Cutting Rules For Real Time Finite Temperature Amplitudes

Outline

- Goal: Imaginary parts of self energies
  - Why? ImΠ → a prod. rate
  - What's different?
- $T = 0$ - one loop ϕ^3 theory
- $T = 0$ at >1 loop → Cutkosky Rules
- $T \neq 0$
  - Basics
  - One Loop
  - Two Loops
  - Generalization

Imaginary Parts of Self Energies

Why? ImΠ → Production Rates

For QED: $\text{Im}\Pi = \text{Im}\Pi^{R}_{\mu}{}^{\mu}(q0, \vec{q})$

1) $Q^2 = 0 \rightarrow$
prod. rate for real photons:

$$\frac{dN}{dtd^3x} = -\frac{d^3q}{(2\pi)^3 2q_0} 2n_B(q_0)\text{Im}\Pi$$

2) $Q^2 > 0 \rightarrow$
prod. rate for l^+l^- with $m = \sqrt{Q^2}$:

$$\frac{dN}{dtd^3x} = -\frac{dq_0 d^3q}{12\pi^4}\frac{\alpha}{Q^2} n_B(q_0)\text{Im}\Pi$$

$$n_B(q_0) = \frac{1}{e^{q_0/T} - 1}$$

Basic Idea

- Imag part puts some sets of internal lines on the mass shell → real particles

  Thermal Field Theory:

  positive mass shell → emitted particle

  negative mass shell → absorbed particle

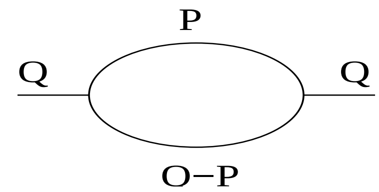
- Prod Rate

  $\sim \mathrm{Im}\Pi$

  $\sim$ Sum of different physical processes

The Goal:
To calculate $\mathrm{Im}\Pi_R$ in a way that makes it easy to isolate contributions from different physical processes to the production rate

Zero T One Loop



$$\Pi \sim i \int dP D(p) D(q-p)$$

$$D(p) = \frac{1}{P^2 - m^2 + i\epsilon}$$

$$D(x-y)$$
$$= \Theta(x_0 - y_0) \underbrace{\langle \phi(x)\phi(y)\rangle}_{D^>(x-y)} + \Theta(y_0 - x_0) \underbrace{\langle \phi(y)\phi(x)\rangle}_{D^<(x-y)}$$

$$D^>(P) = \Theta(p_0) 2\pi\delta(P^2 - m^2)$$

$$D^<(P) = \Theta(-p_0) 2\pi\delta(P^2 - m^2)$$

Imaginary Part:

$$\mathrm{Im}\Pi = \frac{1}{2i}(\Pi - \Pi^*)$$

$$\sim \left(\mathrm{Im}D(P)\mathrm{Im}D(Q-P) + \underbrace{\mathrm{Re}D(P)\mathrm{Re}D(Q-P)}_{=0}\right)$$

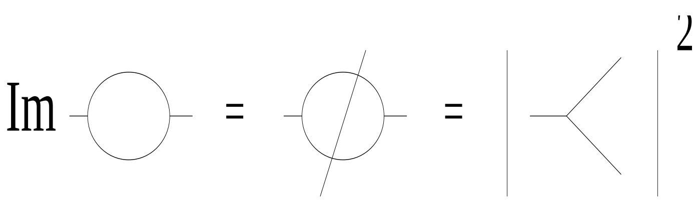
Use:

$$\mathrm{Im}\left(\frac{1}{x+i\epsilon}\right) = -\pi\delta(x)$$

$$\Rightarrow \mathrm{Im}\Pi \sim \int dP\, \delta(P^2 - m^2)\delta((Q-P)^2 - m^2)$$

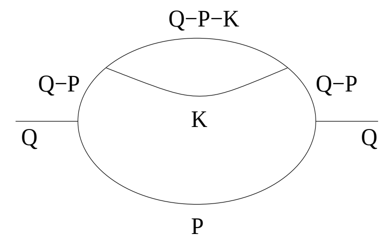
ON SHELL LINES
→ PHYSICAL EMITTED PARTICLES

Notation:



ZERO T - MORE THAN ONE LOOP

Example



$$\Pi(Q) \sim \int dP \int dK$$
$$[D(Q-P))^2 D(P) D(K) D(Q-P-K)]$$

$$\mathrm{Im}\Pi = \frac{1}{2i}(\Pi - \Pi^*)$$

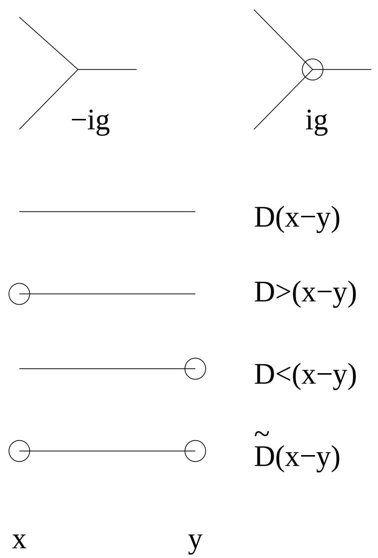
$$\sim \underbrace{(\mathrm{Im}D)^5}_{1\ \mathrm{term}} + \underbrace{(\mathrm{Im}D)^3(\mathrm{Re}D)^2}_{5!/(2!3!)=10\ \mathrm{terms}} + \underbrace{(\mathrm{Im}D)(\mathrm{Re}D)^4}_{5!/(4!1!)=5\ \mathrm{terms}}$$

Result: 3 terms are non-zero

Cutkosky Rules

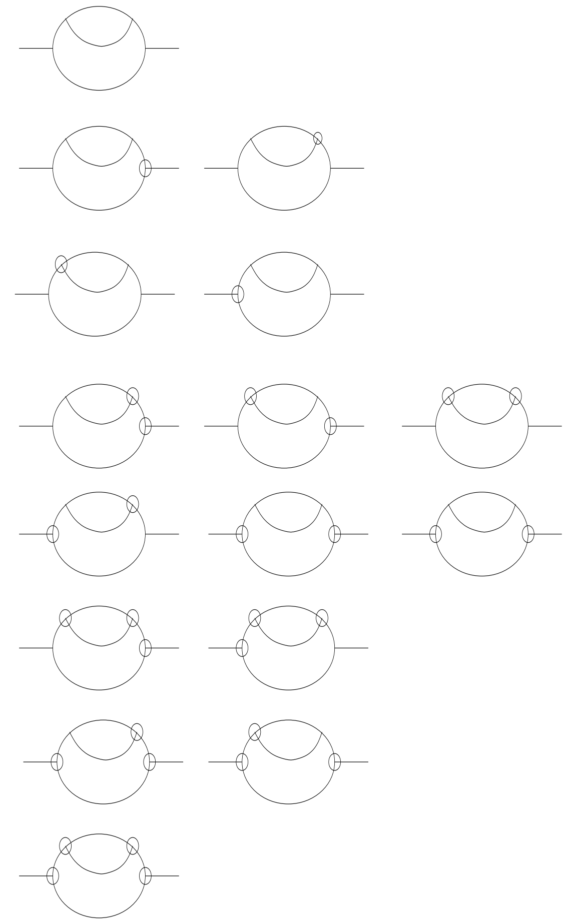
STEP 1: write in co-ordinate space

Θ fcns $\rightarrow$ "Largest Time Equation"
Define circled vertices:



$$\tilde{D}(x-y) = \Theta(y_0 - x_0)\langle\phi(x)\phi(y)\rangle + \Theta(x_0 - y_0)\langle\phi(y)\phi(x)\rangle$$

LTE $\rightarrow \Sigma_{all\ circlings} = 0$
For our example:



STEP 2: go to momentum space

$$D^>(P) = \Theta(p_0)\delta(P^2 - m^2)$$

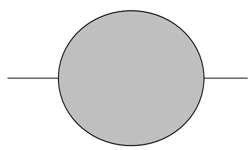
$$D^<(P) = \Theta(-p_0)\delta(P^2 - m^2)$$

$$D(P) = 1/(P^2 - m^2 + i\epsilon)$$

$$\tilde{D}(P) = 1/(P^2 - m^2 - i\epsilon)$$

RESULT:
(all circled) = (zero circled)*

−i Π =



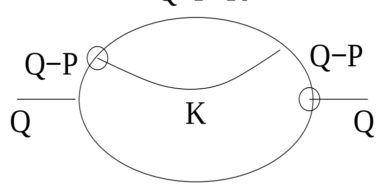
$$\Rightarrow \text{Im}\Pi = -\frac{1}{2}\sum_{circlings}{}' [\text{Diagram}]$$

STEP 3: use momentum space Θ functions:

$\Rightarrow$ CIRCLED AND UNCIRCLED VERTICES FORM CONNECTED SETS

- reduces number of graphs
- gives product of two amplitudes

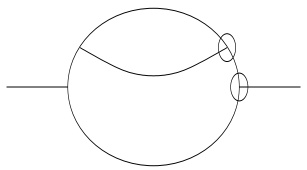
Example:



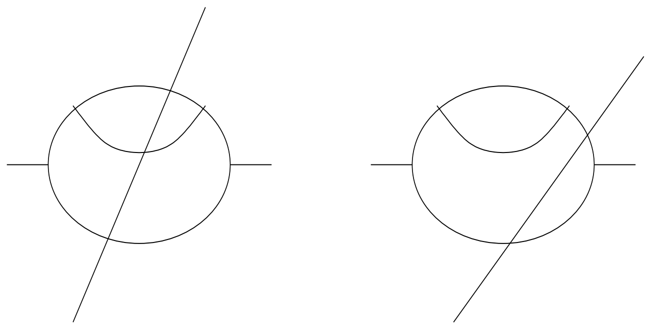
$$D^<(P)[D^<(Q-P)]^2 D^>(K)D^>(Q-P-K)$$

$$\Rightarrow \quad \Theta(p_0 - q_0)\Theta(k_0)\Theta(q_0 - p_0 - k_0) = 0$$

Allowed Graphs:



Notation:



1) uncircled vertices on left side; circled vertices on right side

2) propagators crossed by cut line are on shell

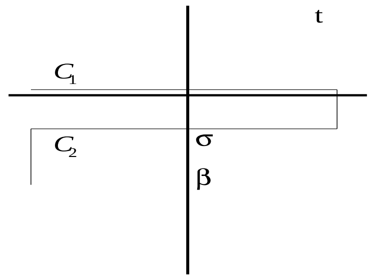
3) diagram is divided into two amplitudes

Finite Temperature

Expectation Value

$$\langle 0|\mathcal{O}|0\rangle \to \sum_n \langle t_n|e^{-\beta H}\mathcal{O}|t_n\rangle$$
$$\sum_n \langle t_n - i\beta|\mathcal{O}|t_n\rangle$$

Real Time Contour



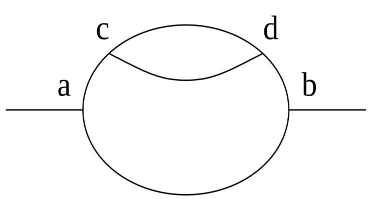
$\Rightarrow$ Propagator Matrix:

$$D(x-y) = \begin{pmatrix} D_{11}(x-y)\ , & D_{12}(x-y) \\ D_{21}(x-y)\ , & D_{22}(x-y) \end{pmatrix}$$

Choose $\sigma = 0 \rightarrow$

$$\begin{aligned} D_{11}(x-y) &= D(x-y) \\ D_{12}(x-y) &= D^{<}(x-y) \\ D_{21}(x-y) &= D^{>}(x-y) \\ D_{22}(x-y) &= \tilde{D}(x-y) \end{aligned}$$

Consider:



$a,\ b$ external indices (Π is a 4×4 matrix)
$\rightarrow$ choose $a = 1\,,\ b = \{1, 2\}$ for Π_R

$c,\ d$ internal indices $\rightarrow$ sum over $\{c, d\} = \{1, 2\}$

$\Rightarrow$ Looks like Zero T Eqn:

$$\mathrm{Im}\Pi = -\frac{1}{2} \sum_{circlings}{}' [Diagram]$$

BUT:

$$D^{>}(P) = \underbrace{\mathrm{sgn}(p_0)(1 + n(p_0))}_{\lim_{T\to 0}\ \Rightarrow\ \theta(p_0)} 2\pi\delta(P^2 - m^2)$$

$$D^{<}(P) = \underbrace{\mathrm{sgn}(p_0)n(p_0)}_{\lim_{T\to 0}\ \Rightarrow\ \theta(-p_0)} 2\pi\delta(P^2 - m^2)$$

$\rightarrow$ E can flow in both drns

$\rightarrow$ disconnected circlings are allowed

$\rightarrow$ some terms don't look like prods of amps

R.L. Kobes and G.W. Semenoff, Nucl. Phys. B **260** *(1985) 714; Nucl. Phys. B.* **272** *(1986) 329.*

First Part of Soln:

R/A or Keldysh representation
$\rightarrow$ retarded and advanced propagators

WHY?

1) Linear response $\rightarrow$ retarded correlators

2) Retarded correlators are in some sense 'more natural' at finite temperature

Zero T:

$\phi \rightarrow$ creation/annihilation ops
$a^\dagger(k)$, $a(k)$ create/destroy particles in the vacuum (the ground state)

$\Rightarrow$

+E states are particles trvlng fwds in t
-E states are antiparticles trvlng bwds in t

TO PROP
= amp for [(+ve E) fwd + (-ve E) bwd]

Finite T:

ground state of the system is the heat bath

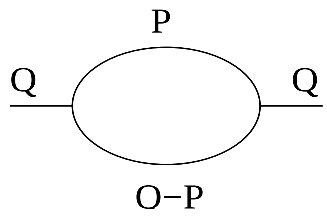
$\Rightarrow$

+E sts are parts. emitted by the medium
-E sts are parts. absorbed by the medium

RET PROP = amp for [(+ve E) - (-ve E)] fwd

Example: Finite T - One Loop

H.A. Weldon, Phys. Rev. **D28** *(1983) 2007.*



$$N_{p_0} := 1 + 2n(p_0); \quad N_{-p_0} = -N_{p_0}$$
$$r_p := D_R(P)$$
$$\rho_p := i(r_p - a_p) = 2\pi \mathrm{sgn}(p_0)\delta(P^2 - m^2)$$

$$\omega_p = \sqrt{p^2 + m^2}$$

$$\Pi_R(Q) \sim i \int dP (N_p + N_{q-p}) r_p r_{q-p}$$

$$\mathrm{Im}\Pi_R(Q) \sim \int dP (N_p + N_{q-p}) \rho_p \rho_{q-p}$$

$$\rho_p = \frac{\pi}{\omega_p}[\delta(p_0 - \omega_p) - \delta(p_0 + \omega_p)]$$

Expand → 4 terms

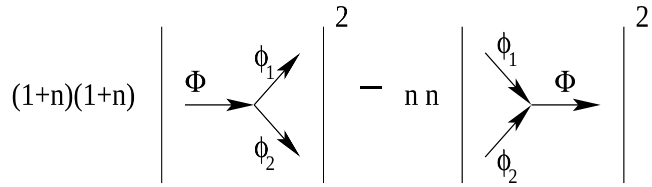
$$(N_{\omega_p} + N_{\omega_{q-p}})[\underbrace{\delta(q_0 - \omega_p - \omega_{q-p})}_{(1)} - \underbrace{\delta(q_0 + \omega_p + \omega_{q-p})}_{=\,0}]$$
$$(N_{\omega_p} - N_{\omega_{q-p}})[\underbrace{\delta(q_0 + \omega_p - \omega_{q-p})}_{(2)} - \underbrace{\delta(q_0 - \omega_p + \omega_{q-p})}_{(3)}]$$

Term (1):

$q_0 = \omega_p + \omega_{q-p} \to$ two emitted particles

$$N_{\omega_p} + N_{\omega_{q-p}}$$
$$\sim (1 + n_{\omega_p})(1 + n_{\omega_{q-p}}) - n_{\omega_p} n_{\omega_{q-p}}$$

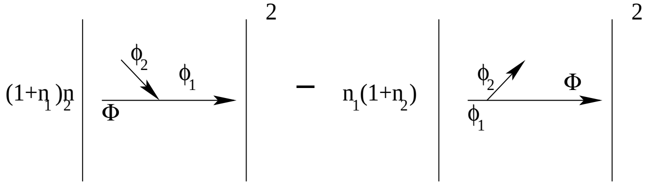
= statistical weight for emission of a pair of particles from the medium minus weight for inverse process → probability for emission



Term (2):
$q_0 + \omega_p = \omega_{q-p}$
$\rightarrow$ one absorbed and one emitted

$$N_{\omega_p} - N_{\omega_{q-p}}$$
$$\sim (1 + n_{\omega_{q-p}}) n_{\omega_p} - (1 + n_{\omega_p}) n_{\omega_{q-p}}$$

$=$ statistical weight for emission of 1st and absorption 2nd minus weight for inverse process



NOTE:

- Zero T limit:
  $N_\omega \rightarrow 1 \Rightarrow$ only first process survives (no medium to absorb from)
- One loop example is 'trivial'
  2 vertices $\rightarrow$ no disconnected sets

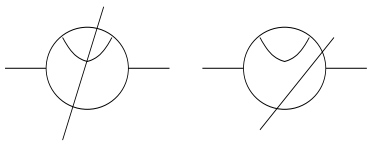
WILL SHOW:

AT >1 LOOP NEED RET/ADV PROPS TO GET CUTTING RULES WITH ZERO T FORM

F. Gelis, Nuc. Phys. B **508** *(1997) 483.*
*S.M.H. Wong, Phys. Rev.***D64** *(2001) 025007.*

Generalized Rules

1) Take all 'zero T' cuts
→ product of amplitudes



Cut propagators:
momentum approaches cut from left/right:

$$2\pi\delta(P^2 - m^2)\,[(1 + n(p_0)) \;\text{ or }\; n(p_0)]$$

Interpretation: cut props are real particles
→ statistical weights for emission/absorption

Note: $n(-p_0) = -(1 + n(p_0))$
→ emission of a positive energy particle
= absorption of a negative energy particle

Construct thermal factor:
(generalization of one loop result)

$$\Pi_l(1 + n_i)\;\Pi_r n_j - \Pi_l n_i\;\Pi_r(1 + n_j)$$

For our example:

Cut 1 :
$[1 + n(l_0)][1 + n(p_0)][1 + n(k_0)] - n(l_0)n(p_0)n(k_0)$

Cut 2 :
$[1 + n(q_0 - p_0)][1 + n(p_0)] - n(q_0 - p_0)n(p_0)$

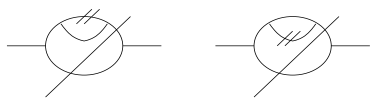
$l_0 = q_0 - p_0 - k_0$

2) Open all loops with 'tic's' $\rightarrow$ tree amplitudes

Single tic-ed propagator:

$$2\pi\delta(P^2 - m^2)\,\frac{1}{2}[1 + 2n(p_0)]$$

Our example:



$$\text{Cut 2a}: \frac{1}{2}(1 + 2n(q_0 - p_0 - k_0))$$
$$\text{Cut 2b}: \frac{1}{2}(1 + 2n(k_0))$$

Conditions:
1) [#cut] + [#tic-ed] = [# δ's] = $L + 1$
2) It must be possible to reach cut line from either drn along off shell props

INTERPRETATION: tic-ed lines represent particles emitted/absorbed (or vica-versa) on the same side of the cut
$\Rightarrow$ Factor $(1 + n(p_0)) + n(p_0) := e_p + a_p$

QUESTION:
What if there is > 1 tic-ed line?

ANSWER:
S/t interact with the medium independently

$$[1 + 2n(p_0^1)]\,[1 + 2n(p_0^2)]\cdots$$
$$= (e_1 + a_1)(e_2 + a_2)\cdots$$

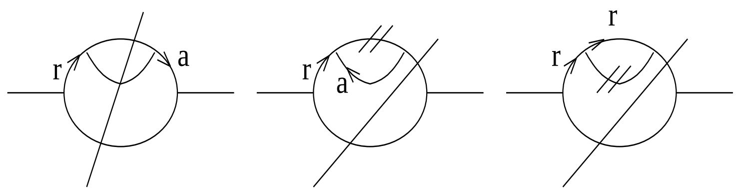
S/t interactions are correlated

$$(1 + n(p_0^1))(1 + n(p_0^2))\cdots + n(p_0^1)n(p_0^2)\cdots$$
$$= e_1 e_2 \cdots + a_1 a_2 \cdots$$

Correlation only at $l \geq 3$ loops

To understand when correlations occur must look at scattering amplitudes
- *more later*

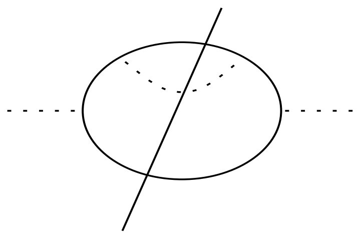
3) All uncut and untic-ed propagators must be either retarded or advanced
The 'epsilon flow' is retarded from either external leg



Note: $r_{-p} = a_p$

Scattering Amplitudes in QED

Example:



4 possibilities are kinematically allowed:

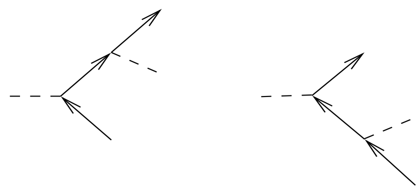
1) γ, l and $\bar{l}$ emtd $\rightarrow$ photon decay

2-3) γ, l emtd and $\bar{l}$ absd $\rightarrow$ compton scattering

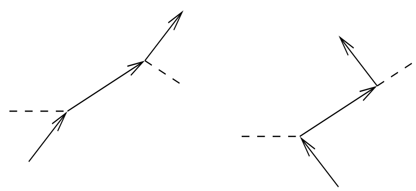
4) γ absd and l, $\bar{l}$ emtd $\rightarrow$ pair production

Notation:
emitted particle $\rightarrow$ line slanting fwds
absorbed particle $\rightarrow$ line slanting bkwds

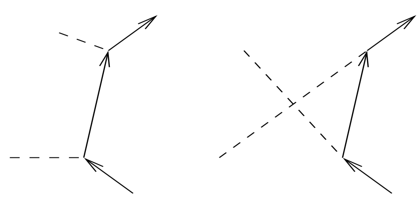
Photon Decay



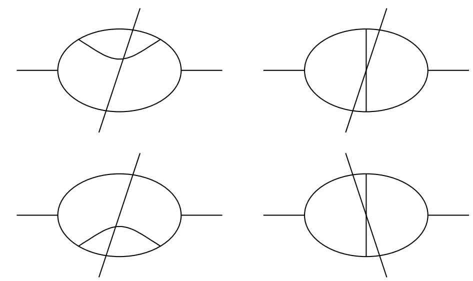
Compton Scattering



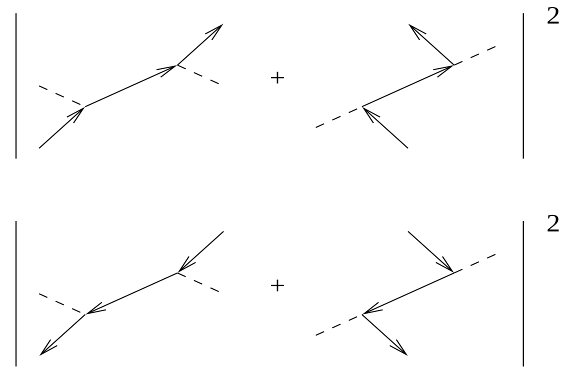
Pair Production



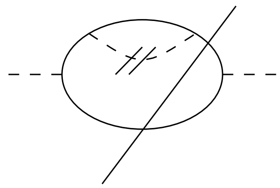
Include both two loop graphs
Central cuts:



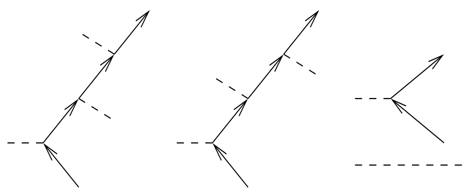
For Compton Scattering:



Example of a non-central cut:



Forward scattering only:
(both cut propagators $\rightarrow$ emitted particles)



Tic-ed γ is absd/emtd without interacting w/ rest of participants
$\rightarrow$ A SPECTATOR PARTICLE

Spectator in 3rd diagram normally not drawn (needed to calc convolution of 2 amps)

Schematic equation:

$$(\gamma\tilde{\gamma} \rightarrow e^+e^-\tilde{\gamma}) \otimes (\gamma e^+e^-; \;\; \tilde{\gamma} \rightarrow \tilde{\gamma})$$

Correlated Spectator Particles

Notation:

a) ϕ^3 theory

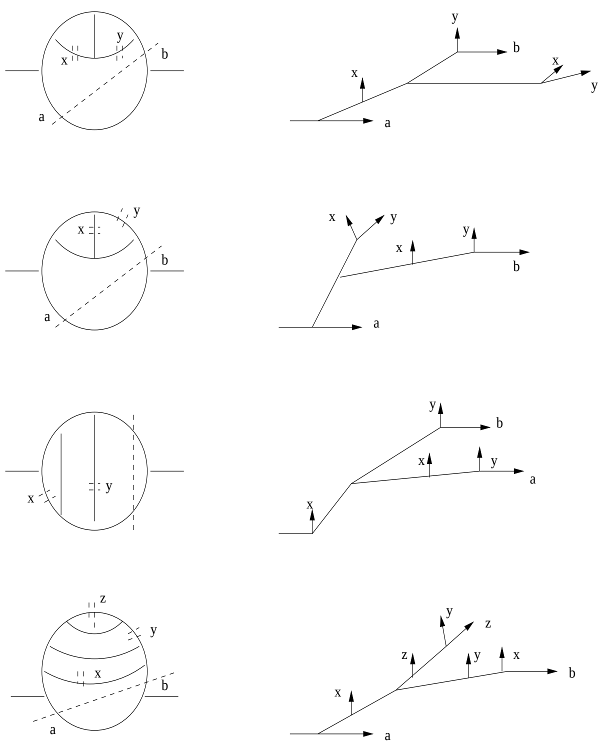
b) Consider only forward scattering for l.h.s. Cut propagators drawn as horizontal arrows

c) Tic-ed propagators drawn as arrows straight up (except if have 2 from same vertex)
$\Rightarrow$ # of diagrams: $2^{\#tic-ed} \rightarrow 1$

d) Uncut and untic-ed propagators slant up to the right (if we define P flow to the right all props are retarded)

I. First type of uncorrelated spectators

$\Rightarrow$ All emissions/absorptions occur before (or after) final state emissions

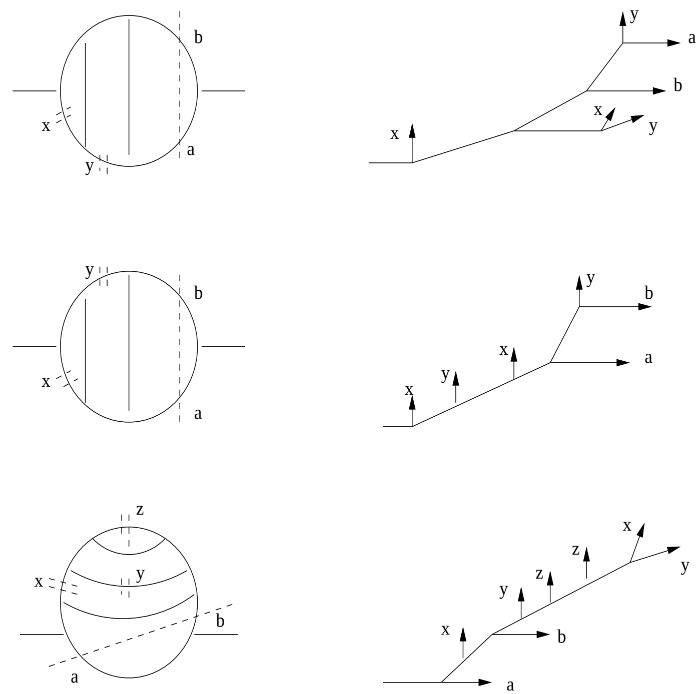


Thermal Factors:

$$\Pi_i(e_i + a_i) = N(x)N(y)[N(z)]$$

II. Second type of uncorrelated spectators

$\Rightarrow$ All but one emission/absorption occurs before (or after) final state emissions

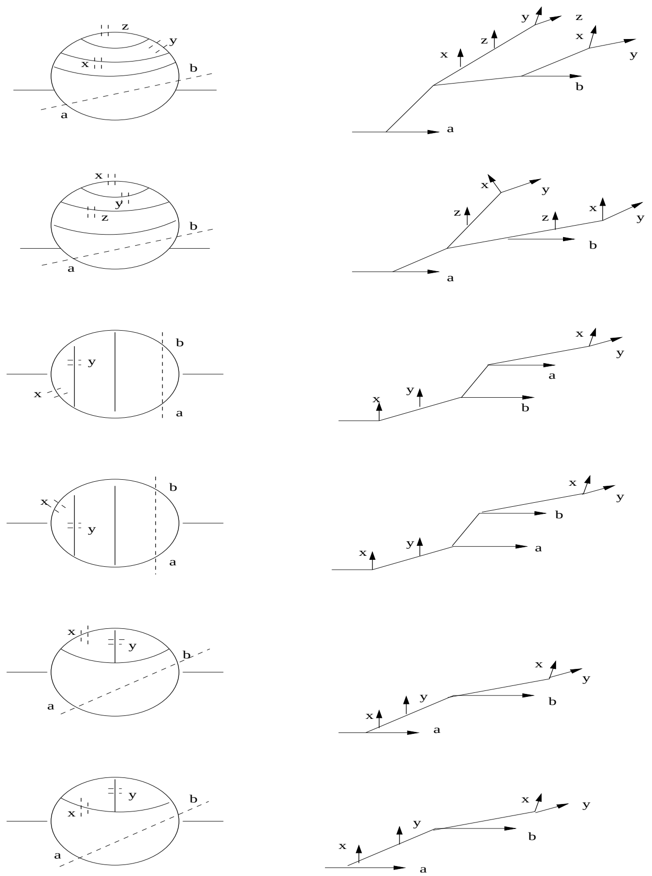


Thermal Factors:

$$\Pi_i(e_i + a_i) = N(x)N(y)[N(z)]$$

III. Correlated spectators

$\Rightarrow$ Emissions/absorptions of spectators occur on both sides of final state emissions



Thermal Factors:

$$\Pi_{us}(e_i + a_i)[\Pi_s e_j + \Pi_s a_j]$$

$$N(z)[(1+n(x))(1+n(y)) + n(x)n(y)]$$

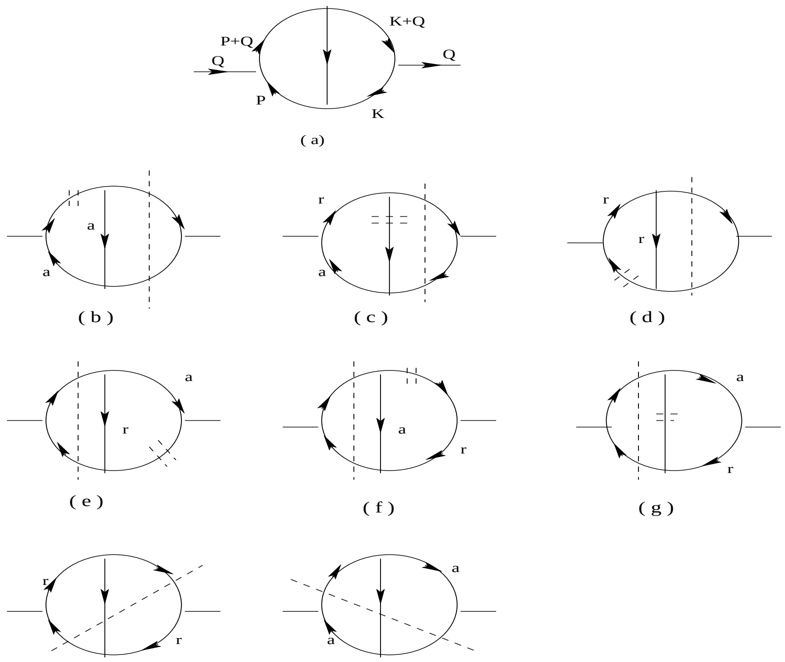
$$(1+n(z))(1+n(x))(1+n(y)) + n(z)n(x)n(y)$$

$$(1+n(x))(1+n(y)) + n(x)n(y)$$

Interpretation

If the e/a pairs from tic-ed lines appear on same side of the e/a lines produced by the cut propagator, they are e/a without under going non-trivial interactions with the medium, and can be treated as independent spectators.

Example 1:



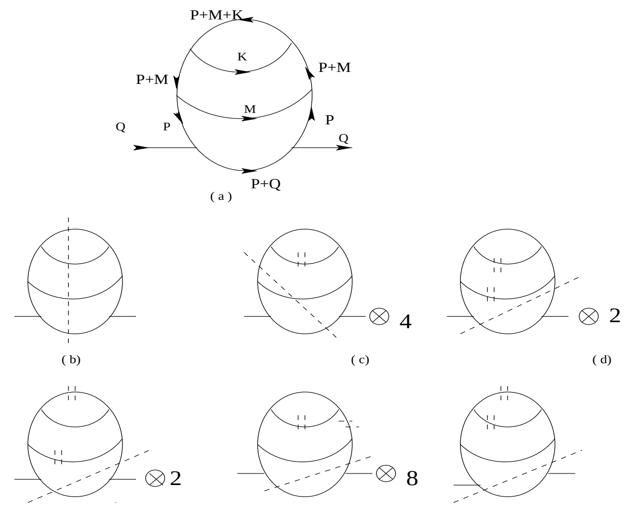
(b) $[(1+n_{k+q})n_k - n_{k+q}(1+n_k)]N_{p+q}$

(e) $[(1+n_{p+q})n_p - n_{p+q}(1+n_p)]N_k$

(h) $(1+n_{q+k})(1+n_{p-k})n_p - n_{k+q}n_{p-k}(1+n_p)$

(i) $(1+n_{q+p})n_{p-k}n_k - n_{p+q}(1+n_{p-k})(1+n_k)$

Example 2:



$$\mathcal{N}(x_1,\cdots x_n; y_1,\cdots y_m)$$
$$= [(1+n(x_1)\cdots(1+x(x_n)n(y_1)\cdots y(x_n) - \text{reverse}]$$

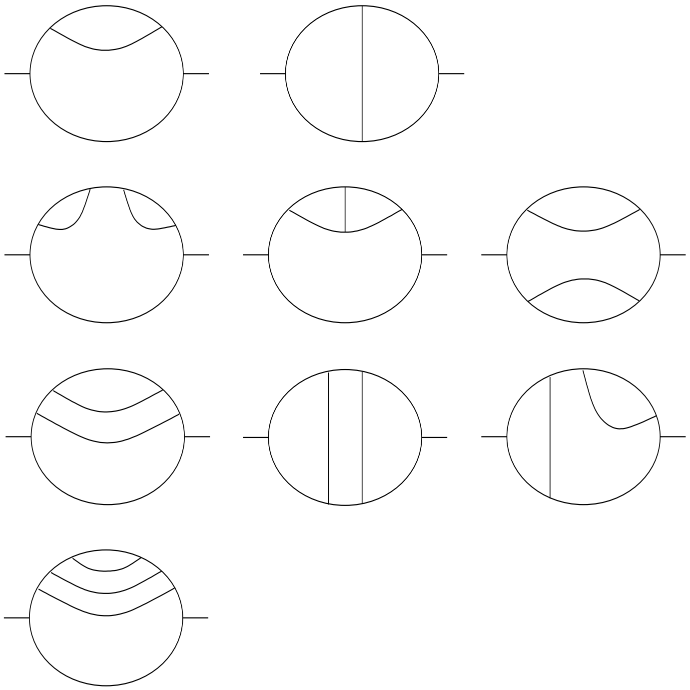
(b) $\mathcal{N}(p+q,m,k;p+m+k)$

(c) $\mathcal{N}(p+q,m;p+m)N(k)$

(d) $\mathcal{N}(p+q;p)N(k)N(m)$

(g) $[(1+n_k)n_{p+m+k} + n_k(1+n_{p+m+k})]$
$\mathcal{N}(p+q;p)$

Other Examples:



Conclusions

Have developed a set of cutting rules for $\mathrm{Im}\Pi$:

- Zero T cuts
- Product of tree amplitudes
  $\rightarrow$ a production rate
- Thermal factors have straightforward physical interpretation