Hadronic Spectral Functions in Lattice QCD at T=0 & T≠0

1. *Hadronic spectral functions*
2. *Principles of the Maximum Entropy Method*
3. *Example with mock data*
4. *Lattice QCD results*
   - *light mesons (π, ρ) at T=0*
   - *light baryons (N, N^*) at T=0*
   - *heavy meson (J/ψ) at T≠0*
5. *Summary and future*

   Other applications, transport coefficients ...

MELQCD Collaboration
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- Asakawa, Nakahara & T. H., Phys. Rev. D60 ('99) 091503. Prog. Part. Nucl. Phys. 46 ('01) 459. [hep-at/0011040]
- Sasaki, Sasaki, Asakawa & T.H., in progress.

1. Hadronic spectral functions

- *Matsubara correlation*

$$D(\tau) = \int \langle \mathrm{T}_\tau J^\dagger(\tau, \vec{x}) J(0) \rangle \, d^3x$$

- *Retarded correlation*

$$D^R(t) = i \int \langle \mathrm{R} \, J^\dagger(t, \vec{x}) J(0) \rangle \, d^3x$$

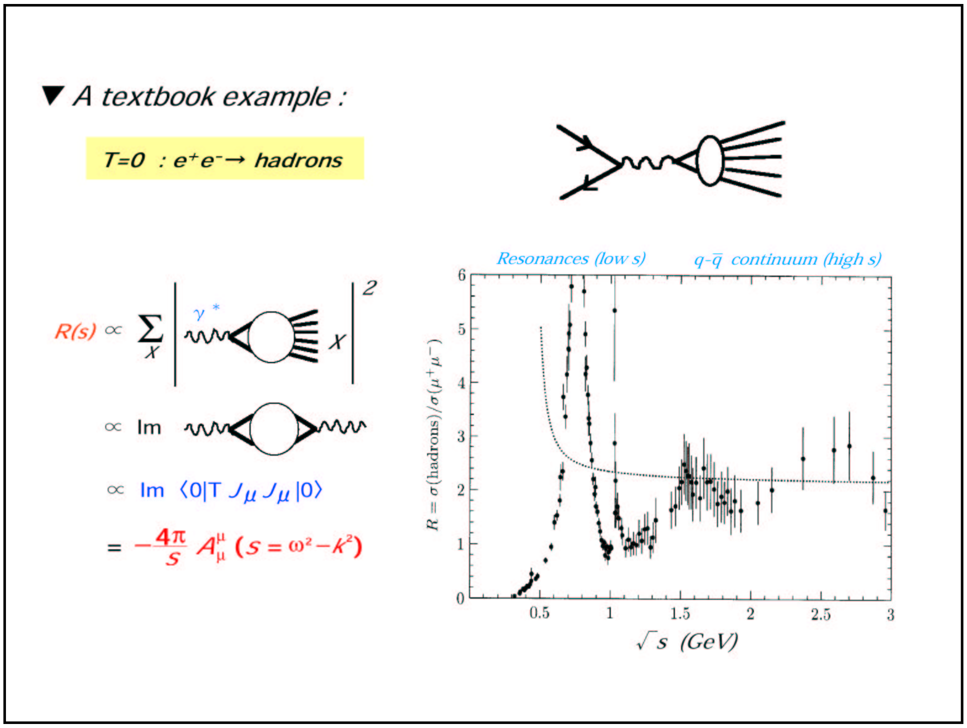
- *Spectral representation*

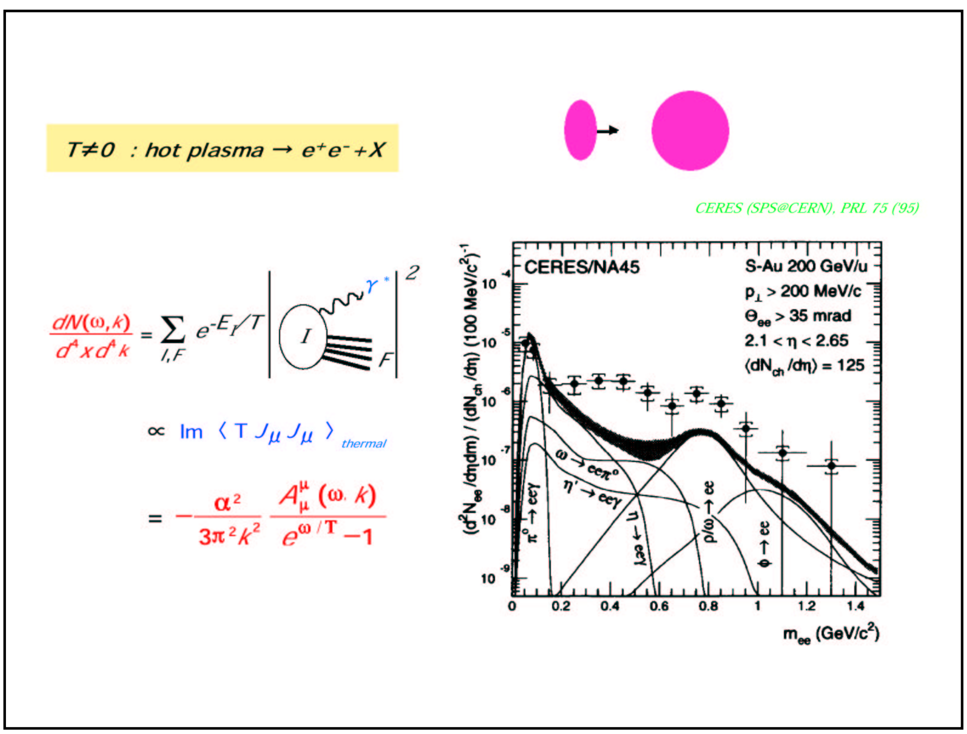
$$D(\tau > 0) = \int_{-\infty}^{\infty} \frac{e^{-\omega\tau}}{1 \mp e^{-\omega/T}} A(\omega) \, d\omega \qquad \bar{D}^R(\omega) = \int_{-\infty}^{\infty} \frac{A(\omega')}{\omega' - \omega - i\varepsilon} d\omega' - (\mathrm{subt.})$$

- *Spectral function*

$$A(\omega) = \frac{1}{\pi} \mathrm{Im} \, D^R(\omega)$$
$$= (2\pi)^3 \sum_{n,m} \frac{e^{-E_n/T}}{Z} \langle n | J^\dagger | m \rangle \langle m | J | n \rangle \times (1 \mp e^{-(E_n - E_m)/T}) \, \delta(\omega - (E_n - E_m)) \, \delta^3(\vec{P}_n - \vec{P}_m)$$

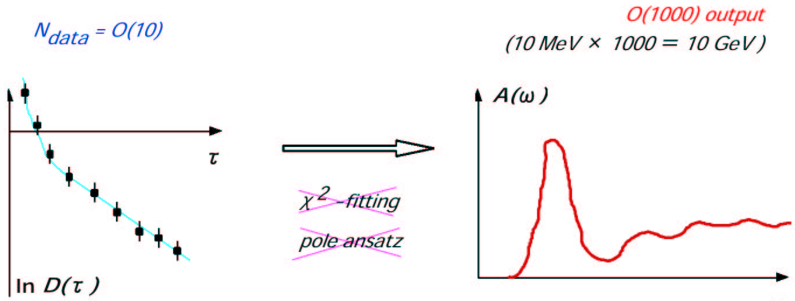
$$A(\omega \geq 0) = \mp A(-\omega) \geq 0 \qquad A_{pQCD}(\omega \gg 1\,GeV, T) \propto \omega^{2\dim[J]-4} \cdot \left(1 + c_1 \frac{\alpha_s}{\pi} + \cdots\right)$$


▼ A textbook example :
T=0 : e⁺e⁻→ hadrons
Resonances (low s)
q-q̄ continuum (high s)
R(s) ∝ Σ_X | γ* X |²
∝ Im
∝ Im ⟨0|T J_μ J_μ |0⟩
= −4π/s A^μ_μ (s = ω² − k²)
R = σ(hadrons)/σ(μ⁺μ⁻)
√s (GeV)


T≠0 : hot plasma → e⁺e⁻+X
CERES (SPS@CERN), PRL 75 ('95)
dN(ω,k)/d⁴x d⁴k = Σ_{I,F} e^{-E_I/T} | I γ* F |²
∝ Im ⟨ T J_μ J_μ ⟩ thermal
= −α²/(3π²k²) A^μ_μ(ω, k)/(e^{ω/T} −1)
CERES/NA45
S-Au 200 GeV/u
p⊥ > 200 MeV/c
Θee > 35 mrad
2.1 < η < 2.65
⟨dNch/dη⟩ = 125
(d²Nee/dηdm) / (dNch/dη) (100 MeV/c²)⁻¹
mee (GeV/c²)

▼ *How to calculate A(ω) in QCD ?*

$$D(\tau > 0) = \int_{-\infty}^{\infty} \frac{e^{-\omega\tau}}{1-e^{-\omega/T}} A(\omega)\, d\omega = \int_{0}^{\infty} \underbrace{\frac{e^{-\omega\tau} + e^{+\omega(\tau - 1/T)}}{1-e^{-\omega/T}}}_{K(\tau,\omega)} A(\omega)\, d\omega$$



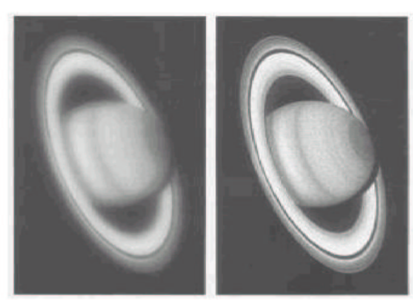
Maximum Entropy Method (MEM) ?

Nakahara, Asakawa and T.H., Phys. Rev. D60 ('99)

MEM Image Reconstruction

The girl's portrait

The Image of Saturn



From N. Wu, "The Maximum Entropy Method" (1997)

2. Principles of MEM

$D(\tau) \rightarrow A(\omega)$: not unique !

What is the most probable $A(\omega)$ for given $D(\tau)$?

T. Bayes 1702-1761

Statistical Inference

Posterior prob. — likelihood func. — Prior prob.

$$P[A|D] = \frac{1}{P[D]} P[D|A]\, P[A]$$

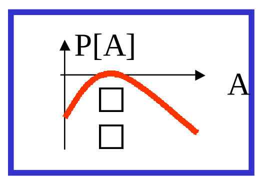
- $\frac{\delta P[A|D]}{\delta A} = 0 \Rightarrow A_{out}(\omega)$
- $\frac{\delta^2 P[A|D]}{\delta A \delta A} \Rightarrow$ reliability of $A_{out}(\omega)$

Bayes Theorem

$$P[XY] = P[X|Y]\, P[Y] = P[Y|X]\, P[X]$$

▶ *Explicit form of the posterior prob.*

$$P[A|D] \propto P[D|A]\, P[A] \propto e^{Q(D,A)}$$



- Central limiting theorem

$$P[D|A] \propto e^{-L(D,A)} = \exp\left(-\frac{1}{2}\sum_{i,j}\left[D(\tau_i) - D_A(\tau_i)\right] C_{ij}^{-1}\left[D(\tau_j) - D_A(\tau_j)\right]\right)$$

- Information entropy (Shanon-Jaynes)

$$P[A] \propto e^{\alpha S(A)} = \exp \alpha \int_0^\infty \left[A(\omega) - m(\omega) - A(\omega)\ln\left(\frac{A(\omega)}{m(\omega)}\right)\right] d\omega$$

- ◆ $Q(D,A) = \alpha S - L$: "- free energy" ⇦ to be maximized with respect to $A(\omega)$
- ◆ α : a fictitious "temperature" ⇦ to be integrated with a weight $P[\alpha|D]$
- ◆ $m(\omega)$: prior estimate of $A(\omega)$ ⇦ to be updated by error analysis

Advantages of MEM

(i) No a priori parametrizations of spectral functions

(ii) Unique solution is obtained

(iii) Statistical significance of the solution can be studied

Shonon-Jaynes Entropy

- Combinatorial construction (Monkey argument) — Friedan ('72); Gull & Daniell ('79); Jaynes ('86); Skilling ('88)

  Product of Poisson distribution → SJ entropy

$$P_\lambda(n) = \prod_{i=1}^{N} \frac{\lambda_i^{\,n_i} e^{-\lambda_i}}{n_i\,!}$$

$$\rightarrow \exp\left[\alpha \sum_{i=1}^{N}\left(A_i - m_i - A_i \ln\left(\frac{A_i}{m_i}\right)\right)\right]$$

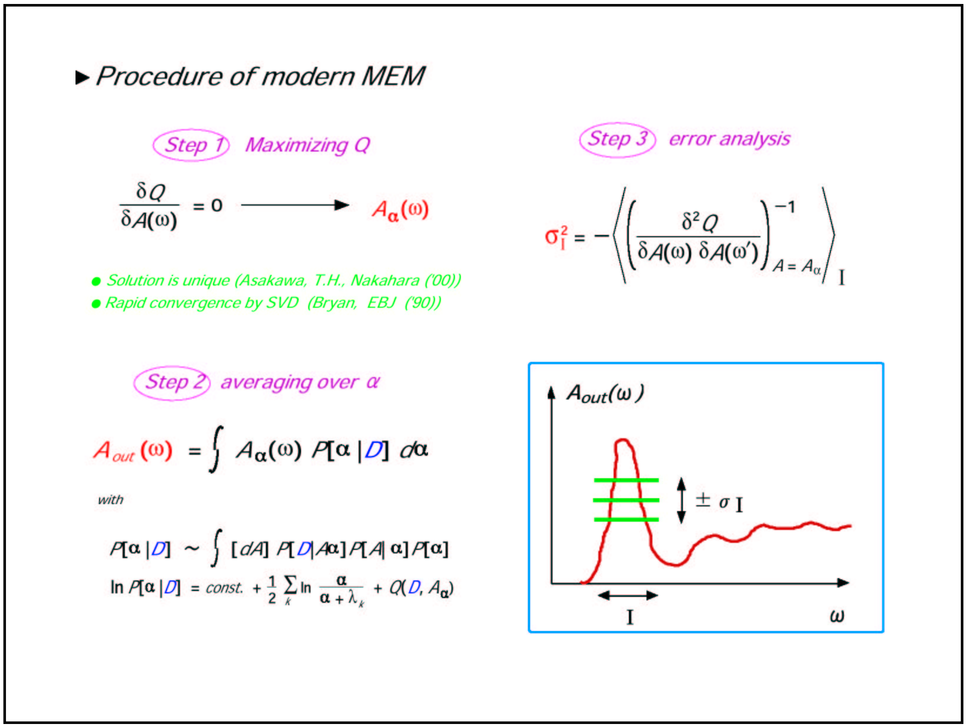
$A_i = n_i/\alpha$
$m_i = \lambda_i/\alpha$

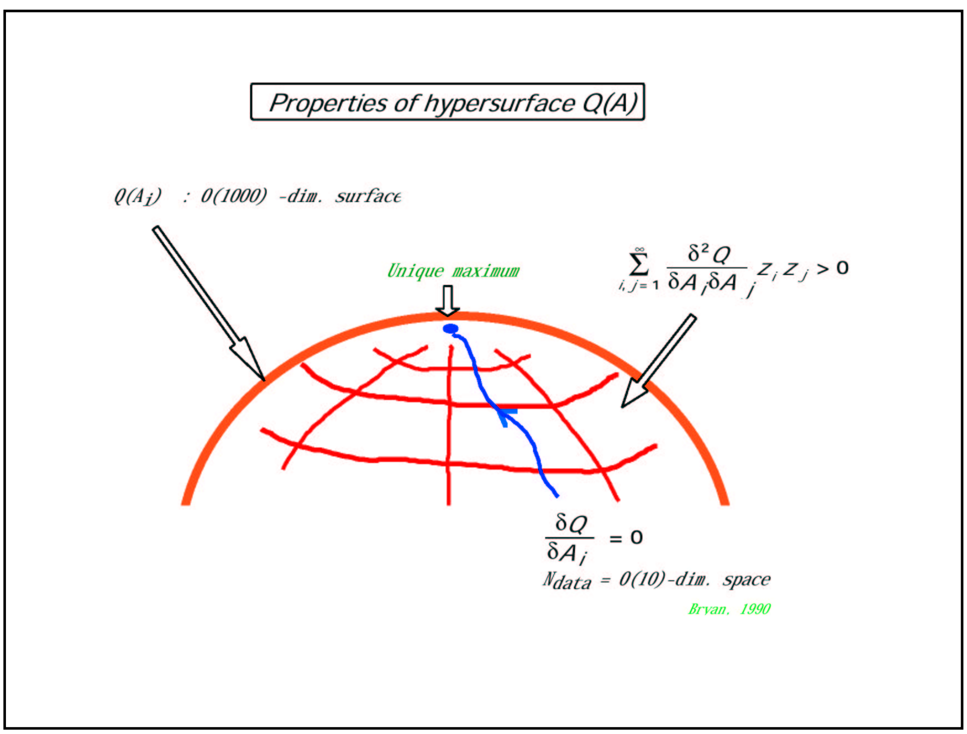
$n! \sim \exp[n \log n - n]$

- Axiomatic construction

  Axom I : Locality
  Axiom II : Coordinate invariance
  Axiom III : System independence
  Axiom IV : Scaling

  Shanon (1948); Jaynes (1957), ……, Skilling ('88); Asakawa, T.H., & Nakahara, ('01)


▶ Procedure of modern MEM
Step 1 Maximizing Q
$\frac{\delta Q}{\delta A(\omega)} = 0 \longrightarrow A_\alpha(\omega)$
● Solution is unique (Asakawa, T.H., Nakahara ('00))
● Rapid convergence by SVD (Bryan, EBJ ('90))
Step 3 error analysis
$\sigma_I^2 = -\left\langle \left(\frac{\delta^2 Q}{\delta A(\omega)\,\delta A(\omega')} \right)^{-1}_{A=A_\alpha} \right\rangle_I$
Step 2 averaging over α
$A_{out}(\omega) = \int A_\alpha(\omega)\, P[\alpha|D]\, d\alpha$
with
$P[\alpha|D] \sim \int [dA]\, P[D|A\alpha]\, P[A|\alpha]\, P[\alpha]$
$\ln P[\alpha|D] = const. + \frac{1}{2}\sum_k \ln \frac{\alpha}{\alpha+\lambda_k} + Q(D, A_\alpha)$
$A_{out}(\omega)$
$\pm\, \sigma_I$
I
ω


Properties of hypersurface Q(A)
$Q(A_i)$: 0(1000) -dim. surface
Unique maximum
$\sum_{i,j=1}^{\infty} \frac{\delta^2 Q}{\delta A_i \delta A_j} z_i z_j > 0$
$\frac{\delta Q}{\delta A_i} = 0$
N_{data} = 0(10)-dim. space
Bryan. 1990

▶ *Kernel K(τ ,ω)*

Free propagator with mass ω

$$D(\tau > 0) = \int_0^\infty K(\tau,\omega)\, A(\omega)\, d\omega$$

- *"continuum" kernel*

$$K_{cont}(\tau,\omega) = e^{-\omega\tau} = 2\omega \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} \frac{e^{i\nu\tau}}{\omega^2+\nu^2}\, d\nu$$

- *"lattice" kernel*

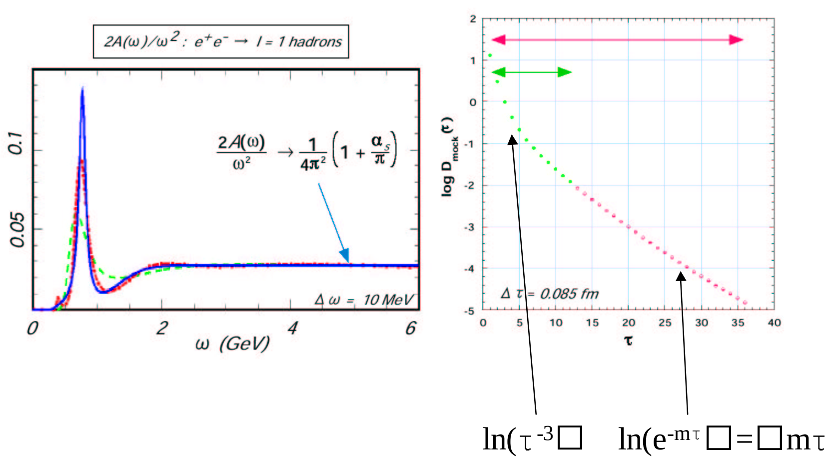
O(a) difference

$$K_{lat}(\tau,\omega) \equiv 2\omega \int_{-\pi/a}^{\pi/a} \frac{d\nu}{2\pi} \frac{e^{i\nu\tau}}{\omega^2+\left(\frac{2}{a}\sin\frac{\nu a}{2}\right)^2}\, d\nu$$

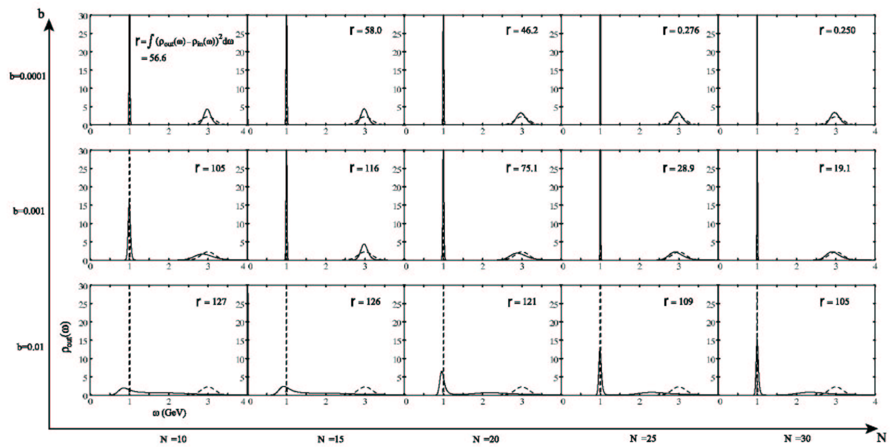
3. Test with mock data

Laplace Trans.	Truncation & Noise	MEM

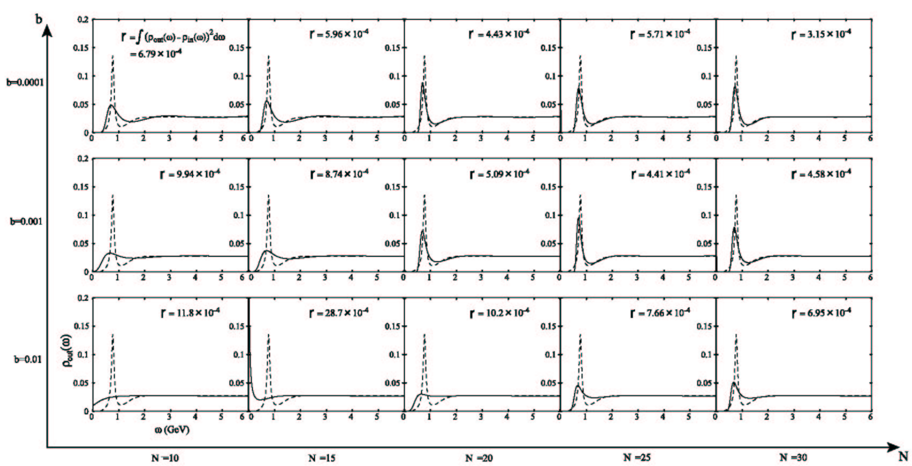
$A_{in}(\omega) \rightarrow D_{in}(\tau) \rightarrow D_{mock}(\tau_i) \rightarrow A_{out}(\omega_n)$



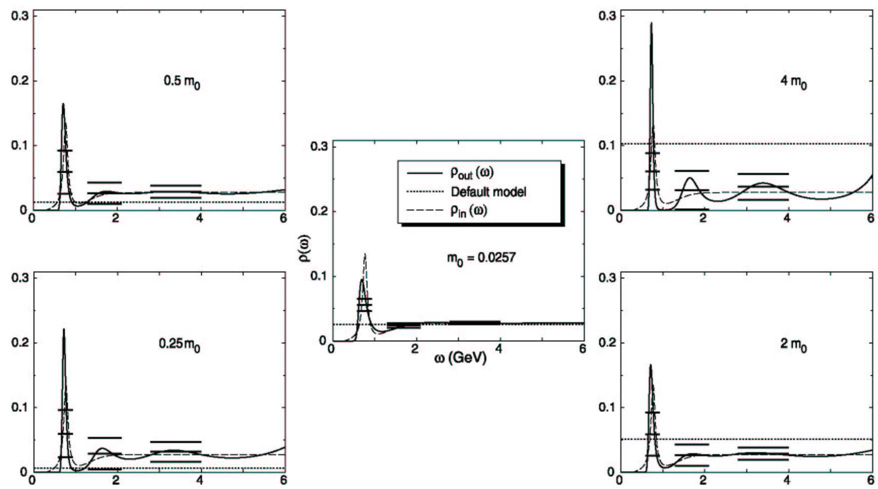
Sharp peak and broad peak



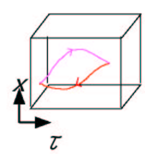
Dependence on b and N_{data}



How to choose m ?



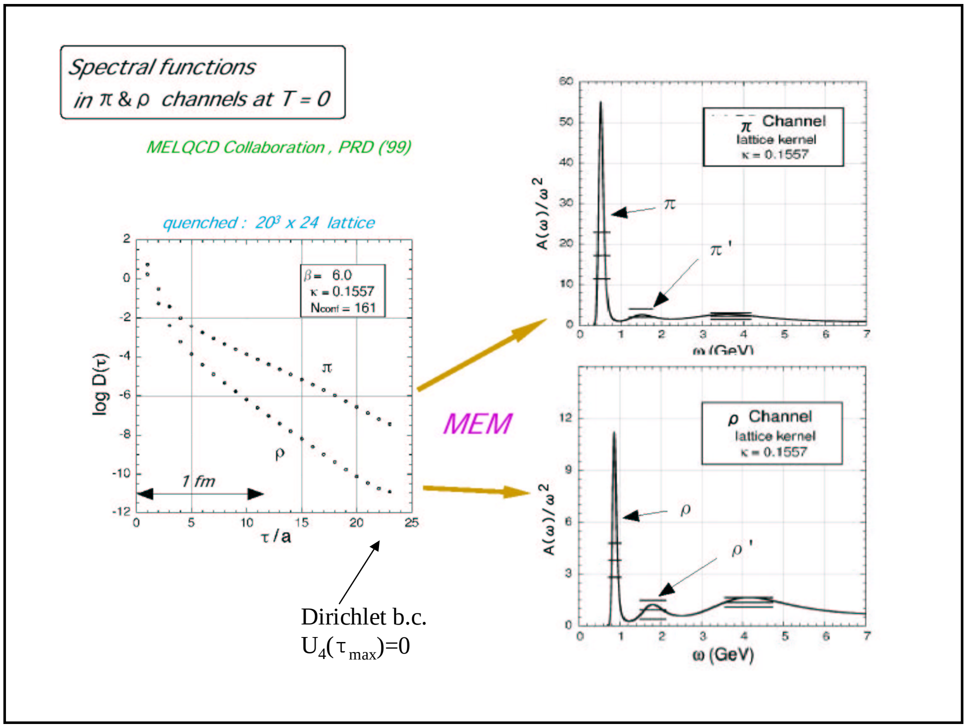
4. Our lattice QCD simulation (T=0, T≠0)

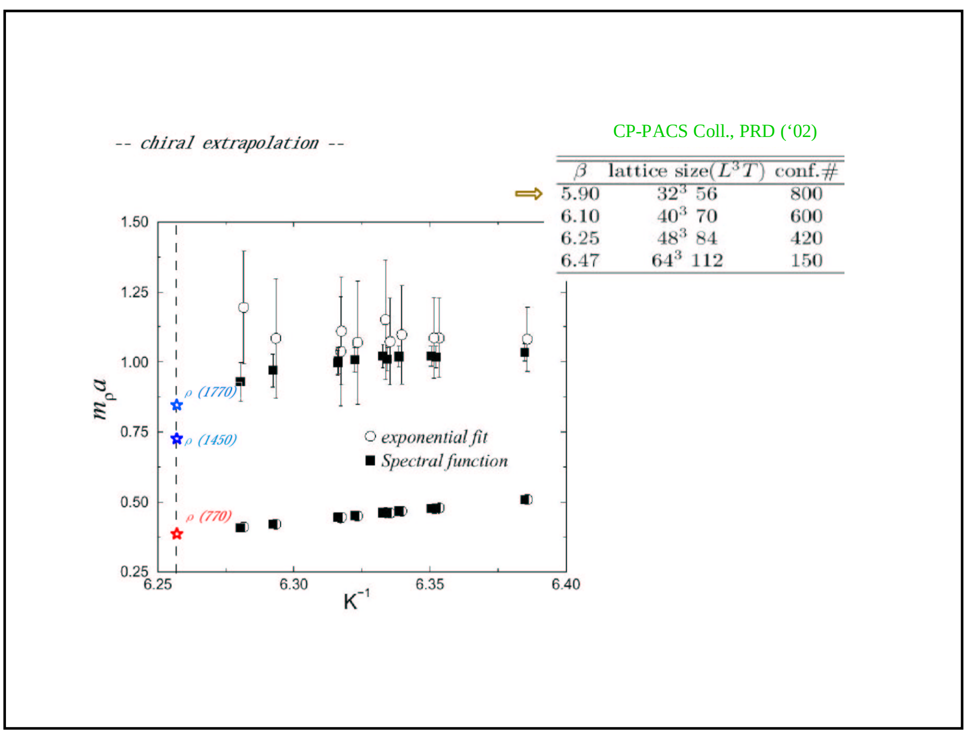


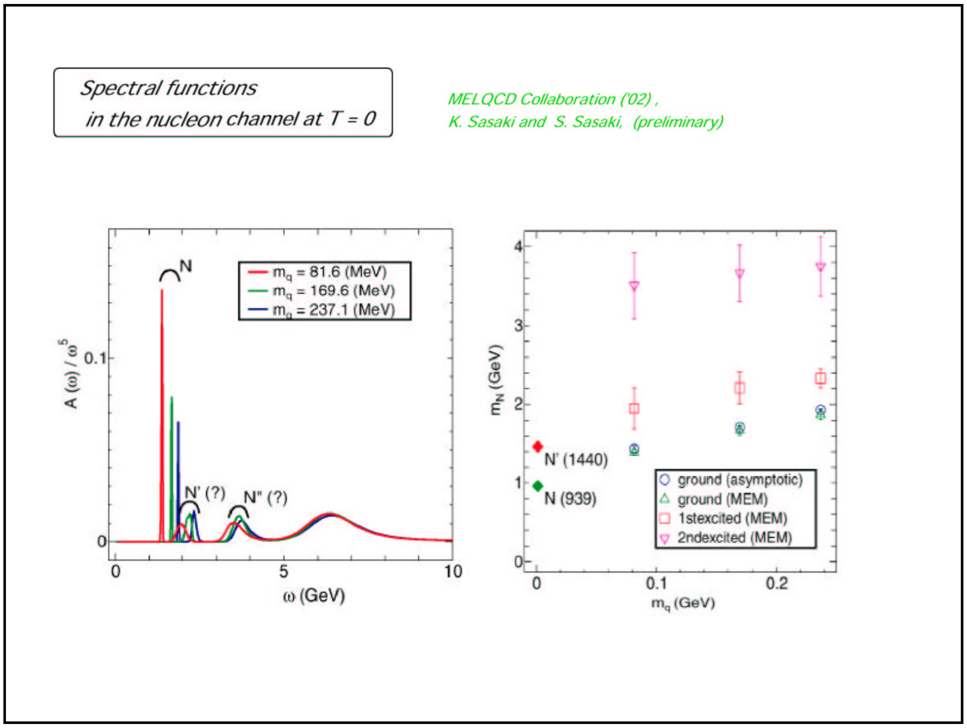
MELQCD Collaboration - Lattice parameters --

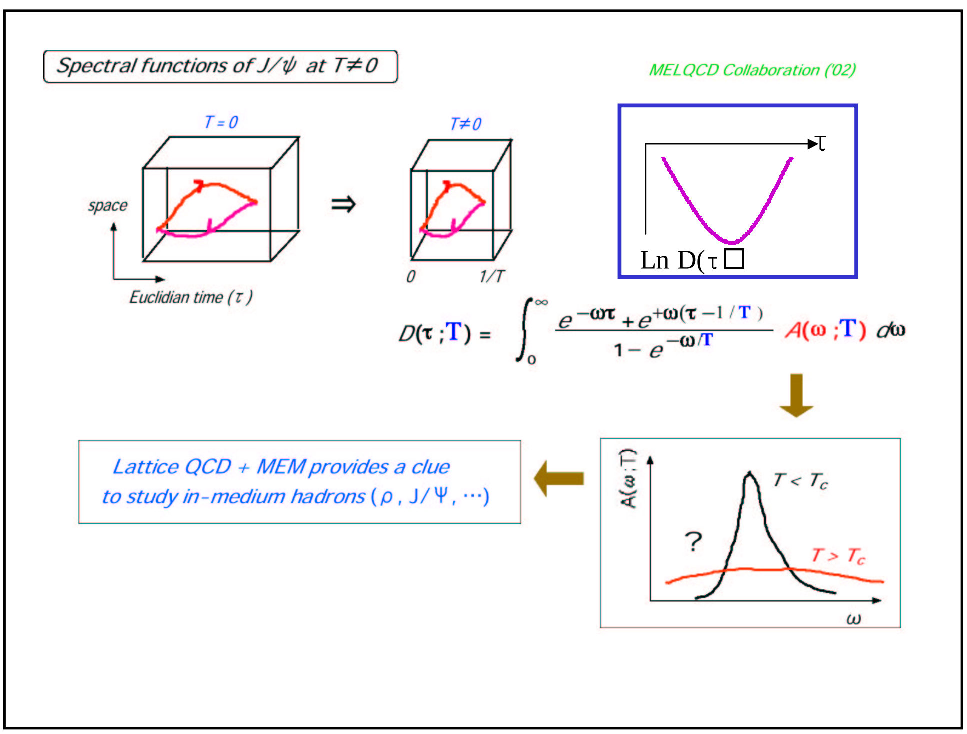
Lattice action	plaquette gluon action, Wilson fermion, quenched approximation			
Source and Sink	point-point			
Machine	SR2201 (JAERI)	SR8000 (KEK)	CP-PACS (RCCP@Tsukuba)	
Lattice size	20^3 x 24 with β = 6.0 a=0.09 fm (1.7 fm)^3 x 2.0 fm	16^3 x 32 with β = 6.0 a=0.09 fm (1.5fm)^3 x 3.0 fm	40^3 x 30 with β = 6.47 a=0.06 fm (2.5fm)^3 x 1.9 fm	32^3 x 32, 40, 72, 96 anisotropic lattice 1:4 with β = 7.0 a_τ =0.01 fm, a_σ =0.04 fm (1.25 fm)^3 x (up to 2.5 T_c)
Boundary condition	Dirichlet	Periodic and Anti-periodic	Anti-periodic	Anti-periodic
Physics	Light mesons at T=0	Light baryons at T=0	Light mesons at T=0	Heavy mesons at T≠0 Transport coefficients

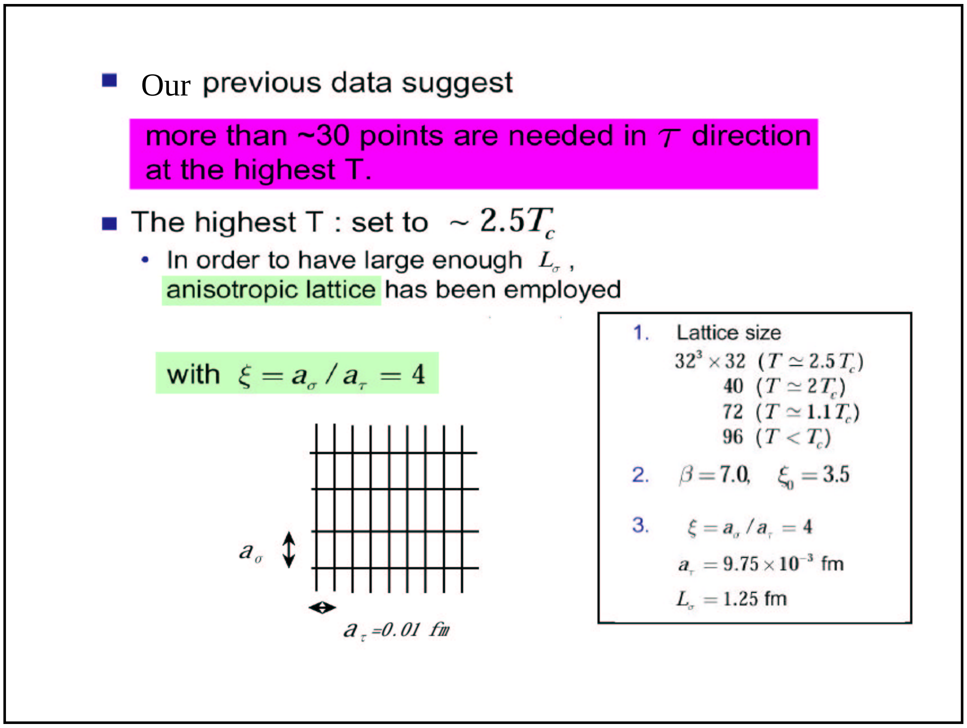
Experience tells: N_τ > 20 (Dirichlet), > 30 (periodic) required

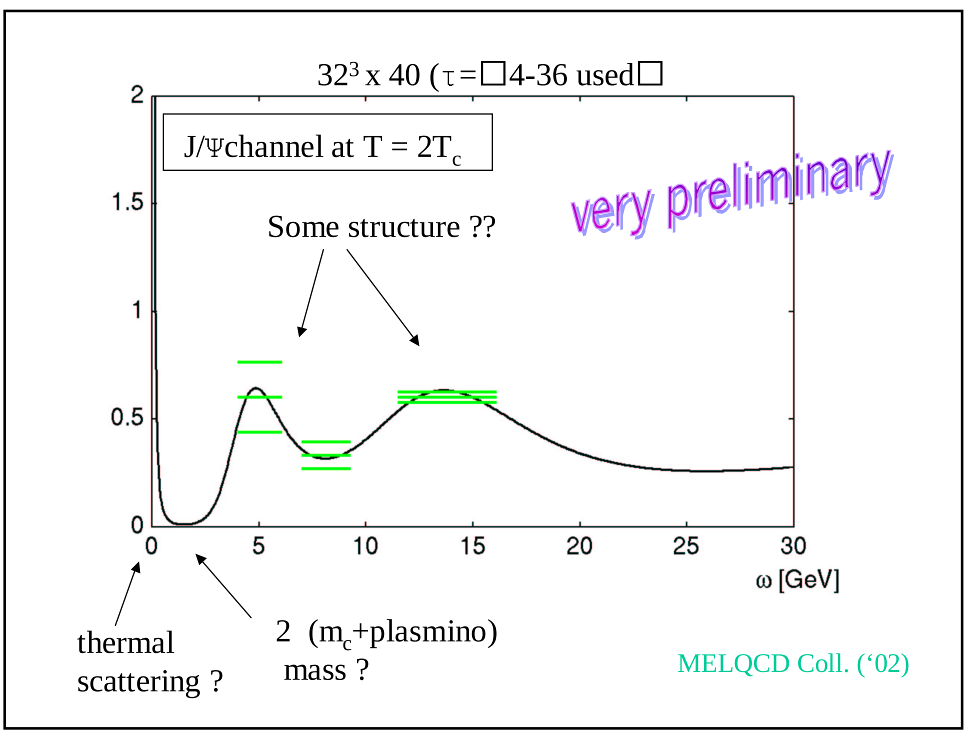

Spectral functions
in π & ρ channels at T = 0
MELQCD Collaboration , PRD ('99)
quenched : 20³ x 24 lattice
β = 6.0
κ = 0.1557
Nconf = 161
log D(τ)
τ / a
1 fm
π
ρ
MEM
Dirichlet b.c.
U4(τmax)=0
π Channel
lattice kernel
κ = 0.1557
π '
A(ω)/ω²
ω (GeV)
ρ Channel
lattice kernel
κ = 0.1557
ρ '


-- chiral extrapolation --
CP-PACS Coll., PRD ('02)
β lattice size(L³T) conf.#
5.90 32³ 56 800
6.10 40³ 70 600
6.25 48³ 84 420
6.47 64³ 112 150
m_ρ a
K⁻¹
ρ (1770)
ρ (1450)
ρ (770)
exponential fit
Spectral function


Spectral functions
in the nucleon channel at T = 0
MELQCD Collaboration ('02) ,
K. Sasaki and S. Sasaki, (preliminary)
N
N' (?)
N" (?)
m_q = 81.6 (MeV)
m_q = 169.6 (MeV)
m_q = 237.1 (MeV)
A (ω) / ω^5
ω (GeV)
m_N (GeV)
N' (1440)
N (939)
ground (asymptotic)
ground (MEM)
1stexcited (MEM)
2ndexcited (MEM)
m_q (GeV)


Spectral functions of J/ψ at T≠0
MELQCD Collaboration ('02)
T = 0
T≠0
space
Euclidian time (τ)
0
1/T
τ
Ln D(τ)
D(τ ;T) = ∫_0^∞ (e^{−ωτ} + e^{+ω(τ −1/T)}) / (1 − e^{−ω/T}) A(ω ;T) dω
Lattice QCD + MEM provides a clue
to study in-medium hadrons (ρ , J/Ψ, …)
A(ω;T)
T < Tc
?
T > Tc
ω


Our previous data suggest
more than ~30 points are needed in τ direction at the highest T.
The highest T : set to ~ 2.5T_c
In order to have large enough L_σ ,
anisotropic lattice has been employed
with ξ = a_σ / a_τ = 4
a_σ
a_τ =0.01 fm
1. Lattice size
32^3 × 32 (T ≃ 2.5 T_c)
40 (T ≃ 2 T_c)
72 (T ≃ 1.1 T_c)
96 (T < T_c)
2. β = 7.0, ξ_0 = 3.5
3. ξ = a_σ / a_τ = 4
a_τ = 9.75 × 10^-3 fm
L_σ = 1.25 fm


32^3 x 40 (τ=4-36 used)
J/Ψchannel at T = 2T_c
very preliminary
Some structure ??
2
1.5
1
0.5
0
0
5
10
15
20
25
30
ω [GeV]
thermal scattering ?
2 (m_c+plasmino) mass ?
MELQCD Coll. ('02)

5. Summary and future

- What we have learned from the ill-posed problem, $D(\tau) \rightarrow A(\omega)$?
  - MEM works pretty well in lattice QCD
    1. O(10) data points → most probable and unique solution $A(\omega)$
    2. error analysis → "plausibility" of $A(\omega)$
    3. better data (smaller a, better statistics, larger L) → better results
    4. O(a) error can be studied by using $K_{lat}(\tau, \omega)$
  - Advantages over standard pole-fit

	MEM	pole-fit
ground states	good	good
excited states	reasonable	unstable
q-$\bar{q}$ continuum	correct height detection of π/a	no way
finite T	straightforward	no way

- Spectral Functions on the Lattice in the Future
  - T = 0 system

under way 1. nucleon resonances, glueballs etc

2. hadron mixings (ρ - ω, Λ - Σ^0 etc)
3. full QCD and hadronic widths ($\rho \rightarrow 2\pi$, $\sigma \rightarrow 2\pi$ etc)
4. parton distribution functions

- $T \neq 0$ system

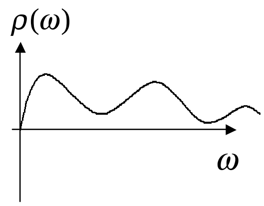
under way 1. $\rho\ \omega\ \phi$ ⇔ chiral restoration

under way 2. J/Ψ, Ψ' ⇔ deconfinement

3. collective modes at $T \sim T_c$, $T > T_c$
4. ⇨ transport coefficients of hot plasma
5. collective modes in N_c=2 color superconductor

Transport coefficient on the lattice:

$$G(\tau) = \int_0^\infty d\omega\, K(\tau,\omega)\, \rho(\omega), \quad \eta \equiv \frac{d}{d\omega}\rho(\omega)\Big|_{\omega=0}$$



-- Hard to extract η in the "standard" MEM --

Karsch, Aarts, ….

"Subtracted" spectral rep. + MEM for p/n distribution: T.H.('02)

$$\tau(\beta-\tau)G(\tau) = \eta + \int_0^\infty d\omega\, \tilde{K}(\tau,\omega)\, \tilde{\rho}(\omega)$$

τ-indep.