Colour Glass, Froissart bound, and all that

What can we say about high energy scattering in QCD from first principles?

- Deep inelastic scattering at small $x$
- The Colour Glass Condensate
- Gluon Saturation
- Froissart bound

E. Iancu
Saclay

What is the high energy limit of QCD scattering?

- Can this be addressed in perturbation theory?
  - $\alpha_s(Q^2)$ and not $\alpha_A(s)$! $[s=(E_{c.m.})^2]$
- Froissart bound: $\sigma(s) \leq \frac{\pi}{M_r^2} \ln^2 s$
- BFKL’s failure to describe the high energy limit
  - $\sigma_{BFKL}(s) \sim S^{w_0(x)}$, $\omega = 4(\ln 2)N_c/N_f$
  - “Infrared diffusion”: $Q^2 \rightarrow 0$ as $s \rightarrow \infty$

- Can we use perturb. theory to study quantum evolution with $s$?

- What are the relevant degrees of freedom?
  - “Small-$x$” partons (mostly gluons) in a state of high density

  “Colour Glass Condensate”

  - the matter mode of the small-$x$ gluons
High-energy Scattering in QCD

- Hadron-Hadron Collision (center-of-mass frame)
  \[
  \frac{1}{k^2} \rightarrow \frac{1}{P} \quad \text{with} \quad x \ll 1
  \]

High energy \rightarrow Small-x tail of the hadron wavefunction

- Deep Inelastic Scattering (Bjorken frame)
  \[
  Q^2 = -2n^2 \mu^2 > 0
  \]

  \[
  x_{Bj} = \frac{Q^2}{2P^2} \sim \frac{Q^2}{s} \ll 1
  \]

  Bjorken frame: \( P^\mu = (P, 0, 0, P) \) and \( \mathbf{z} = (0, 0, 2z) \)

  Feynman \( x = \frac{k^2_{z \perp}}{P} \)

  Kinematics \( x = x_{Bj} \ll 1 \)

  or \( k^2_{z \perp} = \frac{2z}{2} \)

World's data on \( F_2 \) at \( Q^2 = 15 \text{ GeV}^2 \)

\[
F_2(x, Q^2) = \sum_f \frac{e_f^2}{x} \left[ x \, q_f^2(x, Q^2) + x \, \bar{q}_f^2(x, Q^2) \right]
\]

**Figure 1:** World's data on \( F_2 \) at \( Q^2 = 15 \text{ GeV}^2 \) as a function of \( x \). The solid line is a DGLAP fit by the CTEQ group [7].

\[
F_2 \sim \frac{1}{x^\lambda} \quad \text{with} \quad \lambda = 0.4 \div 0.1 \quad \text{(depending upon} \quad Q^2)\]
$x \, g(x, Q^2) \equiv \frac{dN_{\text{gluon}}}{d\ln \frac{1}{x}} \propto \frac{2 F_2(x, Q^2)}{d \ln Q^2}$

for sufficiently high $Q^2$

**Light-Cone Kinematics**

$V^+ = \frac{1}{2} (V_0 + V^3)$; $V^- = \frac{1}{2} (V_0 - V^3)$; $V_L = (V^1, V^2)$

\[ P^\mu \propto (P, 0, 0, P) \quad \Rightarrow \quad P^\mu = (P^+; 0, 0, 0) ; \quad P^+ = \frac{1}{2} P \]

$k \cdot x = k^- x^+ + k^+ x^- - k_L \times x_L$

**Feynman's $x$:** $x = \frac{k^+}{P^+}$ (boost invariant)

**Rapidity:** $z = \ln \frac{1}{x} = \ln \frac{P^+}{k^+}$

For the struck parton: $z \sim \ln \frac{1}{s}$

---

Figure 23: Gluon distribution resulting from the NLO DGLAP QCD fit to H1 $ep$ and BCDMS $\mu p$ cross section data in the massive heavy flavour scheme. The innermost error bands represent the experimental error for fixed $\alpha_s(M_Z^2) = 0.1150$. The middle error bands include in addition the contribution due to the simultaneous fit of $\alpha_s$. The outer error bands also include the uncertainties related to the QCD model and data range. The solid lines inside the error band represent the gluon distribution obtained in the fit to the H1 data alone.
The BFKL cascade

\[ k^+ = x P^+, \quad x \ll 1 \]
\[ P^+ \gg k_1^+ \gg k_2^+ \gg \ldots \gg k_n^+ \approx k^+ \]

\[ d\mathcal{P}_\perp \sim \alpha_s N_c \left( \frac{d\mathcal{P}_\perp}{d^2 k} \right) \rightarrow \ln \frac{1}{x} \]

For the whole cascade:

\[ P_n \sim \frac{1}{n!} \left( \alpha_s N_c \ln \frac{1}{x} \right)^n \]

Gluon distribution:

\[ \frac{dN}{d\ln \frac{1}{x}} \equiv \mathcal{G}(x, Q^2) \]

= # of gluons with transverse size \( \Delta x \sim \frac{1}{Q} \)

\[ \Delta x \equiv \ln \frac{1}{x} \sim \ln \frac{1}{s} \]

\[ \frac{dN}{d\tau} \bigg|_{\text{BFKL}} \sim \alpha_s N_c \sum_{n=0}^{\infty} \mathcal{P}_n = \alpha_s N_c e^{\omega_0} \sum_{n=0}^{\infty} \omega_n N_c \]

\[ \omega = \frac{4 \ln 2}{\pi} \]

D I S in the dipole frame

\[ \rho^n = (P, 0, 0, P) \]

\[ 2\tau = (\sqrt{2\tau - Q^2}, 0, 0, -2\tau) \]

\[ \mathcal{O}_{\text{dipole}}(\tau, \mathbf{k}_\perp) = \int d\mathbf{R}_\perp \mathcal{O}(\tau, \mathbf{R}_\perp^2, Q^2) \mathcal{O}_{\text{dipole}}(\tau, \mathbf{k}_\perp) \]

\[ \mathcal{O}_{\text{dipole}}(\tau, \mathbf{r}_\perp) = 2 \int d^2 b_\perp N_z(\mathbf{r}_\perp, b_\perp) \]

Scattering amplitude

\[ N_z(\mathbf{r}_\perp, b_\perp) = 1 - \frac{1}{N_c} \langle \mathcal{V}_{z, 1}^+ \mathcal{V}_{z, 1}^- \rangle \]

\[ V_{z, 1}^+ = P \exp\{ig \int dx A^+(x, \tau) \} \]

\[ \langle \mathcal{V}_{z, 1}^+ \mathcal{V}_{z, 1}^- \rangle_c = \text{average over the hadron wave function} \]

\[ \text{"evolved" up to rapidity } \tau \]

\( s \)-matrix element
Assume:

i) Two-point function only: \( \langle A^+_a(x) A^+_b(y) \rangle \),

\[ \text{ii) Small-dipole: } r_1 \ll \text{correlation length of } A^+ \]

\[ ig_{1a}^a (A^+_a(x_1) - A^+_a(y_1)) \approx ig_{1a}^a x_{1}^i A^+_a(x_1) \]

\[ N_\pi(r_1, b_1) \approx 1 - \exp \left\{ -\alpha_s \frac{r_1^2 x_{1}^i x_{1}^j G(x, \frac{r_1^2}{x_{1}^i}, b_1) \} \right\} \leq 1 \]

\[ xG(x, Q^2, b) = \text{Wigner transform of } \langle \hat{A}^+ \hat{A}^+ \rangle \]

- Fock space, local, gluon distribution

- Low density / very small dipole ⇒ single scattering

\[ N_\pi(r_1, b_1) \approx \alpha_s \frac{r_1^2 x_{1}^i x_{1}^j G(x, \frac{r_1^2}{x_{1}^i}, b_1) \} \]

\[ \frac{1}{r_1^2} \approx \frac{1}{Q^2(z, b)} \]

- Multiple scattering becomes important when

\[ r_1^2 \gtrsim \frac{1}{Q^2(z, b)} \]

\[ \text{saturation length} ^2 \]

\[ \frac{Q^2}{N_c} \approx \frac{\alpha_s N_c}{N_c - 1} x G(x, Q^2, b) \]

\[ Q^2 = \text{increases with } z \left( \sim e^{\omega_0 z} \right), \text{ decreases with } b_1 \]

- The scattering amplitude at fixed \( b_1 \) rises as a power of \( s \)

\[ N_\pi \leq 1 \]

- Violation of unitarity

Unitarization at fixed \( b_1 \)

Coherent multiple scattering

\[ N_\pi(r_1, b_1) = 1 - \exp \left\{ -\alpha_s \frac{r_1^2 x_{1}^i x_{1}^j G(x, \frac{r_1^2}{x_{1}^i}, b_1) \} \right\} \leq 1 \]

- "all twists"

Multiple scattering becomes important when

\[ r_1^2 \gtrsim \frac{1}{Q^2(z, b)} \]

\[ \text{(saturation length) } ^2 \]

\[ Q^2 = \text{increases with } z \left( \sim e^{\omega_0 z} \right), \text{ decreases with } b_1 \]
$b_{1}$-dependence and the problem of Froissart bound

- The edge of the hadron is not sharp!

\[ Q^2 = \frac{1}{r_{1}^2} : \text{dipole resolution} \]

\[ Q^2 < Q_s^2(r, b_{1}) \Rightarrow N_{c}(Q, b_{1}) \propto 1 \]

"block"

\[ Q^2 = Q^2_s(r, b) \text{ for } b = R(r, Q^2) \]

\[ e^{-2m_{b}b} \]

- Assume quantum evolution is local in $b_{1}$

\[ \Rightarrow N_{c}(Q^2, b_{1}) \sim e^{\omega_{s}x_{s}r} e^{-2m_{b}b} \sim 1 \text{ for } b = R(r, Q^2) \]

\[ \Rightarrow R(r, Q^2) \sim \frac{\omega_{s}x_{s}}{2m_{b}} r \text{ \heisenberg} \]

\[ \Rightarrow T_{\text{block disc}} = 2\pi R^2(r, e^2) \sim (\ln s)^2 \]

- However: perturbative evolution \Rightarrow massless gluons
What happens to the high density gluons?

- At high density (= small $x$), gluons overlap in the transverse plane and interact with each other.

\[ \text{Radiation} = \text{Recombination} \Rightarrow \text{Saturation} \]

(Gribov, Levin, Ryshkin, 83)

- Non-linear effects become important when

\[ \frac{\alpha_s N_c}{Q^2} \cdot \frac{x G(x, Q^2, b_2)}{N_c - 1} \sim 1 \]

i.e. at low transverse momenta: $Q^2 \ll \alpha_s^2 (x, b_2)$

\[ Q_s^2 (x, b_2) \sim \alpha_s \frac{N_c}{N_c - 1} \cdot x G(x, Q_s^2, b_2) \]

- Unitarization of the (local) scattering amplitude

\[ \leftrightarrow \text{Saturation of small-$x$ gluons in the hadron wave function} \]

- **N.B.** In the non-linear regime at saturation, the dipole is **not** computable in terms of the gluon distribution (a 2-point function) alone.
A classical effective theory

- How to compute in the non-linear regime at small $x$?
  Typical momenta: $k_{>z}^2 \sim Q_s^2 \sim e^{x_s/2} A^{1/3}$

$\Rightarrow x_s (Q_s^2) \ll 1$ for high energy and/or large $A$

- Large occupation numbers: $\hat{G}(x_1, Q_s^2, Q_2) \sim \frac{1}{x_s}$

$\Rightarrow$ semi-classical regime (McLerran, Veneugapalan, 94)

\[ p^+ \approx k^+ \]

\[ k^+(x) \Rightarrow \{ p_+ \approx k^+ \] [ \[ p^+ = e^{x_2} P^+ \] ]

$\Rightarrow$ Lorentz contracted: \[ p_+ (x, x_1) \sim S_+ (x) \]

\[ \sim \left( x_1 P^+ \gg x_2 \right) \]

$\Rightarrow$ Time dilated: \[ p_+ \text{ independent of LC-time } x^+ \]

\[ (D_{\perp} F_{\perp \perp})_{\parallel} (x) \sim S^+ + S (x_1) S (x_2) \]

$\Rightarrow$ Classical random variable, with weight from $W_{\parallel} (p)$

\[ \langle F_+ (x) F_+ (y) \rangle_{\perp} = [D_{\perp} W_{\perp} (p) F^+_x (p) F^+_y (p)] \]

$F^+_+ \sim \frac{1}{s}$ at saturation $\Rightarrow$ Exact classical solution

The Classical Solution

A non-Abelian Weiszäcker-Williams field

\[ \begin{cases} E_z = B_z = 0 \\ E_x = B_y ; \quad E_y = -B_x \quad (E \cdot B = 0) \end{cases} \]

\[ E^i(x; x_1) = S(x^0) \quad V(x_1) \quad \Rightarrow V^i(x_2) \quad (i = x, y) \]

\[ V^i(x_2) = P \exp \left( i g \int dx^0 \alpha_0 (x_1, x_2) T^i \right) \]

\[ \nabla^2 \alpha_0 = P \quad : \text{2-dimes Coulomb field} \]

\[ 1 \text{ unit range across } A^+ = 0.7 \]
Why "Colour Glass"?

- Spin glass
  Random distribution of spins $S_i = \pm 1$
  $H = - \sum_{ij} J_{ij} S_i S_j$ with random links $J_{ij}$

  Spin $S_i$: "dynamical"
  Link $J_{ij}$: "quenched"

  $H = - \sum_{ij} J_{ij} S_i S_j$

$\Rightarrow$ The spins thermalize for a fixed configuration of links

$F_T = -T \int [dJ_{ij}] W[J] \ln Z_T[J]$ $\Rightarrow$ The spins thermalize for a fixed configuration of links

$Z_T[J] = \sum_{\{S_i\}} e^{-\beta H[J]}$

- Analogy: $S_i \leftrightarrow F^{+,i}_a(x)$ $J_{ij} \leftrightarrow \rho_a(x)$

- Very different from a plasma!
  Mobile charges $\Rightarrow$ The current $j^H$ is determined by the background field $A^H$, and not vice-versa!

$J^H \neq j^H[A]$
Non-Linear Quantum Evolution

- How to compute the weight from $W_L(x)$ at high energy?
- Renormalization group in $k^+$
  - Ji, Liao, Mariani, Kovner, Leonidov, Weigert, et al.
- Linear evolution $\Leftrightarrow$ BFKL

\begin{align*}
\mathcal{P}_\xi &= \mathcal{P}_\xi \quad \text{for } b^+ < p^+ < k^+ \\
\mathcal{P}_\xi &= \mathcal{P}_\xi \quad \text{for } b^+ < k^+
\end{align*}

- Non-linear evolution

\begin{align*}
\mathcal{P}, W_{k^+} &\rightarrow \mathcal{P}, W_{k^+} \\
\mathcal{P}, W_{k^+} &\rightarrow \mathcal{P}, W_{k^+}
\end{align*}

- Due-loop background field calculation
  - Matching to a modified effective theory

The Renormalization Group Equation

\[ \frac{d}{dx} W_L(x) = \frac{1}{2} \int_{x_1, x_2} \frac{S}{\delta \alpha (x)} \chi_{x_1} \chi_{x_2} \frac{S}{\delta \alpha (y)} W_L(x) \]

\[ \chi_{x_1} \chi_{x_2} = \sum \frac{x_1^{-2i} y_1^{-2i}}{x_2^{-2i} (y_2^{-2i})} (1 + V_{x}^{+} V_{y} - V_{x}^{+} V_{y} - V_{x}^{+} V_{2}) \]

- A functional Fokker-Planck eq. (E.I., Leonidov, Mclerran, 2000)
- Ordinary evolution eqs. for observables

\[ \frac{2}{3} \langle \text{tr } V_{x}^{+} V_{y} \rangle = \int \mathcal{D} \alpha \text{ tr}(V_{x}^{+} V_{y}) \frac{2}{3} W_L(x) \]

- Equivalent to eqs. established within other formalisms
  - by Balitsky (96), Kovchegov (99), Weigert (2000)
- Path-integral solution (Kovchegov, 2002)
- Langevin eq.: a random walk on a group manifold

\[ V_{i}^{+} = e^{i \alpha_{i} V_{i-1}^{+}} \]

\[ \langle \alpha_{i} \alpha_{i} \rangle = \chi [V_{i-1}] ; \quad \langle \alpha_{i} \rangle = \frac{S}{\delta \alpha_{i-1}} \chi [V_{i-1}] \]

- Numerical implementation (in progress)
  - Kowarschuck, Weigert
Approximate solutions (F.I., McLerran, 2001)

- $Q_s(z) \leftrightarrow$ Inverse correlation length for $\langle N_x V_y \rangle_z$
  
  $Q_s^2(z) \sim \Lambda^2 \exp(\kappa s z)$, $\kappa = (4/5) \Lambda_c N_c / \Lambda$

- $k_L \gg Q_s(z)$: $V_x^+ \approx 1 + i g \alpha_s(x) \Rightarrow $ BFKL
- $k_L \ll Q_s(z)$: $g \alpha_s \approx 1 \Rightarrow V_x^+ \approx 0 \Rightarrow "free" diffusion$

0. Initial condition: the valence quark model (NV)

$\langle p(+k) p(-k) \rangle_0 \sim \mu_A = const.$

- $\{\text{no evolution}\}$
- $\{\text{no correlations}\}$

1. $k_L \gg Q_s(z)$: $\langle p(+k) p(-k) \rangle_z \sim e^{\kappa_1 s z} \Phi_2^z(k_L)$

- exponential increase with $z$
- transverse correlation get built up

2. $k_L \ll Q_s(z)$: $\langle p(+k) p(-k) \rangle_z \sim k_L^{-2} (z - 2 \Xi(k_L))$

- colour neutrality: $\langle Q^0 Q^0 \rangle = 0$
- linear scale $\sim 1/k_L$
- linear increase with $z \approx \mu_L$: saturated sources

$\Xi(k_L) = \frac{1}{c_2} \ln \frac{k_L^2}{\Lambda^2}$: critical rapidity above $Q_s = k_L$

- Geometric scaling: $\Phi_2^z(k_L) = f \left( \frac{Q_s^2(z)}{k_L^2} \right)$
- Holds also above $Q_s$, up to $k_L \sim Q_s^2 / \Lambda_{QCD}$.
- Consistent with $F_2$ data at HERA.
The Froissart Bound Revisited
(Ferreiro, E.I., Itakura, McLerran, in preparation)

- Can the perturbative evolution preserve the exponential fall-off of the non-perturbative initial condition?
  \[ Q_\alpha^2(x) = Q_\alpha^2 \left( \frac{x}{x_0} \right)^\lambda \quad ; \quad \lambda = 0.29 \quad \text{from fit} \]
  \[ Q_\alpha = 1 \text{GeV} \quad x_0 = 3 \times 10^{-4} \]
  \[ \alpha \to \text{fits} \quad \alpha < 0.01 \]

- Potential problem: massless gluons (Kovner, Wiedemann, 2001)
  i) At high energy, the cross-section is dominated by the black disk:
  \[ \sigma_{\text{dipole}}(z, r_b) \approx 2\pi R^2 \left( z, \frac{r_b}{\Lambda_{\text{QCD}}} \right) \]
  ii) The expansion of the black disk is controlled by scattering within the grey area:
  \[ b > R(z, r_b) \]
  iii) The scattering in the grey area is dominated by nearby colour sources:
  \[ 1 \ll \frac{1}{R(z, r_b)} \ll 1 \]
Colour neutrality of the saturated gluons is crucial! 

\[ N_g^c (z_1, b_1) \sim \text{"short-range"} + \text{"long-range"} \]

\[ \left| z_1 - b_1 \right| < \frac{1}{R_S} \quad \text{black disk} \]

\[ \alpha_s \frac{b_1^2}{b_1^4} \int_0^\infty \frac{z_1}{R(t)} \text{d}z_1 \]

BFKL solution

\[ N_\sigma^c (z_1, b_1) \sim 1 \quad \text{for} \quad b_1 = R(z, b_1) \]

\[ \Rightarrow \text{"short-range"} = \delta(1) \quad \text{and} \quad \text{"long-range"} < 1 \]

\[ R(z) \sim \frac{\omega x_s}{2 m_\pi} z, \quad \omega = 4 \ln(2) \frac{N_c}{\pi} \]

N.B. \[ \sum_{\text{BFKL}} \sim S^\omega x_s \leftrightarrow \sum_{\text{Froissart}} \sim \left( \frac{\omega x_s}{m_\pi} \ln S \right)^2 \]

Without transverse correlations: \[ \langle p(x) p(-x) \rangle = \mu \]

\[ \Rightarrow \text{"long-range"} = \frac{\mu x^2}{\beta_2^2} \int_0^\infty \frac{z_1}{R(t)} \text{d}z_1 \]

\[ R^2(t) \sim \exp \left( \frac{-\mu x_1^2}{\beta_2^2} \right) \sim S^{\frac{x^2}{\beta_2^2}} \]

\[ \Rightarrow \text{no Froissart bound!} \]
Perspectives and open problems

- Detailed numerical studies of the non-linear evolution e.g.s. (including $b_1$-dependence)
  (e.g., to numerically confirm Froissart bound, and to compare with $F_2$ data at HERA)
- Next-to-leading order formalism
  (non-linear generalization of NLO-BFKL)
  - necessary in order to compare with data
- Extension to hadron-hadron collisions
  (e.g. heavy ions): "factorization formulae"
  Any hadron may be viewed as a collection of "dipoles" (for large $N_c$ and high energy).