

Linear resurgence and topology of Lefschetz thimbles

Maxim Kontsevich

I.H.E.S.

21 July 2020

1. Borel summation for integrals depending on a small parameter

Given:

- ▶ X - complex manifold, $\dim_{\mathbb{C}} X = N$
- ▶ f - holomorphic function on X (a map $f : X \rightarrow \mathbb{C}$ to the **Borel plane**)
- ▶ μ - holomorphic N -form on X , in local coordinates $\mu = \mu(x) d^N x$
- ▶ γ - oriented noncompact N -cycle such that $\operatorname{Re} f|_{\gamma} \rightarrow +\infty$ at “infinity” of γ

Generalization: $D \subset X$ - divisor, γ - relative cycle: $\partial\gamma \subset D$.

↪ integral depending on small parameter $0 < \hbar \ll 1$ (assume absolute convergence):

$$I(\hbar) := \int_{\gamma} e^{-f/\hbar} \mu = \int_{\gamma} e^{-f(x)/\hbar} \mu(x) d^N x$$

Goal: asymptotic expansion at $\hbar \rightarrow +0$. **Example:** $\underbrace{\int_1^{+\infty} e^{-x/\hbar} \frac{dx}{x}}_{\substack{\text{Exponential integral} \\ E_1(1/\hbar)}} \sim e^{-1/\hbar} \sum_{n \geq 0} (-1)^n n! \hbar^{n+1}$

1. Borel summation for integrals depending on a small parameter

Choose the cycle of integration γ such that its projection by map $f : X \rightarrow \mathbb{C}$ is a straight ray $\mathbf{R} := z_{crit} + e^{i\theta}\mathbb{R}_{\geq 0}$ where $z_{crit} \in \mathbb{C}$ is a *critical value* of f , and direction $\theta \in (-\pi/2, +\pi/2)$ is generic, so there is no other critical values on \mathbf{R} .

Noncompact cycle γ can be thought of as a 1-parameter family of *compact* cycles $\gamma_z \in H_{N-1}(f^{-1}(z), \mathbb{Z})$, $z \in \mathbf{R}$ such that $\lim_{z \rightarrow z_{crit}} \gamma_z = 0 \in H_{N-1}(f^{-1}(z_{crit}), \mathbb{Z})$.

$$I(\hbar) = \int_{\gamma} e^{-f/\hbar} \mu = e^{-z_{crit}/\hbar} \int_{\mathbf{R}} e^{-(f-z_{crit})/\hbar} \text{vol}(z) dz, \quad \text{vol}(z) := \int_{\gamma_z} \frac{\mu}{df}$$

Convergent expansion near z_{crit} (the case of semisimple monodromy, no logarithms):

$$\underbrace{\text{vol}(z)}_{\substack{\text{admits endless} \\ \text{analytic continuation}}} = \sum_{\lambda \in \mathbb{Q}_{>0}} c_{\lambda} (z - z_{crit})^{\lambda-1} \rightsquigarrow e^{z_{crit}/\hbar} I(\hbar) \sim \underbrace{\sum_{\lambda} c_{\lambda} \Gamma(\lambda) \hbar^{\lambda}}_{\text{inverse Borel transform}}$$

Proposal (below), - an *alternative* to Borel summation: use asymptotic expansions of integrals near *all* critical points, together with certain Stokes indices $\in \mathbb{Z}$ (equivalent to the monodromy data of the local system $H_{N_1}(f^{-1}(z), \mathbb{Z})$ on $\mathbb{C} - \{\text{critical values}\}$).

2. Lefschetz thimbles, Stokes rays and Riemann-Hilbert problem

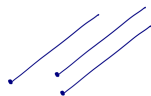
Simplifying assumption: all critical points $x_\alpha \in X$ are *isolated* and *non-degenerate* (i.e. Morse in the holomorphic sense: $f''|_{x_\alpha}$ is a non-degenerate \mathbb{C} -quadratic form on $T_{x_\alpha}X$), and all critical values $z_\alpha = f(x_\alpha)$ are *pairwise distinct*.

There is a unique (up to \pm) vanishing cycle near each x_α , diffeomorphic to S^{N-1} .

Definition: for each Morse critical point x_α and a *generic* direction $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, the **Lefschetz thimble** $th_{\alpha,\theta} \simeq \mathbb{R}^N \subset X$ is the continuation of the vanishing cycle near x_α along the path $z_\alpha + e^{i\theta}\mathbb{R}_{\geq 0} \subset \mathbb{C}$. Ill-defined only if $\theta = \arg(z_\beta - z_\alpha)$ for some $z_\beta \neq z_\alpha$.

For a *complex* value $\hbar \in \mathbb{C} - 0$ and a critical point x_α , the **normalized integral** is

$$I_\alpha^{norm}(\hbar) := \frac{1}{(2\pi\hbar)^{N/2}} e^{z_\alpha/\hbar} \int_{th_{\alpha,\arg \hbar}} e^{-f/\hbar} \mu$$



By general theory, it has a divergent asymptotic expansion $\sim \sum_{n \geq 0} c_n^{(\alpha)} \hbar^n$.
Coefficients $c_0^{(\alpha)}, c_1^{(\alpha)}, \dots$ *do not depend* on the direction $\theta = \arg(\hbar)$.

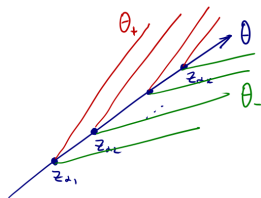
2. Lefschetz thimbles, Stokes rays and Riemann-Hilbert problem

Function I_α^{norm} is defined and *analytic* outside Stokes rays $\{\hbar \mid \arg \hbar = \arg(z_\beta - z_\alpha)\}$ in the complex plane \mathbb{C}_\hbar , extends analytically a little bit across the cuts.

Let $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ be such that on an oriented line in \mathbb{C} with slope θ lie $k \geq 2$ distinct critical values $z_{\alpha_1}, \dots, z_{\alpha_k}$ in the increasing order. For values $\theta_- < \theta < \theta_+$ close to θ the normalized integrals are well-defined.

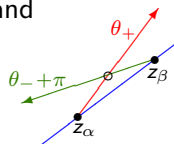
Jump formula :

$$\begin{pmatrix} I_{\alpha_1,+}^{norm}(\hbar) \\ \vdots \\ I_{\alpha_k,+}^{norm}(\hbar) \end{pmatrix} = S_\theta \cdot \begin{pmatrix} I_{\alpha_1,-}^{norm}(\hbar) \\ \vdots \\ I_{\alpha_k,-}^{norm}(\hbar) \end{pmatrix}$$



where $S_\theta = (S_{\alpha,\beta})_{\alpha,\beta \in \{\alpha_1, \dots, \alpha_k\}}$ is an upper-triangular matrix with $S_{\alpha,\alpha} = 1$ and exponentially small off-diagonal terms $S_{\alpha,\beta} = S_{\alpha,\beta}(\hbar) := n_{\alpha,\beta} e^{-(z_\beta - z_\alpha)/\hbar}$ where $n_{\alpha,\beta}$ are *integers* (**Stokes indices**) given by intersection numbers

$$n_{\alpha,\beta} := th_{\alpha,\theta_+} \cdot th_{\beta,\theta_- + \pi} \in \mathbb{Z}$$



2. Lefschetz thimbles, Stokes rays and Riemann-Hilbert problem

Data $(z_\alpha \in \mathbb{C}), (n_{\alpha,\beta} \in \mathbb{Z})$ defines a modification of the trivial vector bundle with fiber $\mathbb{C}^{\{z_\alpha\}}$ over complex plane \mathbb{C}_{\hbar} , by gluing via **linear**¹ automorphisms $S_\theta(\hbar)$ along Stokes rays $e^{i\theta}\mathbb{R}_{\geq 0}$. Let solve somehow (e.g. Cecotti-Vafa method) *Riemann-Hilbert problem*, i.e. find a holomorphic trivialization of the glued bundle. Equivalently, find matrix-valued function G on $\mathbb{C}_{\hbar} - \{\text{Stokes rays}\}$ satisfying jump equations on cuts

$$G_{e^{i\theta},+}(\hbar) = S_\theta(\hbar) \cdot G_{e^{i\theta},-}(\hbar), \quad \text{with boundary condition: } \lim_{\hbar \rightarrow 0} G(\hbar) = Id_{\mathbb{C}^{\{z_\alpha\}}}$$

Then vector-valued function $\vec{J}(\hbar) := G(\hbar)^{-1} \cdot \vec{I}^{norm}(\hbar)$ has *trivial* jumps, hence is *holomorphic* in \hbar . The **convergent** Taylor expansion of \vec{J} is the product of the **divergent** asymptotic expansion of G at $\hbar = 0$ (does not depend on the choice of sector), and of the **divergent** asymptotic expansion of $\vec{I}^{norm}(\hbar) = (I_\alpha^{norm}(\hbar))_{all\{z_\alpha\}}$.

$\vec{I}^{norm} = G \cdot J$ - an alternative to Borel summation: divergent series \rightsquigarrow actual values.

¹This is the origin of “linear resurgence” in the title. WKB problems lead to *non-linear* glueings.

3. Example: classical special functions

- ▶ (Airy function): $X = \mathbb{C}$, $f = \frac{x^3}{3} - x$, $\mu = dx$. Critical points: $x = \pm 1$,
critical values: $z_1 = -2/3, z_2 = +2/3$. Stokes indices: $n_{12} = +1, n_{21} = -1$.

z_1

z_2

$$\text{For } \hbar \in \mathbb{C} - \mathbb{R}_{\leq 0} : I_2^{norm} = \sqrt{2\pi} \frac{e^{2/(3\hbar)}}{\hbar^{1/6}} \text{Ai}(\hbar^{-2/3}) \sim \frac{1}{\sqrt{2}} \sum_{n \geq 0} \frac{(6n-1)!!}{(2n)! 72^n} \hbar^n$$

$$\text{For } \hbar \in \mathbb{C} - \mathbb{R}_{\geq 0} : I_1^{norm}(\hbar) = -i I_2^{norm}(-\hbar)$$

where $\text{Ai}(y) := 1/(2\pi) \int_{\mathbb{R}} \cos(t^3/3 + yt) dt$ is the classical Airy function.

Jumps:

$$I_1^{norm}(\hbar + i0) - I_1^{norm}(\hbar - i0) = +1 \cdot e^{-\frac{4}{3\hbar}} I_2^{norm}(\hbar), \quad \hbar \in \mathbb{R}_{\geq 0}$$

$$I_2^{norm}(\hbar - i0) - I_2^{norm}(\hbar + i0) = -1 \cdot e^{+\frac{4}{3\hbar}} I_1^{norm}(\hbar), \quad \hbar \in \mathbb{R}_{\leq 0}$$

modified, of second kind

- ▶ (Bessel function): $X = \mathbb{C}^\times$, $f = x + \frac{1}{x}$, $\mu = \frac{dx}{x}$. Critical points: $x = \pm 1$,
critical values: $z_1 = -2, z_2 = +2$. Stokes indices: $n_{12} = +2, n_{21} = -2$.

$$\text{Bessel integral: } \int_0^\infty e^{-\frac{1}{\hbar}(x+\frac{1}{x})} \frac{dx}{x} = 2K_0\left(\frac{2}{\hbar}\right) \sim \sqrt{\pi\hbar} e^{-\frac{2}{\hbar}} \sum_{n \geq 0} \frac{(-1)^n ((2n-1)!!)^2}{n! 16^n} \hbar^n$$

4. Example: Gamma function, infinitely many critical values

Let $X = \mathbb{C}$, $f = e^x - x$ (not algebraic), $\mu = dx$.

$$f(x + 2\pi in) = f(x) - 2\pi in$$

Function f has *infinitely many* critical points $x_k = 2\pi ik$, $k \in \mathbb{Z}$.

\rightsquigarrow **critical values**: $z_k = 1 - 2\pi ik$. The normalized integral for $\operatorname{Re} \hbar > 0$, any $k \in \mathbb{Z}$ is

$$I_k^{norm}(\hbar) \stackrel{k=0}{:=} \frac{e^{1/\hbar}}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{(-e^x+x)/\hbar} dx \stackrel{t:=e^x}{=} \frac{e^{1/\hbar}}{\sqrt{2\pi\hbar}} \int_0^{\infty} e^{-t/\hbar} t^{1/\hbar} \frac{dt}{t} = \frac{e^{1/\hbar} \hbar^{1/\hbar} \Gamma(1/\hbar)}{\sqrt{2\pi\hbar}}$$

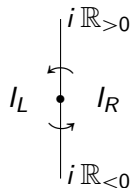
For $\operatorname{Re} \hbar < 0$ one has $I_{\bullet}^{norm}(\hbar) = 1/I_{\bullet}^{norm}(-\hbar)$ (does not depend on $\bullet = k \in \mathbb{Z}$)

Asymptotic expansion: $I_{\bullet}^{norm}(\hbar) \stackrel{\hbar \rightarrow +0}{\sim} 1 + \frac{1}{12}\hbar + \frac{1}{288}\hbar^2 - \frac{139}{51840}\hbar^3 + \dots$ (Stirling formula).

4. Example: Gamma function, infinitely many critical values

We get basically two functions:

$$\begin{cases} I_R(\hbar) & := \frac{e^{1/\hbar} \hbar^{1/\hbar} \Gamma(1/\hbar)}{\sqrt{2\pi\hbar}} & \text{for } \operatorname{Re} \hbar > 0 \\ I_L(\hbar) & := 1/I_R(-\hbar) & \text{for } \operatorname{Re} \hbar < 0 \end{cases}$$



$$\begin{aligned} I_L(\hbar) &= I_R(\hbar) \cdot (1 - \exp(-\frac{2\pi i}{\hbar})) = \\ &= I_R(\hbar) + (-\mathbf{1}) \exp(-\frac{2\pi i}{\hbar}) I_R(\hbar) & \text{for } \hbar \in i\mathbb{R}_{>0} \end{aligned}$$

Jump formulas:

$$\begin{aligned} I_R(\hbar) &= I_L(\hbar) \cdot (1 - \exp(+\frac{2\pi i}{\hbar}))^{-1} \\ &= I_L(\hbar) + \sum_{n \geq 1} (+\mathbf{1}) \exp(+\frac{2\pi i n}{\hbar}) I_L(\hbar) & \text{for } \hbar \in i\mathbb{R}_{<0} \end{aligned}$$

$$\rightsquigarrow \text{Stokes indices } n_{kk'} = \begin{cases} -\mathbf{1} & k' = k - 1 \\ +\mathbf{1} & k' > k \\ 0 & \text{otherwise} \end{cases} \quad \forall k, k' \in \mathbb{Z}$$

5. Example: Infinite-dimensional integral (heat kernel)

Let (M, g) be a real-analytic Riemannian manifold which admits a “reasonable” complexification $(M_{\mathbb{C}}, g_{\mathbb{C}})$. Fix two points $p_0, p_1 \in M$, and consider the *path integral*

$$I(\hbar) := \int_{\substack{\text{Paths } \phi: [0,1] \rightarrow M \\ \phi(0)=p_0, \phi(1)=p_1}} e^{-\frac{S(\phi)}{\hbar}} \mathcal{D}\phi, \quad \text{where } S(\phi) = \frac{1}{2} \int_0^1 \left| \frac{d\phi(t)}{dt} \right|_g^2 dt$$

The integration domain is as a totally real infinite-dimensional contour γ in $X :=$ infinite-dimensional *complex* manifold of paths connecting p_0 and p_1 in $M_{\mathbb{C}}$. Dirichlet functional $\phi \in \gamma \mapsto S(\phi) \in \mathbb{R}$ extends to a *holomorphic* function $f : X \rightarrow \mathbb{C}$, the fictitious “Lebesgue measure” $\mathcal{D}\phi$ “extends” to a holomorphic volume form μ on X .

Mathematically rigorous interpretation (Feynman-Kac formula):

$$I(\hbar) = \text{heat kernel } K(p_0, p_1; t) = \langle p_0 | e^{-\frac{1}{2}\Delta t} | p_1 \rangle$$

where *time* $t > 0$ (for the Brownian motion) is identified with the *Planck constant* \hbar .

5. Example: Infinite-dimensional integral (heat kernel)

Short-time asymptotic expansion of the heat kernel:

$$K(p_0, p_1; t) \stackrel{t \rightarrow +0}{\sim} (2\pi t)^{-\dim M/2} e^{-\frac{\text{dist}(p_0, p_1)^2}{2t}} \sum_{n \geq 0} c_n t^n$$

Conjecture: sequence c_n has factorial growth, is resurgent, and the Borel transform has singularities at numbers $\ell_\alpha^2/2 \in \mathbb{C}$ where ℓ_α are lengths of *complex* geodesics connecting p_0, p_1 (formal solutions of Euler-Lagrange equation for $\phi : [0, 1] \rightarrow (M_{\mathbb{C}}, g_{\mathbb{C}})$).

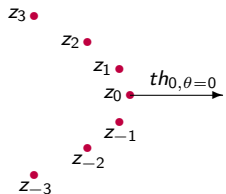
McKean, 1970

$X = \mathbb{H}^2$:

$$K(p_0, p_1; t) = \frac{\sqrt{2} e^{-t/8}}{(2\pi t)^{3/2}} \int_{\ell^2/2}^{\infty} \frac{e^{-z/t}}{(\cosh \sqrt{2z} - \cosh \ell)^{1/2}} dz \quad \ell := \text{dist}(p_0, p_1)$$

hyperelliptic curve of infinite genus

Also works for spheres, flat tori, compact surfaces with hyperbolic metric, ...



Critical values

$$z_n = \frac{(\ell + 2\pi i n)^2}{2}$$

6. Example: complex Chern-Simons, paths in K_2 -geometry, sums of dilogs

3 different series of examples, hypothetically give the *same* class of data $(z_\alpha), (n_{\alpha,\beta})$:

- ▶ $X_3 = \mathbb{Z}$ -cover of $X'_3 := \left\{ C^\infty\text{-connections on G-bundle } \begin{array}{c} P \\ \downarrow \\ M \end{array} \right\} / \text{gauge equivalences,}$

where M is an oriented manifold of dimension 3, G is a complex semisimple group.

- ▶ $X_1 = \mathbb{Z}$ -cover of $X'_1 := \left\{ C^\infty\text{-paths } \phi : [0, 1] \rightarrow (\mathbb{C}^\times)^{2n} \mid \phi(0) \in L_0, \phi(1) \in L_1 \right\}$

where L_0, L_1 are algebraic K_2 -lagrangian submanifolds of $(\mathbb{C}^\times)^{2n}$:

$$\sum_{i=1}^n [z_i]_{|L_\epsilon} \wedge [z_{n+i}]_{|L_\epsilon} = 0 \in K_2(L_\epsilon), \quad \epsilon = 0, 1 \quad (K_2\text{-condition will explain later})$$

- ▶ $X_0 = \mathbb{Z}$ -cover of $X'_0 :=$ a \mathbb{Z}^m -cover of $(\mathbb{C}^\times)^m - \bigcup_{\vec{\nu} \in B} \{ \vec{x} \in (\mathbb{C}^\times)^m \mid \vec{x}^{\vec{\nu}} = 1 \}$

where $B \subset \mathbb{Z}^m - 0$ is a finite subset,

$$\vec{x}^{\vec{\nu}} := \prod_{i=1}^m x_i^{\nu_i} \text{ for } \vec{x} = (x_1, \dots, x_m) \in (\mathbb{C}^\times)^m \text{ and } \vec{\nu} = (\nu_1, \dots, \nu_m) \in \mathbb{Z}^m.$$

6. Example: complex Chern-Simons, paths in K_2 -geometry, sums of dilogs

Case of connections on a 3-dimensional manifold M :

On $X_3 = \{\text{Connections}\}$ the functional f is the *Chern-Simons action*:

for a $SL(N, \mathbb{C})$ -connection $\nabla = d + A$ in the trivialized bundle

$$f(\nabla) := \int_M \text{Tr} \left(\frac{AdA}{2} + \frac{A^3}{3} \right), \quad A \in \text{Mat}(N \times N, \Omega^1(M)), \text{Tr}(A) = 0$$

Ambiguity under gauge transformations: $f(g^{-1}\nabla g) - f(\nabla) \in (2\pi i)^2\mathbb{Z}$.

For special “quantized” values of \hbar :

$$\hbar = \frac{2\pi i}{k}, \quad k = \pm 1, \pm 2, \dots$$

\rightsquigarrow well-defined function $\exp(-f/\hbar)$ on \mathbb{Z} -quotient X'_3 . Canonical “compact” cycle of integration = {unitary connections} \rightsquigarrow quantum Chern-Simons theory at level k (for $k > 0$)

6. Example: complex Chern-Simons, paths in K_2 -geometry, sums of dilogs

Case of paths connecting K_2 -lagrangian subvarieties $L_0, L_1 \in (\mathbb{C}^\times)^{2n}$:

Examples of K_2 -lagrangian subvarieties in $(\mathbb{C}^\times)^{2n}$ endowed with the standard symplectic form $\omega = \sum_{i=1}^n d \log(z_i) \wedge d \log(z_{n+i})$:

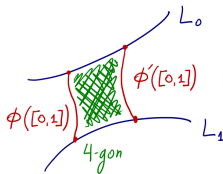
$$L = \{(z_1, \dots, z_{2n}) \in (\mathbb{C}^\times)^{2n} \mid \forall i = 1, \dots, n : z_{n+i} = 1 \text{ or } (1 - z_i)\}$$

and its images under $Sp(2n, \mathbb{Z})$ -action.

For any K_2 -lagrangian L and any $\delta \in H_2((\mathbb{C}^\times)^{2n}, L; \mathbb{Z})$ one has $\int_\delta \omega \in (2\pi i)^2 \mathbb{Z}$.

Functional f on $\{\text{paths connecting two } K_2\text{-lagrangian } L_0, L_1\}$ is defined up to $(2\pi i)^2 \mathbb{Z}$:

$$f(\phi) - f(\phi') = \int_{4\text{-gon}} \omega$$



6. Example: complex Chern-Simons, paths in K_2 -geometry, sums of dilogs

Case of sums of dilogarithms:

DATA:

- ▶ a finite subset $B \subset \mathbb{Z}^m - 0$
- ▶ a collection of integer non-zero weights $w_{\vec{\nu}} \in \mathbb{Z} - 0$ for all $\vec{\nu} \in B$
- ▶ an even quadratic form $b = (b_{ij})_{1 \leq i, j \leq m}$, $b_{ij} = b_{ji} \in \mathbb{Z}$, $b_{ii} \in 2\mathbb{Z}$

↪ multivalued function

$$f := \sum_{\vec{\nu} \in B} w_{\vec{\nu}} \operatorname{Li}_2(\vec{x}^{\vec{\nu}}) + \frac{1}{2} \sum_{i, j} b_{ij} \log(x_i) \log(x_j) \quad \text{recall: } \operatorname{Li}_2(x) := \sum_{k \geq 1} \frac{x^k}{k^2}$$

Its differential $\eta := df$ is well-defined on $X' := \mathbb{Z}^m$ -cover of $(\mathbb{C}^\times)^m - \cup_{\vec{\nu} \in B} \{\vec{x}^{\vec{\nu}} = 1\}$

$$df = \sum_{i=1}^m \left(- \sum_{\vec{\nu} \in B} \nu_i w_{\vec{\nu}} \log(1 - \vec{x}^{\vec{\nu}}) + \sum_{j=1}^m b_{ij} \log(z_j) \right) d \log(x_i)$$

Periods of η belong to $(2\pi i)^2 \mathbb{Z}$, function f is well-defined on a \mathbb{Z} -cover X of X' .

6. Example: complex Chern-Simons, paths in K_2 -geometry, sums of dilogs

In all 3 situations the set of critical values is a *finite* union of arithmetic progressions in \mathbb{C} each with the step $(2\pi i)^2 = -39.4784\dots$



The reason is that in each of 3 situations, critical points (up to \mathbb{Z} -action) are solutions of a system of algebraic equations, with coefficients in \mathbb{Q} , of expected dimension 0.

- ▶ case **3**: representations $\pi_1(M) \rightarrow G$ (the group can be defined over \mathbb{Q})
- ▶ case **1**: intersection $L \cap L'$ of two algebraic subvarieties defined over \mathbb{Q}
- ▶ case **0**: solutions of a system of *algebraic* equations $\exp(x_i \partial_{x_i} f) = 1, i = 1, \dots, m$

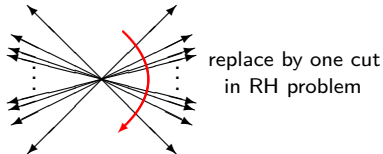
Same critical values, - image of Beilinson-Borel regulator $K_3^{ind}(\overline{\mathbb{Q}}) \rightarrow \mathbb{C}/(2\pi i\mathbb{Z})^2$.

6. Example: complex Chern-Simons, paths in K_2 -geometry, sums of dilogs

Conjecture: one can identify situations **3,1,0** not only matching the critical values $z_{(\underline{\alpha}, k)}$, but also the *Stokes indices* $n_{(\underline{\alpha}, k), (\underline{\alpha}', k')} =: n_{\underline{\alpha}, \underline{\alpha}'; k' - k}$.

Analogy: Stokes indices of an ∞ -dim. path integral (heat kernel) = those of a finite-dim. exponential integral.

It is convenient to encode Stokes data into an angle-ordered product $\overset{\curvearrowright}{\prod}$ of ∞ many Stokes matrices S_θ (like DT invariants of affine Dynkin quivers)

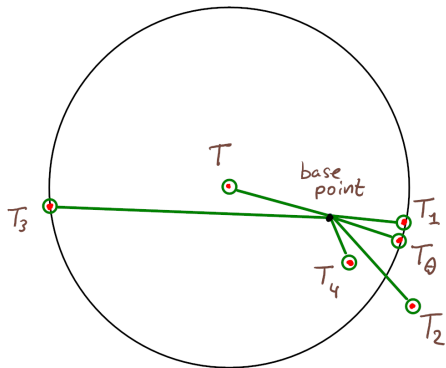


6. Example: complex Chern-Simons, paths in K_2 -geometry, sums of dilogs

An ad hoc example in the case $\mathbf{0}$: $f = \text{Li}_2(x) + 2 \log(x)^2 \rightsquigarrow X'_0 = \{(x, t) \in \mathbb{C}^2 \mid \frac{x^4}{1-x} = e^t\}$

Map $\Phi := \exp(\frac{f}{2\pi i}) : X'_0 \rightarrow \mathbb{C}^\times$ is ramified over $\{0, (\Phi(x, 0))_{x: x^4+x=1}, \exp(\text{Li}_2(1)/2\pi i)\}$

Fiber at generic base point = $\{i_j \mid i \in \mathbb{Z}, 1 \leq j \leq 7\}$



$$T : i_j \mapsto (i+1)_j \quad \forall i, j$$

$$T_1 : 0_1 \leftrightarrow 0_5$$

$$T_2 : 0_2 \leftrightarrow 0_5$$

$$T_3 : 0_3 \leftrightarrow 0_6$$

$$T_4 : 0_4 \leftrightarrow 0_7$$

$$\begin{aligned} T_\theta : \dots &\rightarrow 18_1 \rightarrow 15_2 \rightarrow 12_3 \rightarrow 10_4 \rightarrow \\ &\rightarrow 8_1 \rightarrow 6_2 \rightarrow 4_3 \rightarrow 3_4 \rightarrow 2_1 \rightarrow 1_2 \rightarrow 0_3 \rightarrow 0_4 \rightarrow \\ &\rightarrow 0_5 \rightarrow 0_6 \rightarrow 1_7 \rightarrow 2_5 \rightarrow 3_6 \rightarrow 5_7 \rightarrow \\ &\rightarrow 7_5 \rightarrow 9_6 \rightarrow 12_7 \rightarrow 15_5 \rightarrow 18_6 \rightarrow 22_7 \rightarrow \dots \\ &\text{"quadratic progressions", e.g. } (2n^2)_1 \end{aligned}$$

Problem with the critical value $e^{\text{Li}_2(1)/2\pi i} = e^{-\pi i/12}$: strange asymptotics $\int_0 y^{y/\hbar} dy$