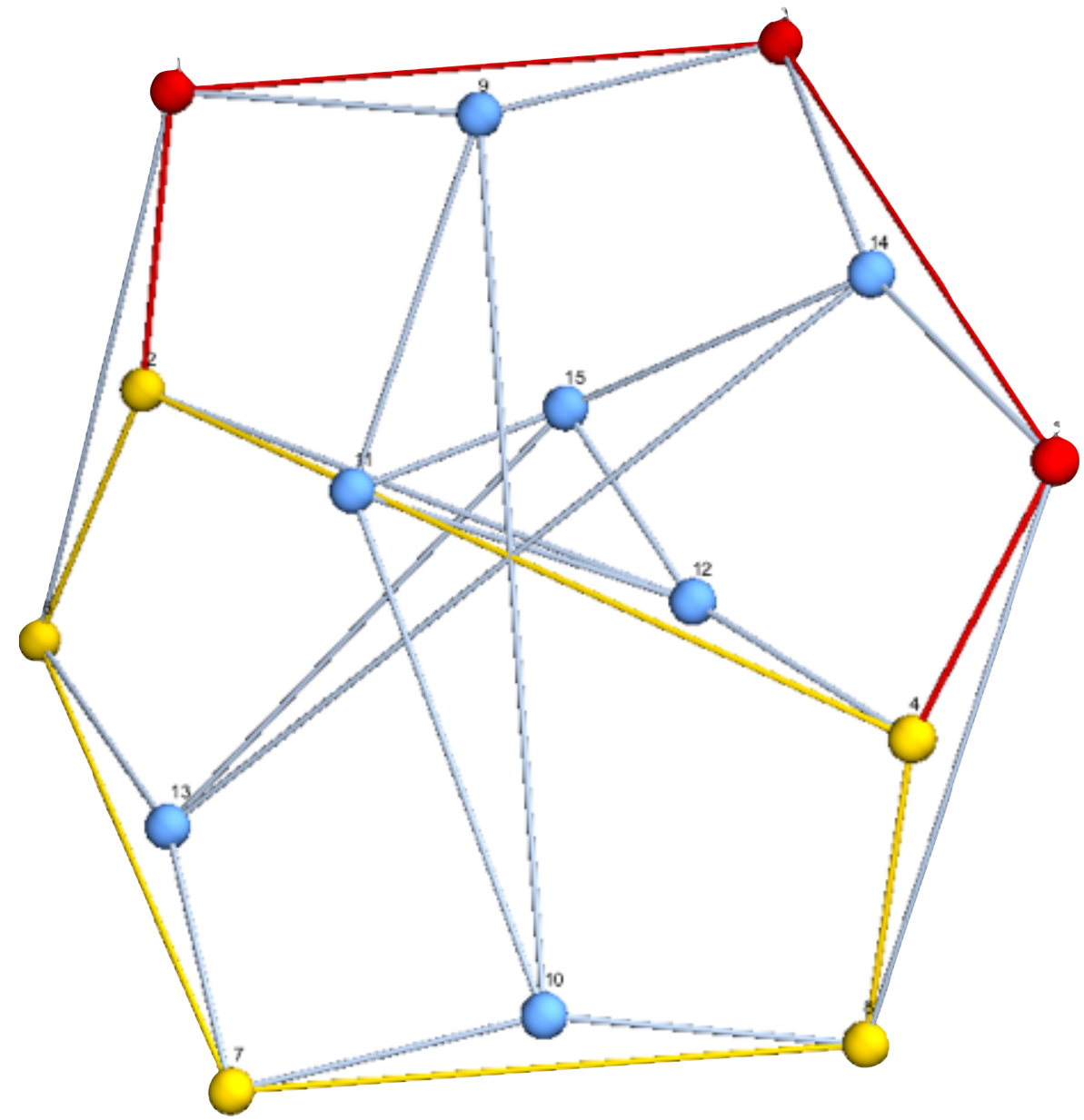
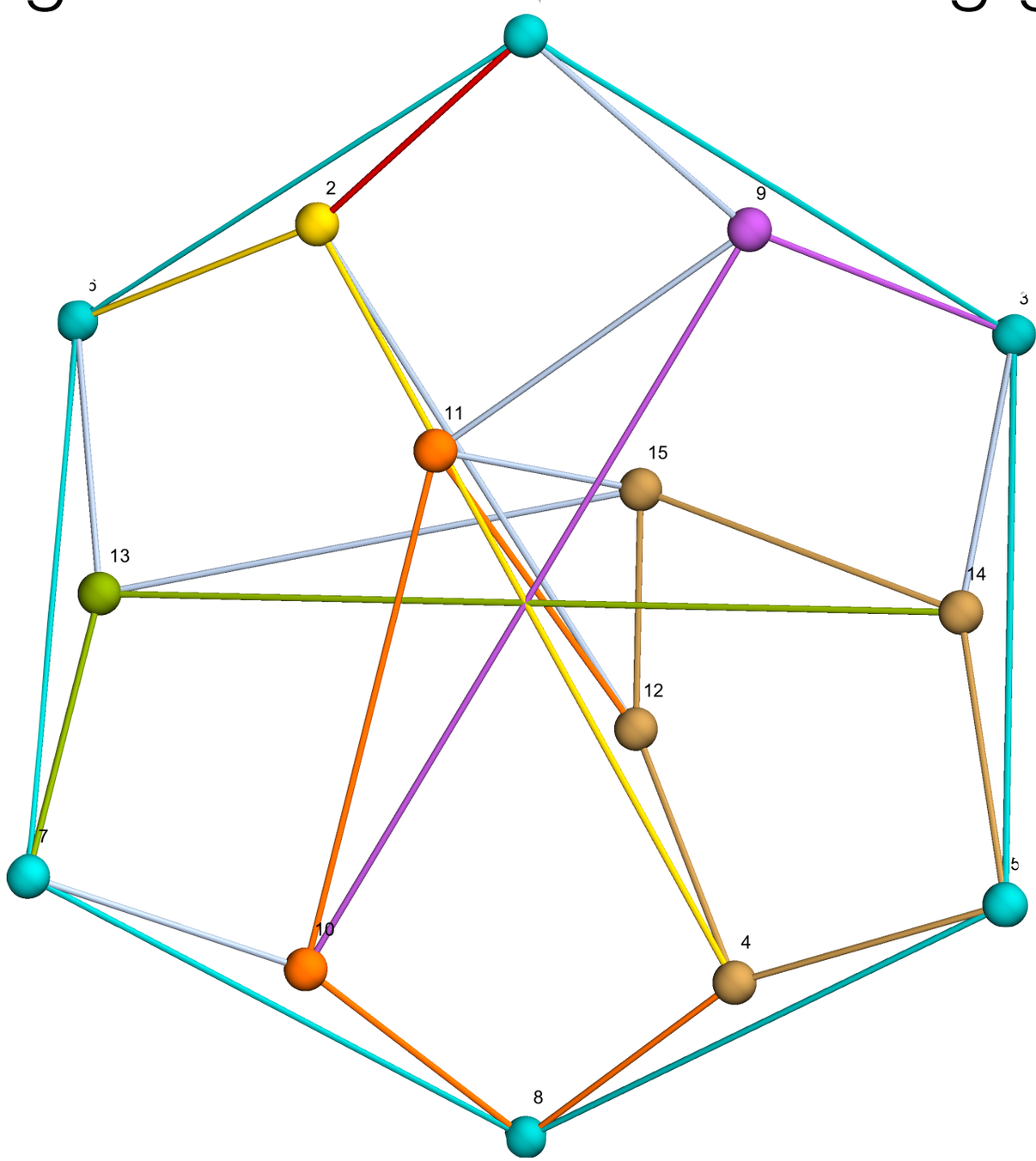


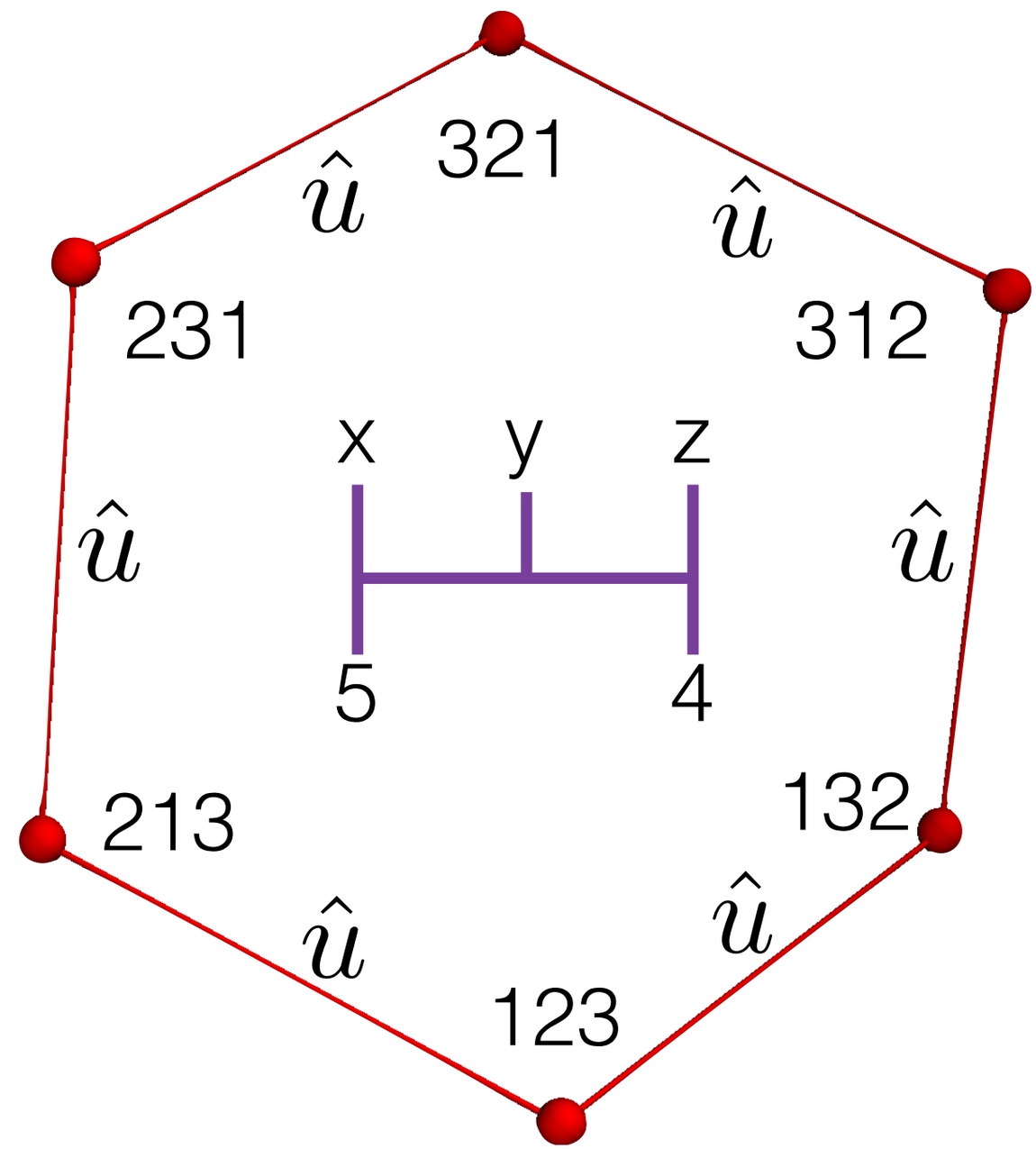


This reduces the set of necessary color-ordered amplitudes (associahedra) to (m-3)! : “BCJ” relations

JJMC

At every multiplicity the masters can be chosen to form the 1-skeleton of a polytope related by $\hat{u}$ on every internal edge of the relevant scattering graphs

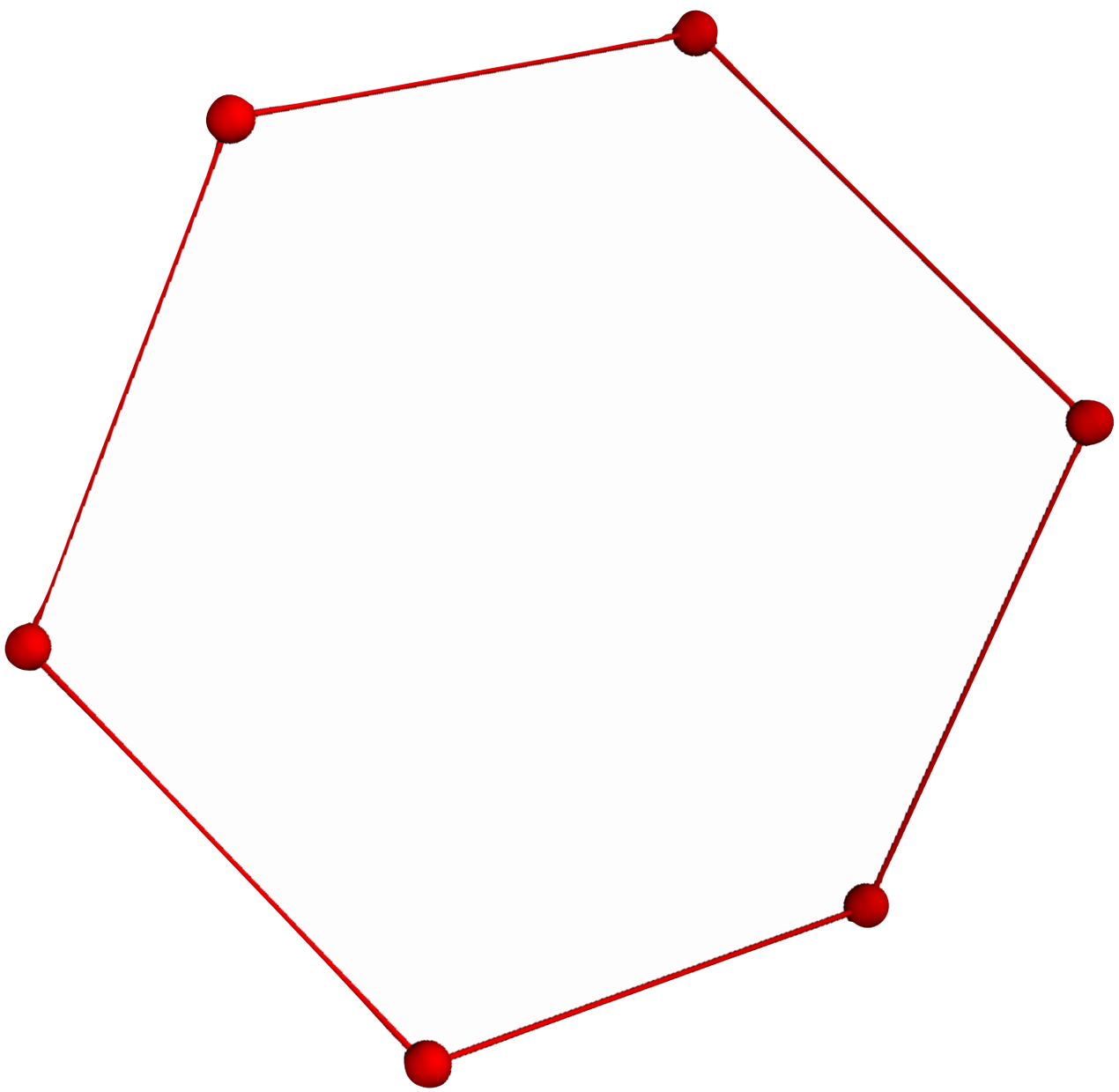


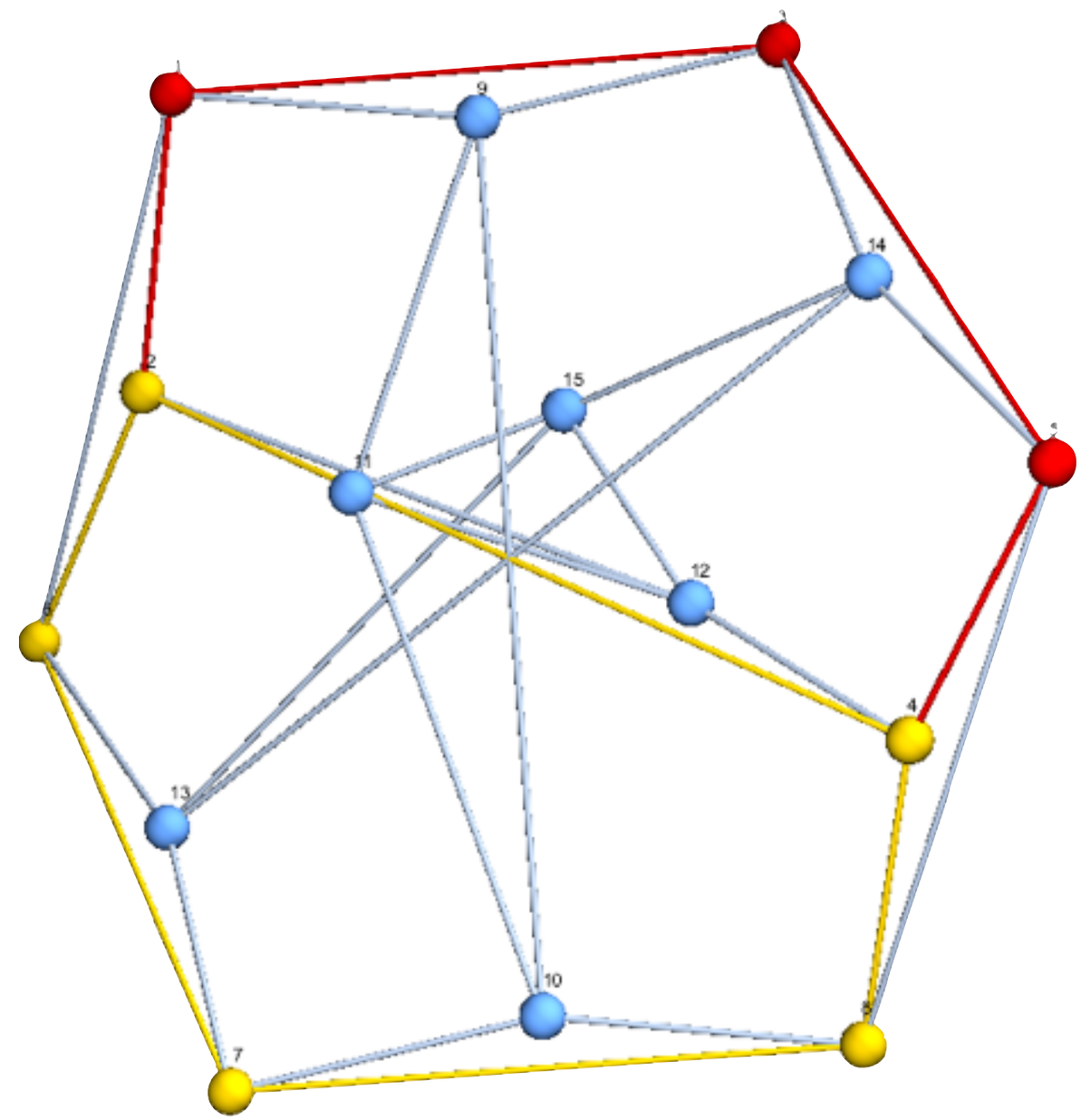


(these polytopes are called **permutahedra**)

Can linearly solve for the (m-2)! numerators of the masters in terms of the (m-3)! “BCJ” independent color-ordered amplitudes. In fact you get (m-3)! numerators in terms of the color-ordered amplitudes and (m-3)(m-3)! free functions.

JJMC

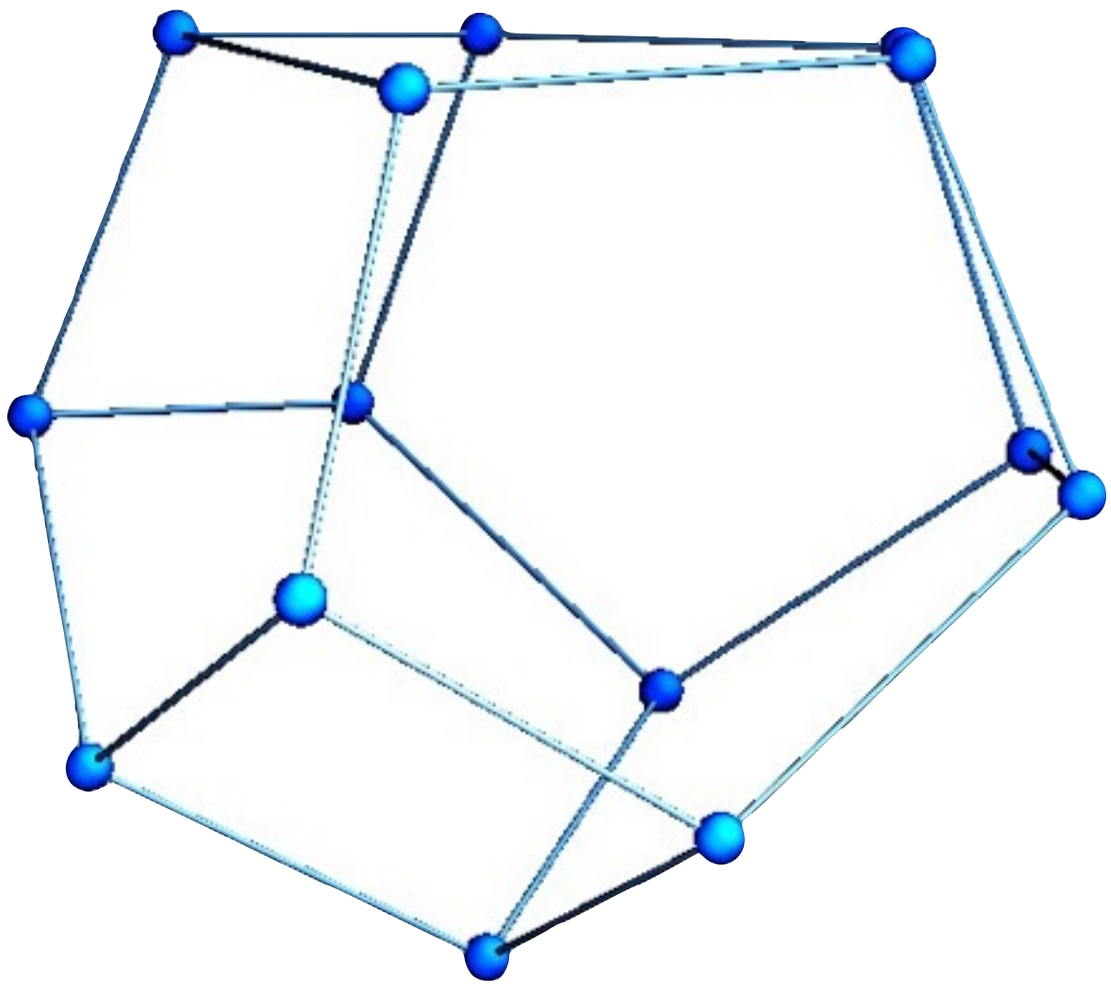




(generalized gauge freedom)

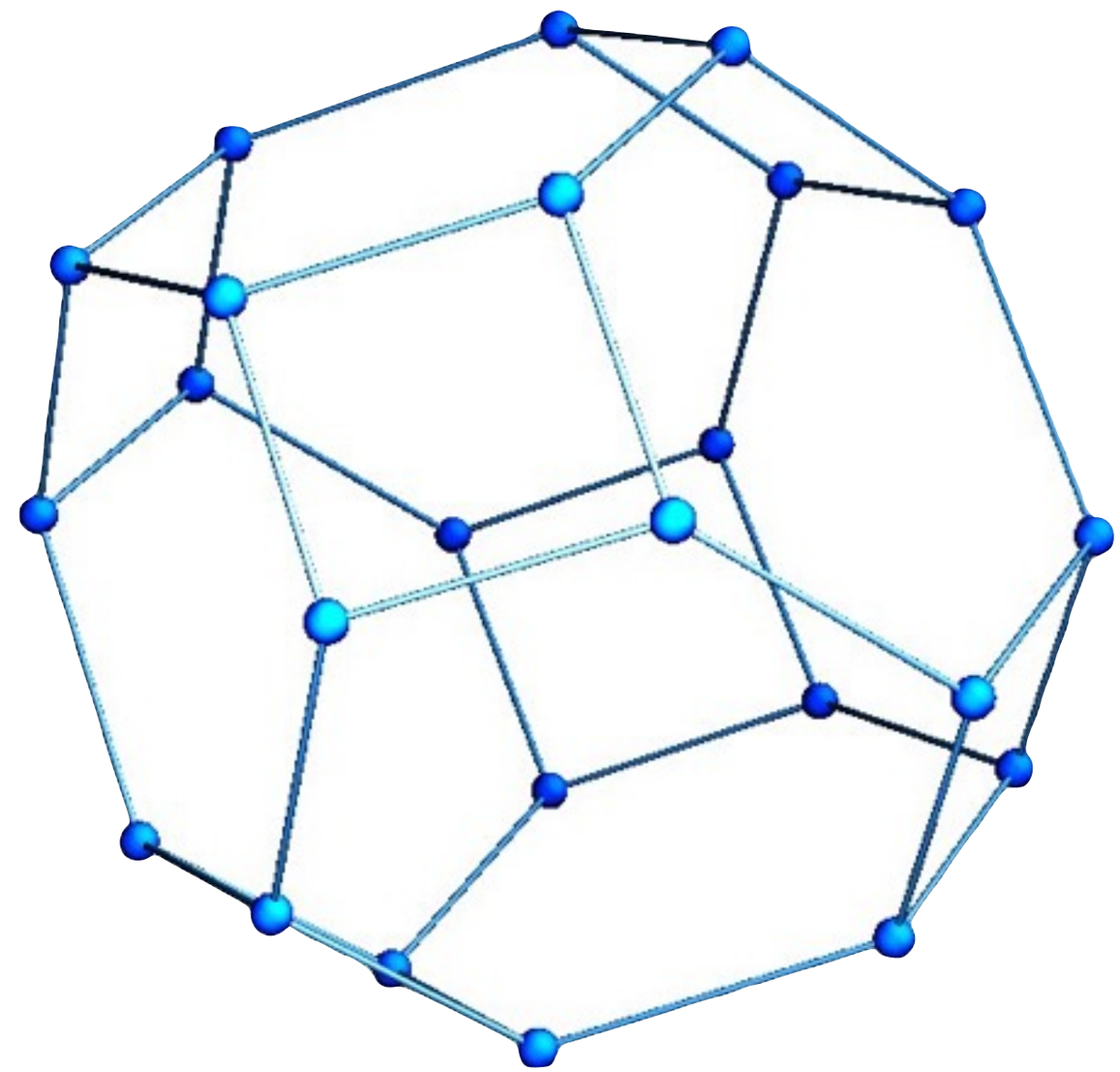
Building blocks at 6-points:

color-ordered amplitude



associahedron

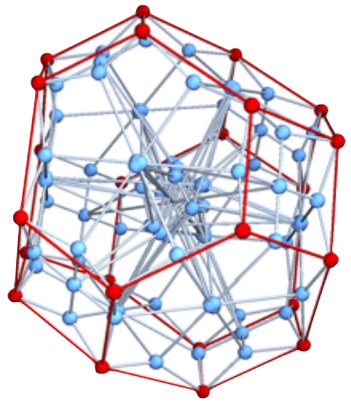
set of masters

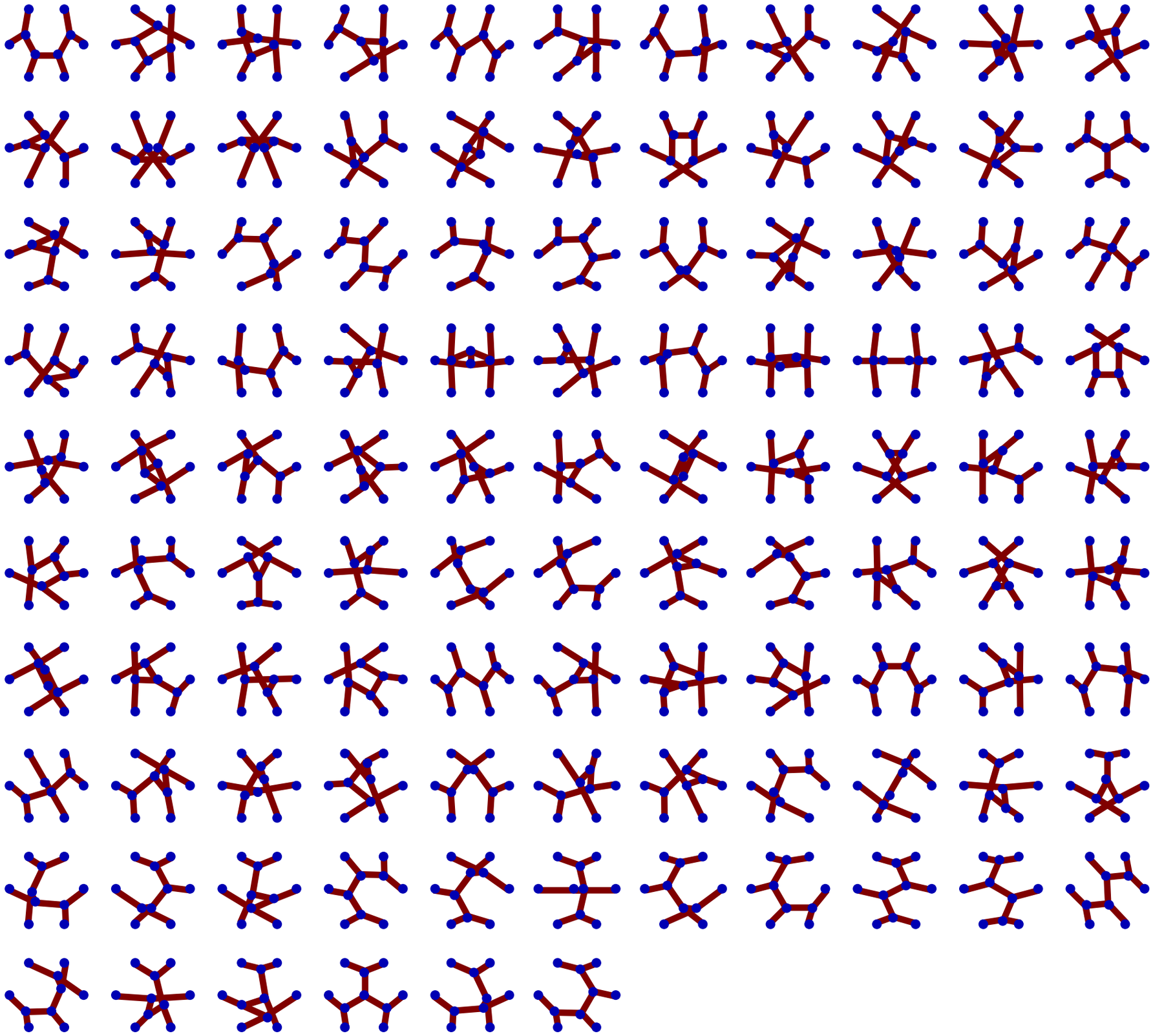


permutahedron

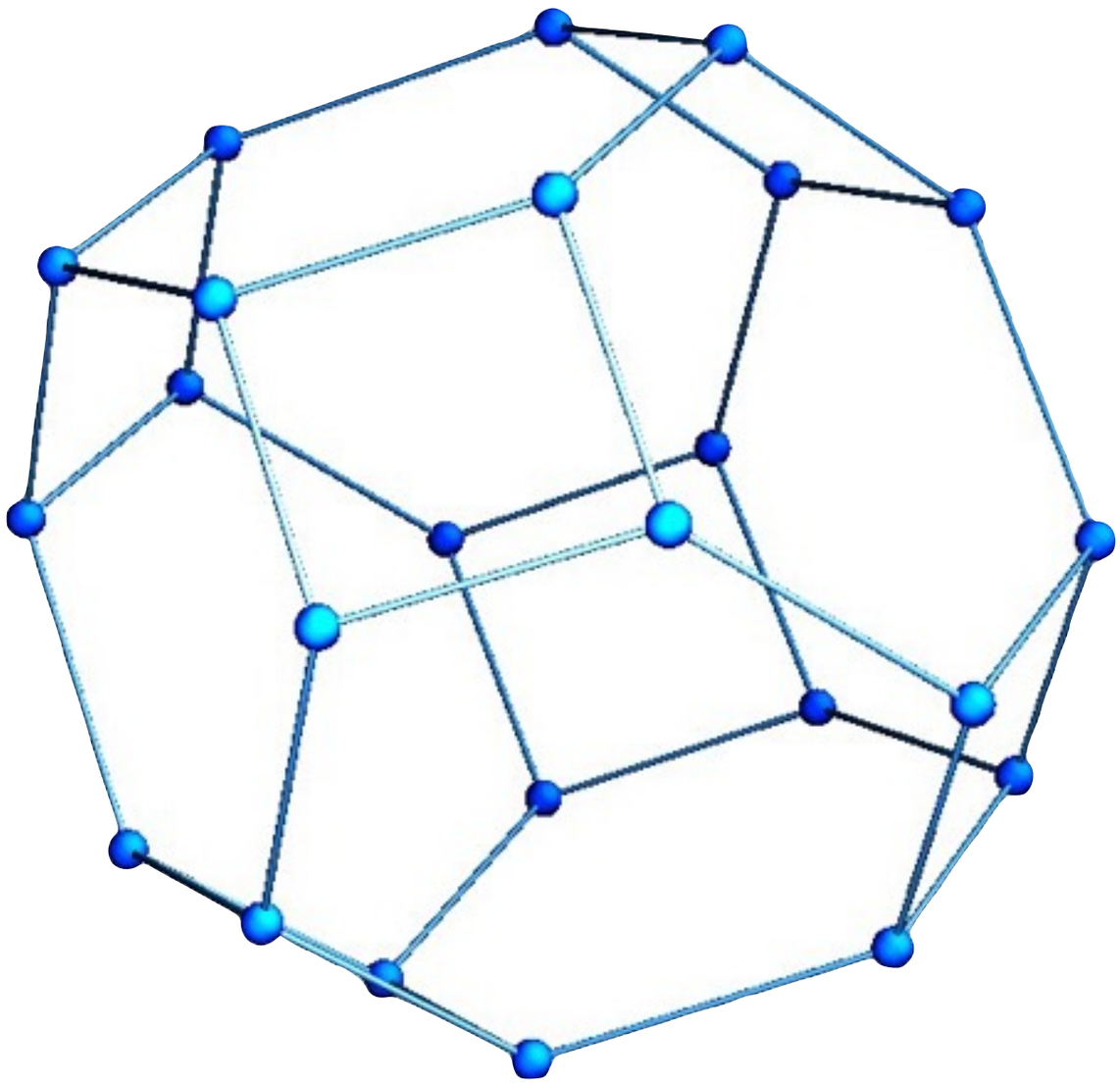
JJMC

105
cubic graphs at 6 pt

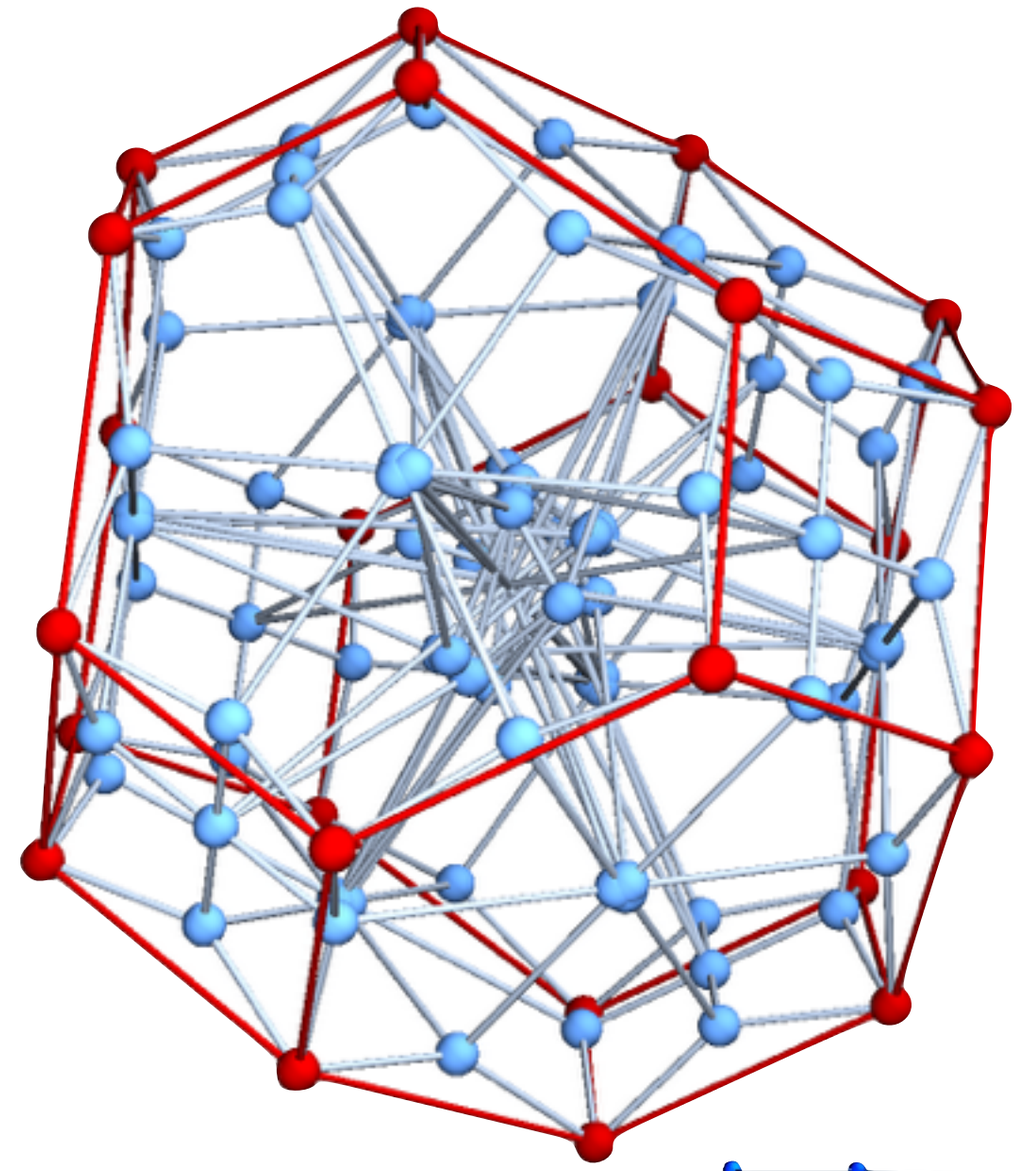


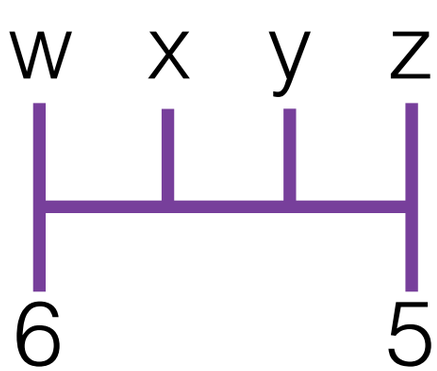


set of masters

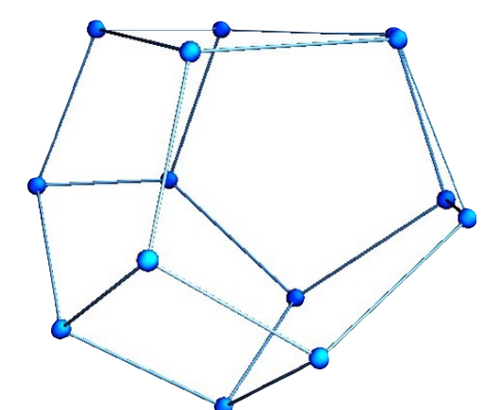


full amplitude



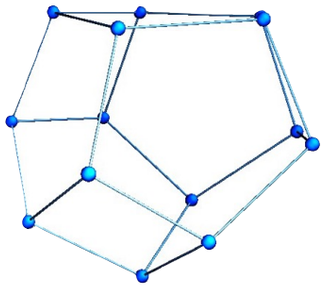


masters fixed by 6

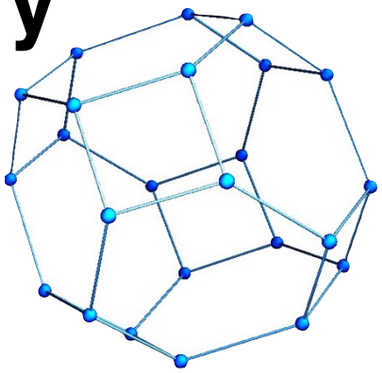


TREE-LEVEL SUMMARY

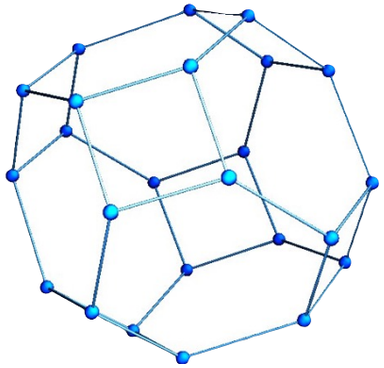
1. **Gauge invariant building blocks that speak to the theory:** color-ordered amplitudes, *associahedra*



2. **CK means only need to specify the boundary data**: the master graphs, given by the relevant *permutahedron*



3. **Can solve for the *full amplitude efficiently* in terms of the (n-3)! independent *associohedra***



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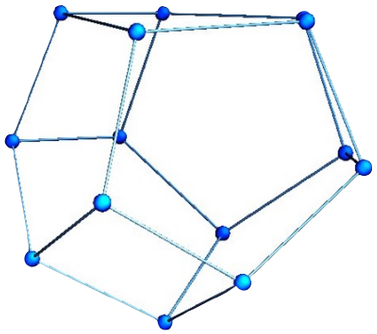
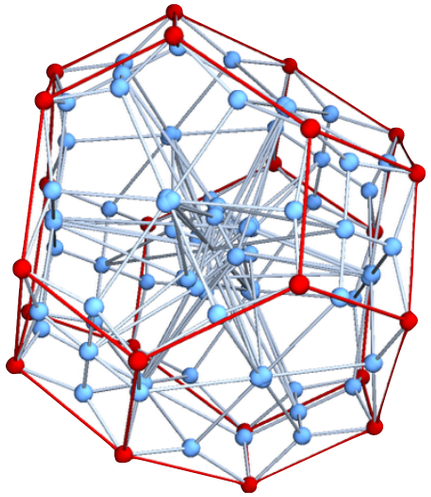


)

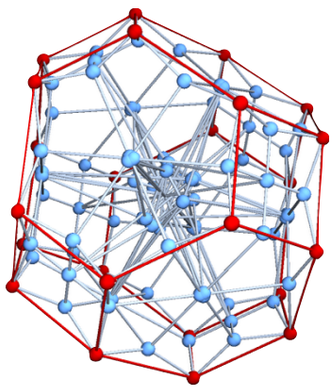
(linear)



physics <—> geometry

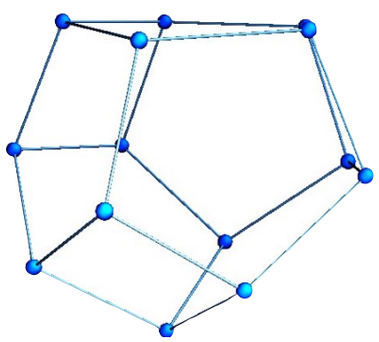
Full YM:

color $\otimes$ spin-1



$$\mathcal{A}_m^{\text{tree}} = \sum_{\mathcal{G}\in\text{cubic}} \frac{c(\mathcal{G})n(\mathcal{G})}{D(\mathcal{G})}$$

color-stripped YM

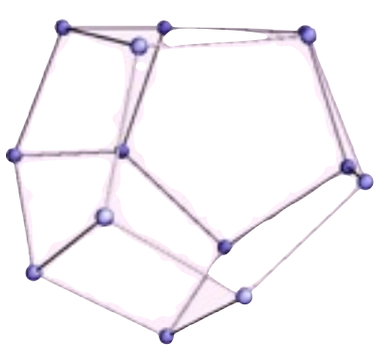


$$\mathbf{A}_{\mathbf{m}}^{\text{tree}}(\rho) = \sum_{\mathcal{G}\in\rho} \frac{\mathbf{n}(\mathcal{G})}{\mathbf{D}(\mathcal{G})}$$

(*same* as kinematic stripped gravity

$$-iM_n^{\text{tree}} = \sum_{\mathcal{G}\in\text{cubic}} \frac{n(\mathcal{G})\tilde{n}(\mathcal{G})}{D(\mathcal{G})}$$

kinematic-stripped YM



$$\mathbf{C}_{\mathbf{m}}^{\text{tree}}(\rho) = \sum_{\mathcal{G}\in\rho} \frac{\mathbf{c}(\mathcal{G})}{\mathbf{D}(\mathcal{G})}$$

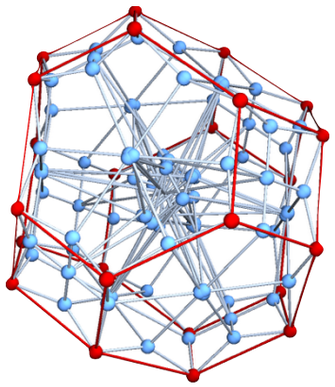
(*same* as color-stripped)

Bi-Adjoint Scalar

$$\mathcal{C}_{\mathbf{m}}^{\text{tree}}(\rho) = \sum_{\mathcal{G}\in\text{cubic}} \frac{\mathbf{c}(\mathcal{G})\tilde{\mathbf{c}}(\mathcal{G})}{\mathbf{D}(\mathcal{G})}$$

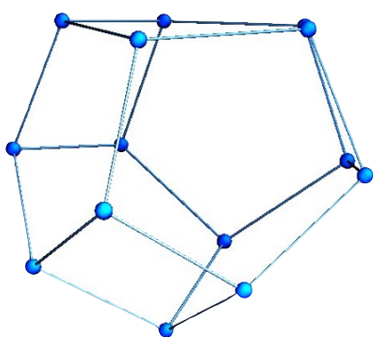
)

Full YM: color $\otimes$ spin-1



$$\mathcal{A}_m^{\text{tree}} = \sum_{\mathcal{G}\in\text{cubic}} \frac{c(\mathcal{G})n(\mathcal{G})}{D(\mathcal{G})}$$

color-stripped YM

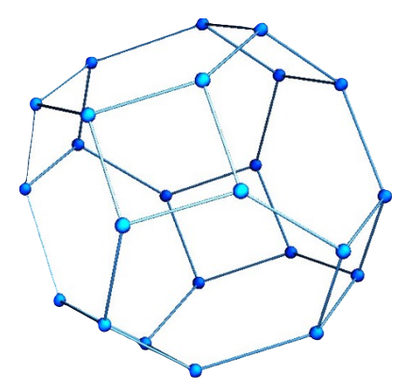


$$\mathbf{A}_{\mathbf{m}}^{\text{tree}}(\rho) = \sum_{\mathcal{G}\in\rho} \frac{\mathbf{n}(\mathcal{G})}{\mathbf{D}(\mathcal{G})}$$

BCJ '08

Can (pseudo) invert:

$$\mathbf{n}(\mathcal{G}) = \sum_{\rho} \mathbf{D}(\mathcal{G}|\rho)^{-1}\mathbf{A}(\rho)$$



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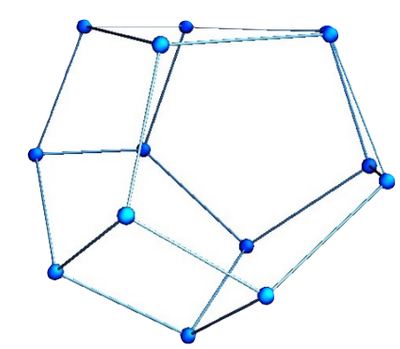

)

(linear)

Bern, JJMC, Johansson (2008)

color-kinematics $\longrightarrow$ KLT-type relations

$$\mathcal{M}_m^{\text{tree}} = \sum_{\mathcal{G}\in\text{cubic}} \frac{n(\mathcal{G})\tilde{n}(\mathcal{G})}{D(\mathcal{G})}$$

$$= \sum_{\mathbf{g}\in\text{cubic},\rho,\tau} \frac{(\mathbf{D}^{-1}(\mathbf{g},\rho)\mathbf{A}(\rho))(\mathbf{D}^{-1}(\mathbf{g},\tau)\tilde{\mathbf{A}}(\tau))}{\mathbf{D}(\mathbf{g})}$$

$$= \sum_{\rho,\tau} \mathbf{A}(\rho) \left(\sum_{\mathbf{g}\in\text{cubic}} \frac{\mathbf{D}^{-1}(\mathbf{g},\rho)\mathbf{D}^{-1}(\mathbf{g},\tau)}{\mathbf{D}(\mathbf{g})} \right) \tilde{\mathbf{A}}(\tau))$$

$$= \sum_{\rho,\tau} \mathbf{A}(\rho)\mathbf{S_0}(\rho|\tau)\tilde{\mathbf{A}}(\tau)$$

Field theory KLT-type matrix / momentum kernel

Bern, Dixon, Perelstein, Rozowsky (1999)

Bjerrum-Bohr, Damgaard, Feng, Sondergaard (2010)

Bjerrum-Bohr, Damgaard,Sondergaard,Vanhove (2011)

Bern, JJMC, Johansson (2008)

color-kinematics $\longrightarrow$ KLT-type relations

can see the same decomposition in YM

$$\mathcal{A}_m^{\text{tree}} = \sum_{\mathcal{G}\in\text{cubic}} \frac{n(\mathcal{G})c(\mathcal{G})}{D(\mathcal{G})}$$

$$= \sum_{\mathbf{g}\in\text{cubic},\rho,\tau} \frac{(\mathbf{D^{-1}}(\mathbf{g},\rho)\mathbf{A}(\rho))(\mathbf{D^{-1}}(\mathbf{g},\tau)\mathbf{C}(\tau))}{\mathbf{D}(\mathbf{g})}$$

$$= \sum_{\rho,\tau} \mathbf{A}(\rho) \left(\sum_{\mathbf{g}\in\text{cubic}} \frac{\mathbf{D^{-1}}(\mathbf{g},\rho)\mathbf{D^{-1}}(\mathbf{g},\tau)}{\mathbf{D}(\mathbf{g})} \right) \mathbf{C}(\tau))$$

$$= \sum_{\rho,\tau} \mathbf{A}(\rho)\mathbf{S_0}(\rho|\tau)\mathbf{C}(\tau)$$

Bern, JJMC, Johansson (2008)

color-kinematics ⟶ KLT-type relations

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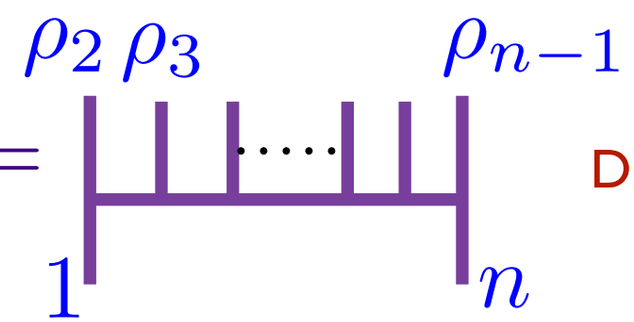
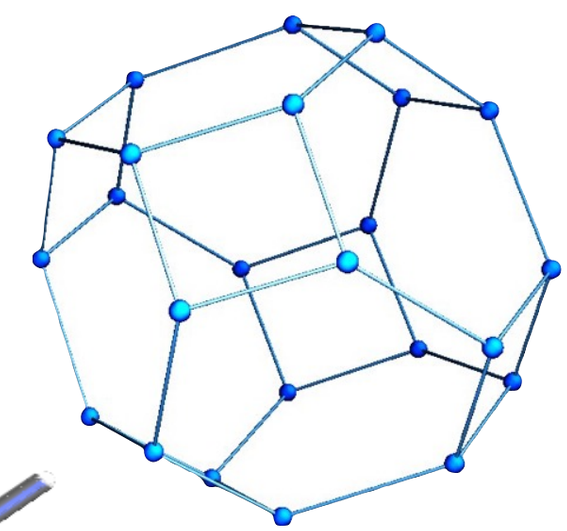
$$= \sum_{\rho,\tau} \mathbf{A}(\rho)\mathbf{S_0}(\rho|\tau)\mathbf{C}(\tau)$$

$$= \sum_{\rho} A(\rho)c(\rho) \qquad c(\rho) =$$

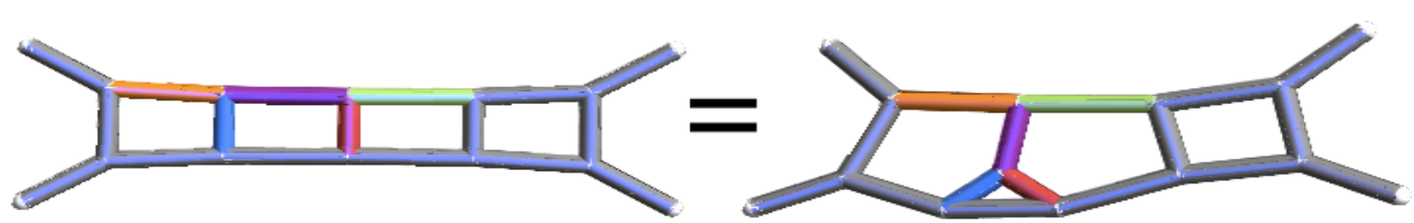


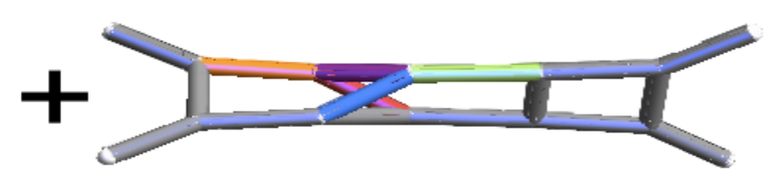
Del Duca, Dixon, Maltoni (1999)



color weights of permutahedron:

relies only on color-Jacobi satisfaction





Bern, JJMC, Johansson (2008)

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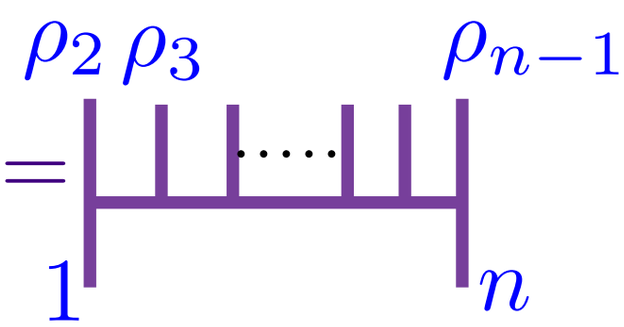
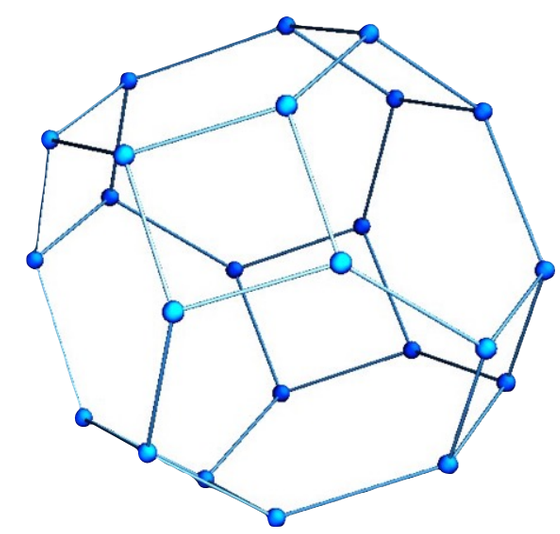
$$= \sum_{\rho,\tau} \mathbf{A}(\rho)\mathbf{S_0}(\rho|\tau)\tilde{\mathbf{A}}(\tau)$$

DDM basis for Gravity!

$$= \sum_{\rho} A(\rho)\tilde{n}(\rho) \quad \tilde{n}(\rho) =$$





kinematic weights of permutahedron: relies only on kinematic-Jacobi satisfaction

Closed form (non-local) color-dual numerators:

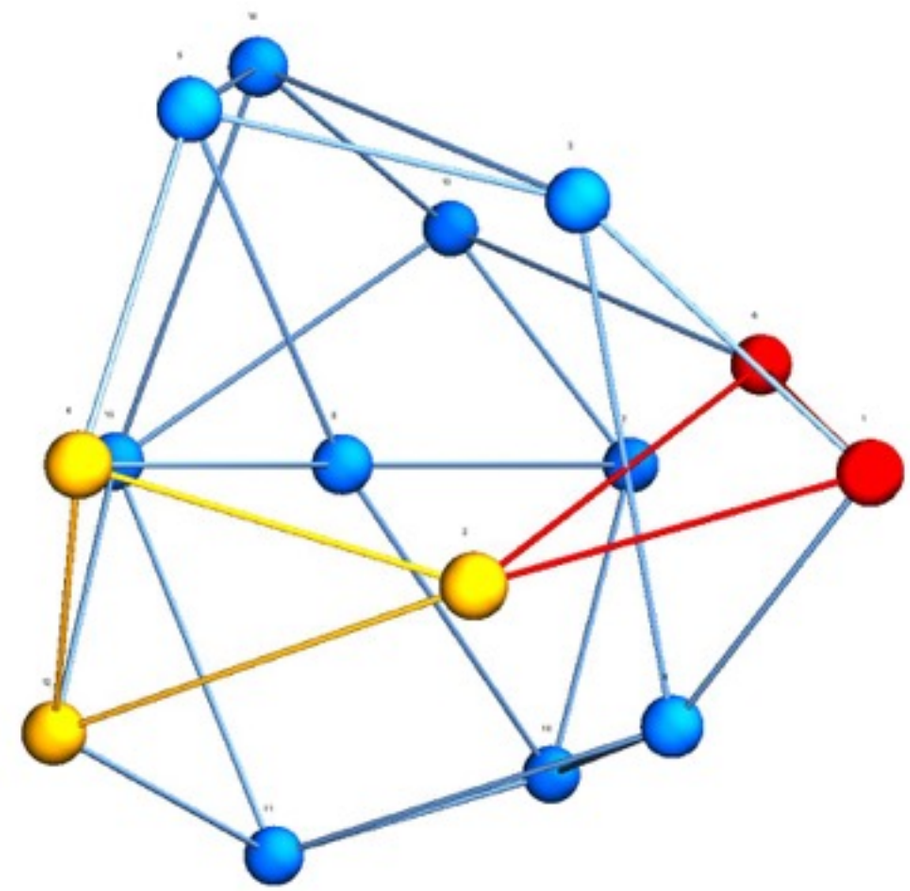
$$\tilde{n}(\rho) = \sum_{\tau} S_0(\rho|\tau)\tilde{A}(\tau)$$

Kiermier; Bjerrum-Bohr, Damgaard, Sondergaard, Vanhove (2010)

Can do this on loop-level cuts. Can generalize to the off-shell integrand either by introducing ansatze or with a massive over-redundancy of graphs (the pre-Integrand).

JJMC

Natural question, given a generic (non color-dual) representation for a **gauge amplitude**, and all you want is the related **gravity amplitudes**.



Is there a simple path forward?

See Zvi's talk friday!

the game of Scattering Amplitudes

QFT, NR-QM, String Theory, Classical Physics…

IN
free states
(no interactions)

$$[\mathrm{S-matrix}]$$

(all the interactions!)

OUT
free states
(no interactions)

the game of Scattering Amplitudes

QFT, NR-QM, String Theory, Classical Physics…

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$$[\mathrm{S-matrix}]$$

(all the interactions!)

OUT
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Symmetries, Scaling, Geometry

Playful Construction Using Double-Copy as a Principle

$$U = V \otimes W$$

1) Take theories that exhibit Double-Copy, strip one “factor” replace with something else that obeys the same algebra.

2) Start with generic ansatze, constrain engineering weight, impose algebra.

Example of playful construction

Open String:
Broedel, Schlotterer, Stieberger

$\alpha' \bigotimes$ spin-1

Chan-Paton Stripped open string

$$\mathrm{OS}(P(1,\ldots,n)) = Z_P \otimes A$$

Doubly-ordered Z-functions: obey monodromy relations on P

But obey field theory (n-3)! relations on it's field theory KLT with Yang-Mills A.

$$Z_P(q_1,q_2,\ldots,q_n) \equiv \alpha'^{n-3} \int_{-\infty \le z_{P(1)} \le z_{P(2)} \le \ldots \le z_{P(n)} \le \infty} \frac{\mathrm{dz}_1\ \mathrm{dz}_2\ \cdots\ \mathrm{dz}_n}{\mathrm{vol}(SL(2,\mathbb{R}))} \frac{\prod_{i<j}^n |z_{ij}|^{\alpha' s_{ij}}}{z_{q_1 q_2} z_{q_2 q_3} \ldots z_{q_{n-1} q_n} z_{q_n q_1}} .$$

Take seriously Z-functions as encoding predictions for some (effective) field theory.

JJMC, Mafra, Schlotterer (2016)

Replace sYM in OS with a color-stripped bi-adjoint Scalar

$$\mathrm{OS}(P(1,\ldots,n)) = Z_P \otimes A$$

$$\mathbf{Z}(P(1,\ldots,n)) = Z_P \otimes C$$

Dressing with Chan-Paton factors renders something that can has the possibility of being interpreted as doubly-colored field-theory scattering amplitudes: we call it Z theory.

Color-Ordered tree-level Z-amplitude:

$$\mathcal{Z}\left(\tau(1,2,\ldots,n)\right) \equiv \sum_{P\in S_{n-1}} \mathrm{Tr}(t^1 t^{P(2)} \cdots t^{P(n)}) Z_{1,P}(\tau(1,2,\ldots,n))$$

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Now look at: $\mathcal{Z} \otimes C$

"Low energy limit" -> bi-adjoint scalar: $\sum_g \frac{\tilde{c}(g)c(g)}{D(g)}$

Higher order in α': $\sum_g \frac{z(g)c(g)}{D(g)}$

both CP-weights and kinematics conspire in z(g) to obey algebraic identities.

Color-Ordered tree-level Z-amplitude

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Play with CP factors. Abelian CP generators means no-longer a bi-colored scalar.

$$\mathcal{Z}_\times \otimes C = \sum_g \frac{z_\times(g)c(g)}{D(g)}$$

Color-Ordered tree-level Z-amplitude

$$\mathcal{Z}\left(\tau(1,2,\ldots,n)\right) \equiv \sum_{P\in S_{n-1}} \mathrm{Tr}(t^1 t^{P(2)}\cdots t^{P(n)}) Z_{1,P}(\tau(1,2,\ldots,n))$$

Play with CP factors. Abelian CP generators means no-longer a bi-colored scalar.

$$\mathcal{Z}_\times \otimes C = \sum_g \frac{z_\times(g)c(g)}{D(g)}$$

Low energy limit: $\lim_{\alpha'\to 0} \mathcal{Z}_\times \otimes C \to \mathrm{NLSM}$

JJMC, Mafra, Schlotterer (2016)

$$\mathcal{L}_{\mathrm{NLSM}} = \frac{1}{2}\mathrm{Tr}\left\{\partial_\mu\varphi \frac{1}{1-\varphi^2}\partial^\mu\varphi \frac{1}{1-\varphi^2}\right\}$$

Color-Ordered tree-level Z-amplitude

$$\mathcal{Z}\left(\tau(1,2,\ldots,n)\right) \equiv \sum_{P\in S_{n-1}} \mathrm{Tr}(t^1 t^{P(2)}\cdots t^{P(n)}) Z_{1,P}(\tau(1,2,\ldots,n))$$

Abelian Z: $\lim_{\alpha'\to 0} \mathcal{Z}_\times \otimes C \to \mathrm{NLSM}$

JJMC, Mafra, Schlotterer (2016)

$$\mathcal{L}_{\mathrm{NLSM}} = \frac{1}{2}\mathrm{Tr}\left\{\partial_\mu\varphi \frac{1}{1-\varphi^2}\partial^\mu\varphi \frac{1}{1-\varphi^2}\right\}$$

(Cayley Parameterization)

Completely different story for the same prediction.

Chen, Du '13 showed obeyed (n-3)! relns. Cheung,Shen '16 found an action that directly gives the color-dual kinematic story.

$$\mathcal{L}_{\mathrm{NLSM}} = Z^{a\mu}\Box X^a_\mu + \frac{1}{2}Y^a\Box Y^a - f^{abc}\left(Z^{a\mu}Z^{b\nu}X^c_{\mu\nu} + Z^{a\mu}(Y^b\overleftrightarrow{\partial}_\mu Y^c)\right)$$

Color-Ordered tree-level Z-amplitude

$$\mathcal{Z}\left(\tau(1,2,\ldots,n)\right) \equiv \sum_{P\in S_{n-1}} \mathrm{Tr}(t^1 t^{P(2)}\cdots t^{P(n)}) Z_{1,P}(\tau(1,2,\ldots,n))$$

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Somehow abelianization is encoding a story related to SSB

Color-Ordered tree-level Z-amplitude

$$\mathcal{Z}\left(\tau(1,2,\ldots,n)\right) \equiv \sum_{P\in S_{n-1}} \mathrm{Tr}(t^1 t^{P(2)}\cdots t^{P(n)}) Z_{1,P}(\tau(1,2,\ldots,n))$$

Abelian Z: $$\lim_{\alpha'\to 0} \mathcal{Z}_\times \otimes C \to \mathrm{NLSM}$$

JJMC, Mafra, Schlotterer (2016)

Let's look at it's other copy, back to the superstring:

Abelian Open Superstring: $$\left[\left(\lim_{\alpha'\to 0} \mathcal{Z}_\times\right) \otimes A\right] \to [\mathrm{NLSM}\otimes A]$$

He, Liu, Wu '16; Cachazo, Cha, Mizera '16 told us:

$$[\mathrm{NLSM}\otimes A] = \mathrm{SDBIVA}$$

For maximal sYM, 16 linearly realized, 16 nonlinearly realized,

Bergshoeff, Coomans, Kallosh, Shahbazi Van Proeyen '13

$$U = V \otimes W$$

Order by order in higher derivatives can play all these constructive games and more using ansatze with the correct ingredients.

Open question as to what theories can be understood as nontrivial double copies and what their dual-stories are.

The amplitudes can still be interesting even if crazy from some perspectives.

(See Henrik's talk)

Clearly lots of fun games yet to be played — very much an open field.