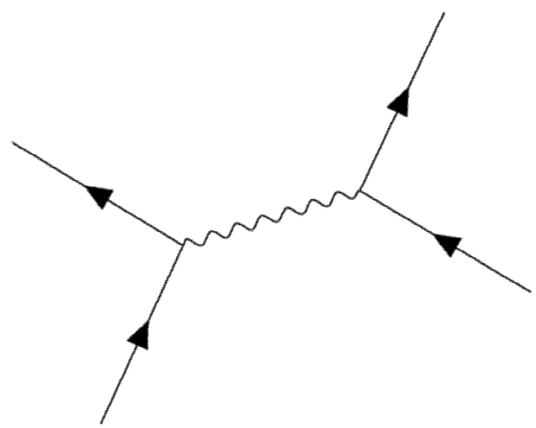
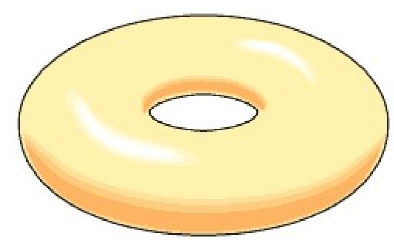
Chern-Simons theory = QFT + Topology





Resources

$$\sum \frac{1000^n}{n!} \quad \text{vs} \quad \sum \frac{n!}{1000^n}$$

G. G. Stokes (1864)
⋮
J. Ecalle (1981)
F. Pham (1983)
M.Berry, C.Howls (1990-91)
S.Marmi, D.Sauzin (2003)
⋮

⋮
M.Marino (2012)
D.Dorigoni (2014)
D.Sauzin (2014)
G.Dunne, M.Unsal (2012, 2015)
M.Kontsevich (lectures 2012, 2014, 2015, 2016)
A.Cherman, D.Dorigoni, G.Dunne, M.Unsal (2013)
D.Sauzin, G.Tiozzo (2017)

Before the 2017 KITP program

Perturbative	$\longleftrightarrow$	Non-perturbative
$\hbar$		$q = e^{\hbar}$
formal power series		q-series

:

R.Lawrence, D.Zagier (1999)
K.Hikami (2004)
O.Costin, S.Garoufalidis (2006)
S.Garoufalidis (2007)

:

:

S.Garoufalidis, T.Le, M.Marino (2008)
T.Dimofte, S.G., J.Lenells, D.Zagier (2009)
E.Witten (2010, 2011)
M.Kontsevich (2014)
S.G., M.Marino, P.Putrov (2016)

After … three years later

:

J.-B.Bae, D.Gang, J.Lee
S.G., D.Pei, P.Putrov, C.Vafa
V.Mikhaylov
D.Gang, Y.Hatsuda
J.Andersen, W.Mistegaard (2018, 2018)
M.Cheng, S.Chun, F.Ferrari, S.G., S.Harrison ← at KITP
H.-J.Chung (2018, 2019, 2020)
S.G., C.Manolescu
P.Kucharski (2019, 2020)
S.Park (2019, 2020)
S.Chun, S.G., S.Park, N.Sopenko
M.Cheng, F.Ferrari, G.Sgroi
W.Mistegaard
S.Garoufalidis, J.Gu, M.Marino

:

A NOTE ON PERTURBATIVE CHERN–SIMONS THEORY

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Received 7 August 1989

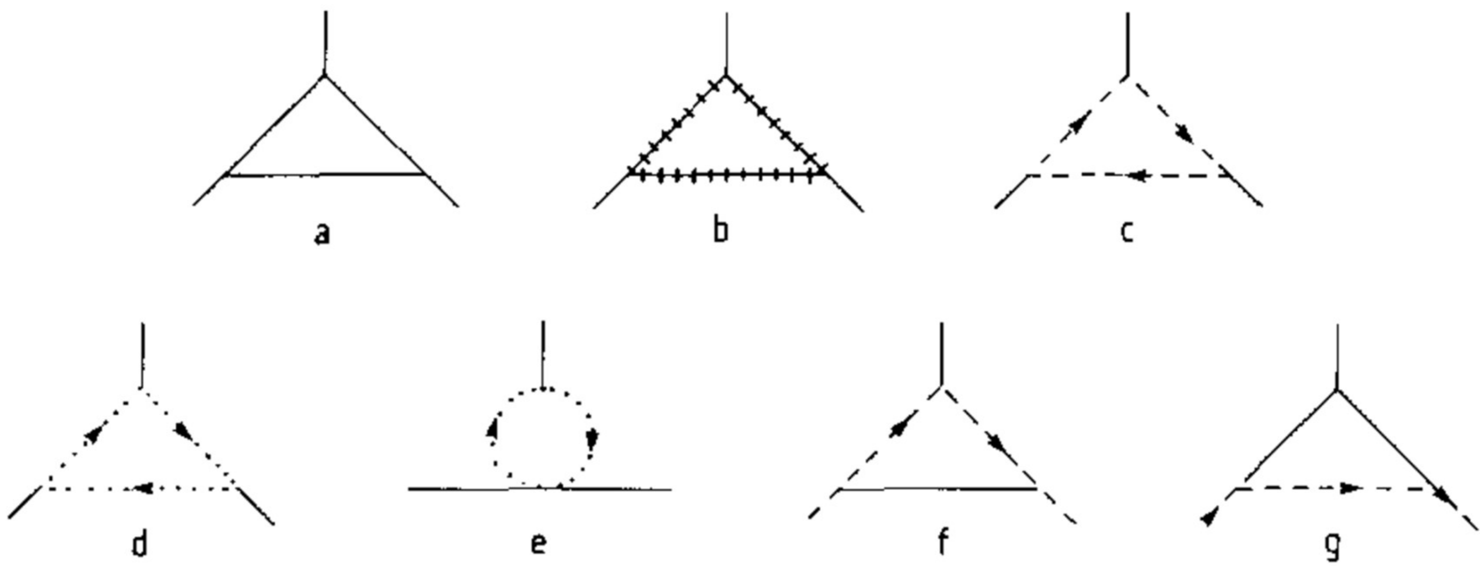
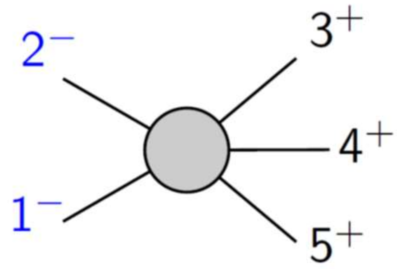
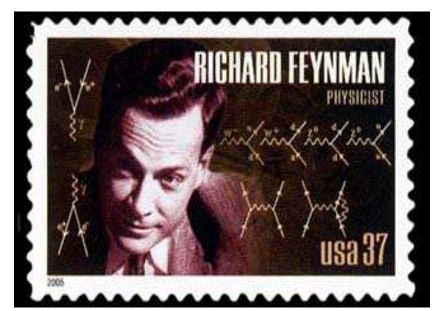


Fig. 3. One-loop diagrams corresponding to the three-point functions of the theory. Diagrams (a), (b), (c), (d) and (e) contribute to the gauge-field two-point function. Diagrams (f) and (g) contribute to the ghost–gauge-field three-point function.



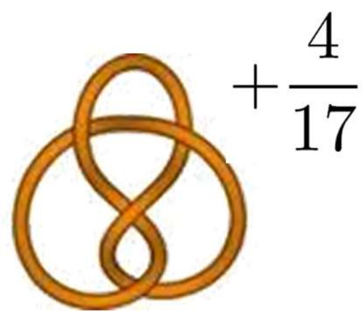
~ 24 pages of calculations $= \dfrac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}$

$$k_1 \cdot k_4\, \varepsilon_2 \cdot k_1\, \varepsilon_1 \cdot \varepsilon_3\, \varepsilon_4 \cdot \varepsilon_5$$



Theorem [Lickorish, Wallace, Kirby]:

Every connected oriented closed 3-manifold arises by performing an integral Dehn surgery along a link $K \hookrightarrow S^3$ (*i.e.* a surgery along a framed link)



$$M_3 = S^3_{-1/r}(K):$$

$$Z_{\text{pert}}(\hbar) = \mathcal{L}_{1/r}\left[\left(x^{\frac{1}{2}} - x^{-\frac{1}{2}}\right)\left(x^{\frac{1}{2r}} - x^{-\frac{1}{2r}}\right) \sum_{m=0}^{\infty} C_m[K]\,(qx)_m(q/x)_m \right]_{q=e^\hbar}$$

$$\mathcal{L}_{1/r} \; : \; x^n \; \mapsto \; q^{rn^2}$$

$$C_m[\mathbf{3_1^\ell}] = q^m$$

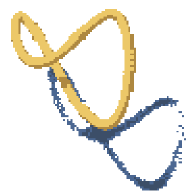
$$C_m[\mathbf{3_1^r}] = q^{-m(m+2)}$$

$$C_m[\mathbf{4_1}] = (-1)^m q^{-\frac{m(m+1)}{2}}$$

$M_3 = S^3_{-1/2}($  $):$

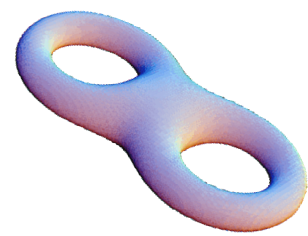
$$Z_{\text{pert}}(\hbar) = 1 + \frac{97}{8}\hbar + \frac{33985}{128}\hbar^2 + \frac{24726817}{3072}\hbar^3 + \frac{30753823105}{98304}\hbar^4 + \ldots$$

Trigonometric (a.k.a. hyperbolic) integrable systems

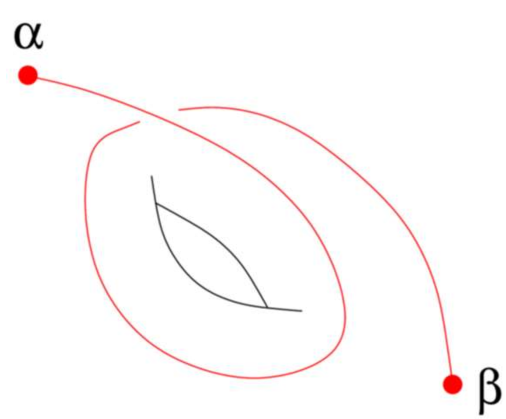


Open topological strings

Problems described by "spectral curves" in $\mathbb{C}^* \times \mathbb{C}^*$



$$\frac{dy}{y} \wedge \frac{dx}{x}$$



Gauge theory in $d \geq 3$ *(incl. Chern-Simons)*

$$\exp\left(\frac{\partial W}{\partial \log x}\right) = 1$$

УДК 513.835 МАТЕМАТИКА

Член-корреспондент АН СССР С.П. НОВИКОВ

МНОГОЗНАЧНЫЕ ФУНКЦИИ И ФУНКЦИОНАЛЫ. АНАЛОГ ТЕОРИИ МОРСА

I. Рассмотрим многообразие M, конечномерное или бесконечномерное, и замкнутую 1-форму ω, где $d\omega = 0$. Интегралы от ω по путям определяют "многозначную функцию" S, которая становится однозначной на некоторой накрывающей $\hat{M} \xrightarrow{\pi} M$ со свободной абелевой группой монодромии: $dS = \pi^* \omega$. Число образующих группы монодромии равно числу рационально независимых интегралов 1-формы ω по целочисленным циклам в M.

З а д а ч а. Построить аналог теории Морса для многозначных функций S, т.е. найти связи чисел стационарных точек $dS = 0$ различных индексов с топологией многообразия M.

II. Естественные примеры многозначных функционалов были рассмотрены и использованы в работах [1, 2] при исследовании периодических решений уравнений типа Кирхгофа, волчка в гравитационном поле и др., а также заряженной частицы в магнитном поле, если имеется "монополь Дирака". Многозначный функционал S в

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Поступило
8 IV 1981

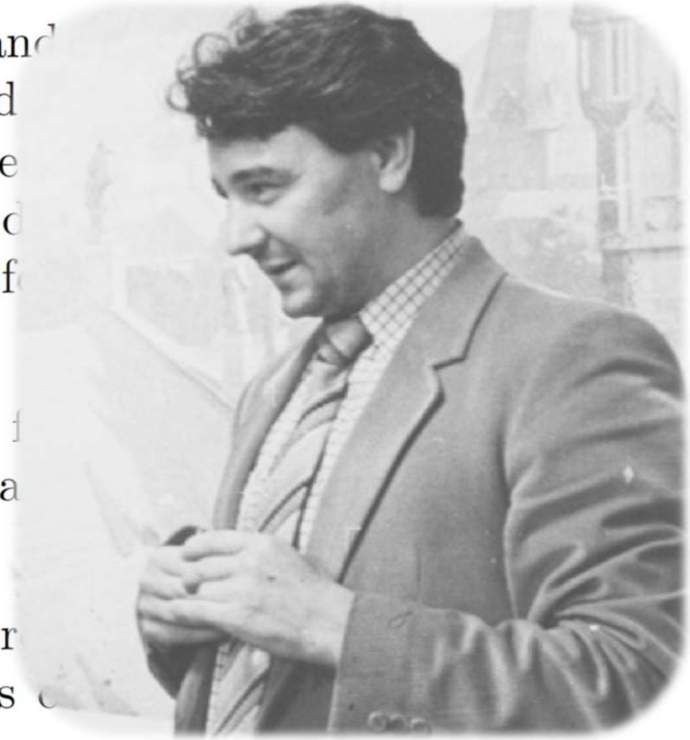
MULTIVALUED FUNCTIONS AND FUNCTIONALS. AN ANALOGUE OF THE MORSE THEORY

S. P. NOVIKOV

I. Let M be a finite or infinite dimensional manifold an
Integrating ω over paths in M defines a "multivalued
single valued on some covering $\hat{M} \xrightarrow{\pi} M$ with a free
$dS = \pi^*\omega$. The number of generators of the monod
number of rationally inde- pendent integrals of the 1-f
M.

Problem. To construct an analogue of Morse theory f
S. That is, to find a relationship between the stationa
index and the topology of the manifold M.

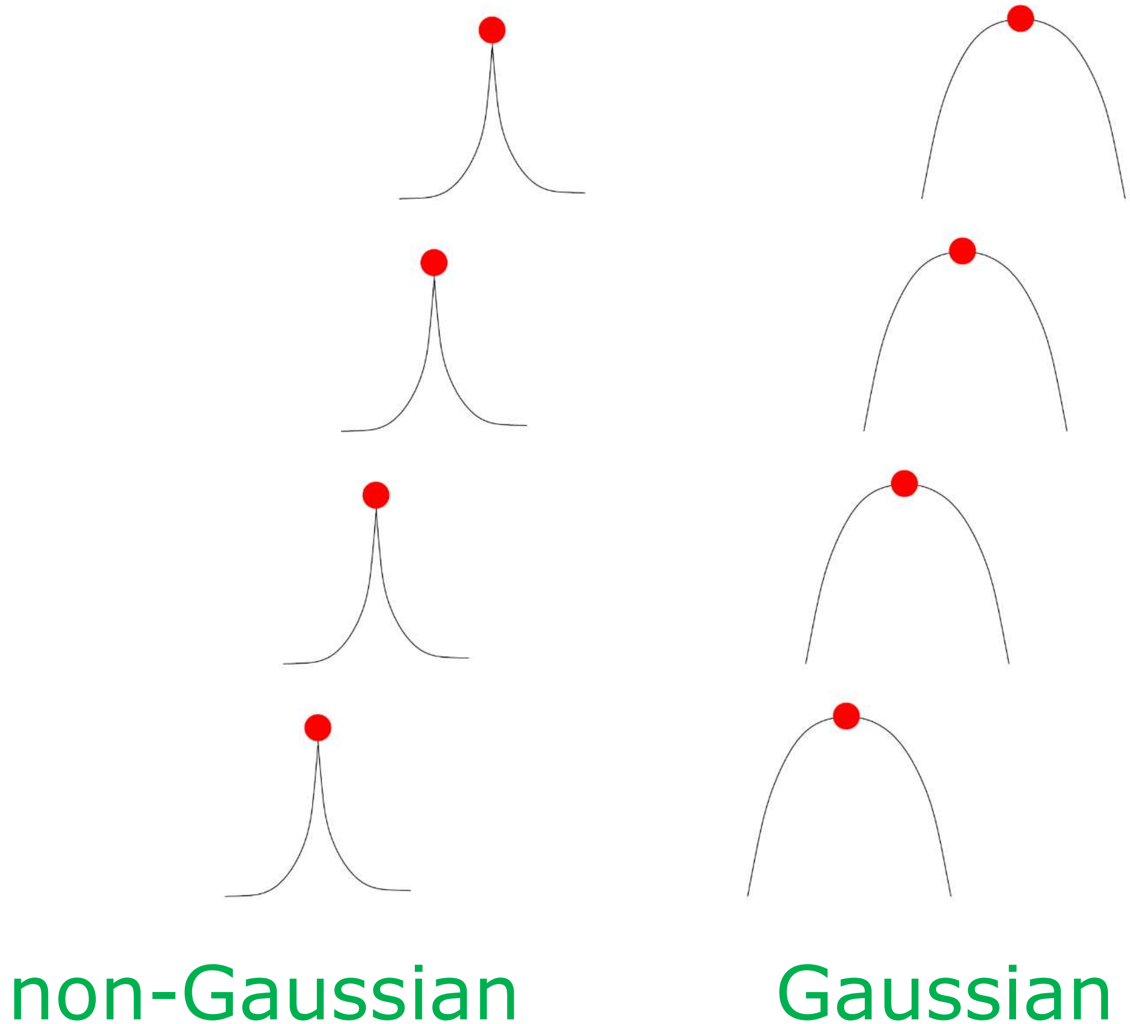
II. Natural examples of multivalued functionals wer
and [2] to investigate periodic solutions of equations o



Landau Institute of Theoretical Physics, Chernogolovka, Moscow Region

$$1 + 1 + 1 = \text{integer}$$

$$1 + 1 + 1 + 1 + \ldots = \text{non-integer}$$

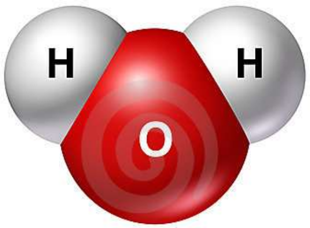


Prospect #1: The role of abelian flat connections and Spin^c structures

$$\widehat{Z}_b(M_3;q) \xrightarrow{q \to 1} \frac{1}{\text{Turaev torsion}}$$

$$b \in \text{Spin}^c(M_3)/Z_2$$

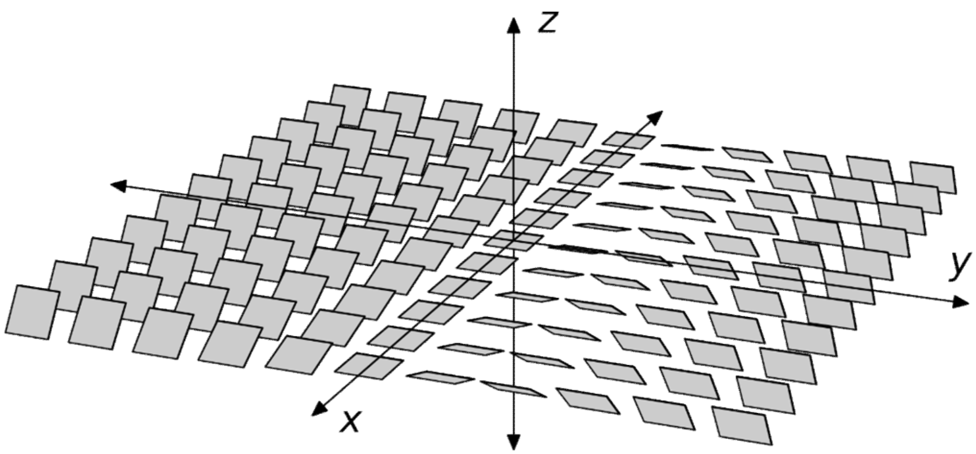
$$\text{WRT}(M_3,k) = \sum_b \ldots \widehat{Z}_b(q)\Big|_{q \to e^{\frac{2\pi i}{k}}}$$



Question #1: Connections to Heegaard Floer homology? Contact geometry?

$$\widehat{Z}_b = q^{d_b}\left(c_0 + c_1 q + c_2 q^2 + \ldots\right) \in q^{d_b}\mathbb{Z}[[q]]$$

"correction term" (Heegaard Floer, Seiberg-Witten theory)



Prospect #2: Superconformal Index

$$\mathcal{I}(q) = \sum_b \widehat{Z}_b(M_3, q)\, \widehat{Z}_b(-M_3, q)$$

$$= \underbrace{\mathcal{I}_{\text{DGG}}(q)}_{\text{non-abelian}} + \text{ abelian}$$

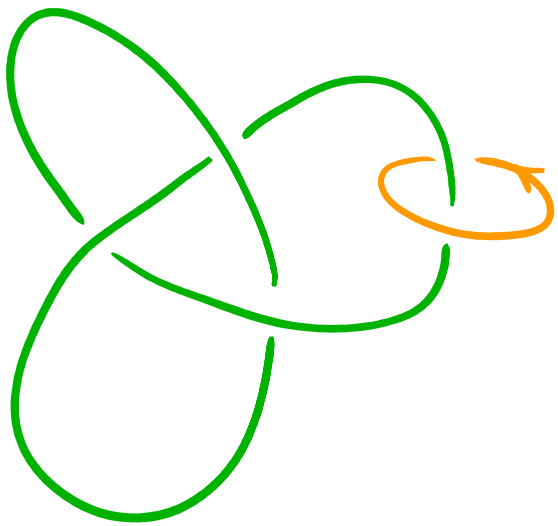
see Marcos' talk

$M_3 = S^3_{+1}($  $)$:

$$\mathcal{I}(q) = 1 - q^2 + q^3 - q^5 - 2q^6 + 2q^7 - 2q^8 - q^{10} + \ldots$$

Prospect #3: Resurgence at $b_1 > 0$

(= parametric resurgence)

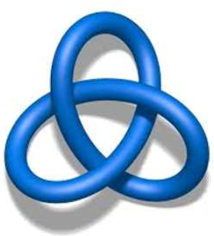


$$\mathrm{Hol} = \begin{pmatrix} x & * \\ 0 & x^{-1} \end{pmatrix}$$

$$F_K(x,q) \;:=\; \widehat{Z}\left(S^3 \setminus K\right)$$

$$
\begin{aligned}
Z^{(\mathrm{ab})}_{\mathrm{pert}}(\mathbf{3_1}) &= (x^{1/2} - x^{-1/2} - x^{5/2} + x^{-5/2} - x^{7/2} + x^{-7/2} + \ldots) \\
&+ \hbar(x^{1/2} - x^{-1/2} - 2x^{5/2} + 2x^{-5/2} - 3x^{7/2} + 3x^{-7/2} + \ldots) \\
&+ \frac{\hbar^2}{2}(x^{1/2} - x^{-1/2} - 4x^{5/2} + 4x^{-5/2} - 9x^{7/2} + 9x^{-7/2} + \ldots) \\
&+ \frac{\hbar^3}{6}(x^{1/2} - x^{-1/2} - 8x^{5/2} + 8x^{-5/2} - 27x^{7/2} + 27x^{-7/2} + \ldots) \\
&+ \frac{\hbar^4}{24}(x^{1/2} - x^{-1/2} - 16x^{5/2} + 16x^{-5/2} - 81x^{7/2} + 81x^{-7/2} + \ldots) \\
&+ \ldots \\
&= \sum_{m \geq 1} f_m(q) \cdot \left(x^{\frac{m}{2}} - x^{-\frac{m}{2}}\right)
\end{aligned}
$$

$$f_m = \epsilon_m q^{\frac{m^2+23}{24}}$$



$$f_1 = q, \quad f_3 = 0, \quad f_5 = -q^2, \quad f_7 = -q^3, \quad f_9 = 0, \quad \ldots$$

Question #2: Logarithmic/non-semisimple MTCs

$$\widehat{Z}_b^{SU(2)}(M_3) \xrightarrow{q \to \text{root of } 1} \text{CGP invariants}$$

Costantino, Geer, Patureau-Mirand

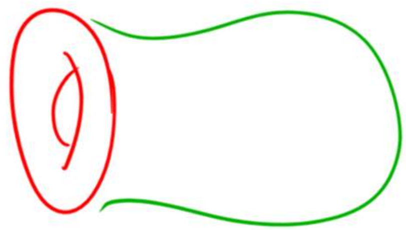
$$\widehat{Z}^{SU(2)}(S^3 \setminus K) \xrightarrow{q \to \text{root of } 1} \text{ADO invariants}$$

Akutsu, Deguchi, Ohtsuki

$$\widehat{Z}^{SU(m|n)}(S^3 \setminus K) \xrightarrow{q \to \text{root of } 1} \text{Links-Gould invariants}$$

Prospect #4: Quantization of Chern-Simons theory ("from A to Z")

$\mathcal{H}(T^2)$



$\|$

quantization of

$(\mathbb{C}^* \times \mathbb{C}^*)/\mathbb{Z}_2$

$\rightsquigarrow$

$\mathcal{H}(T^2)$ with basis

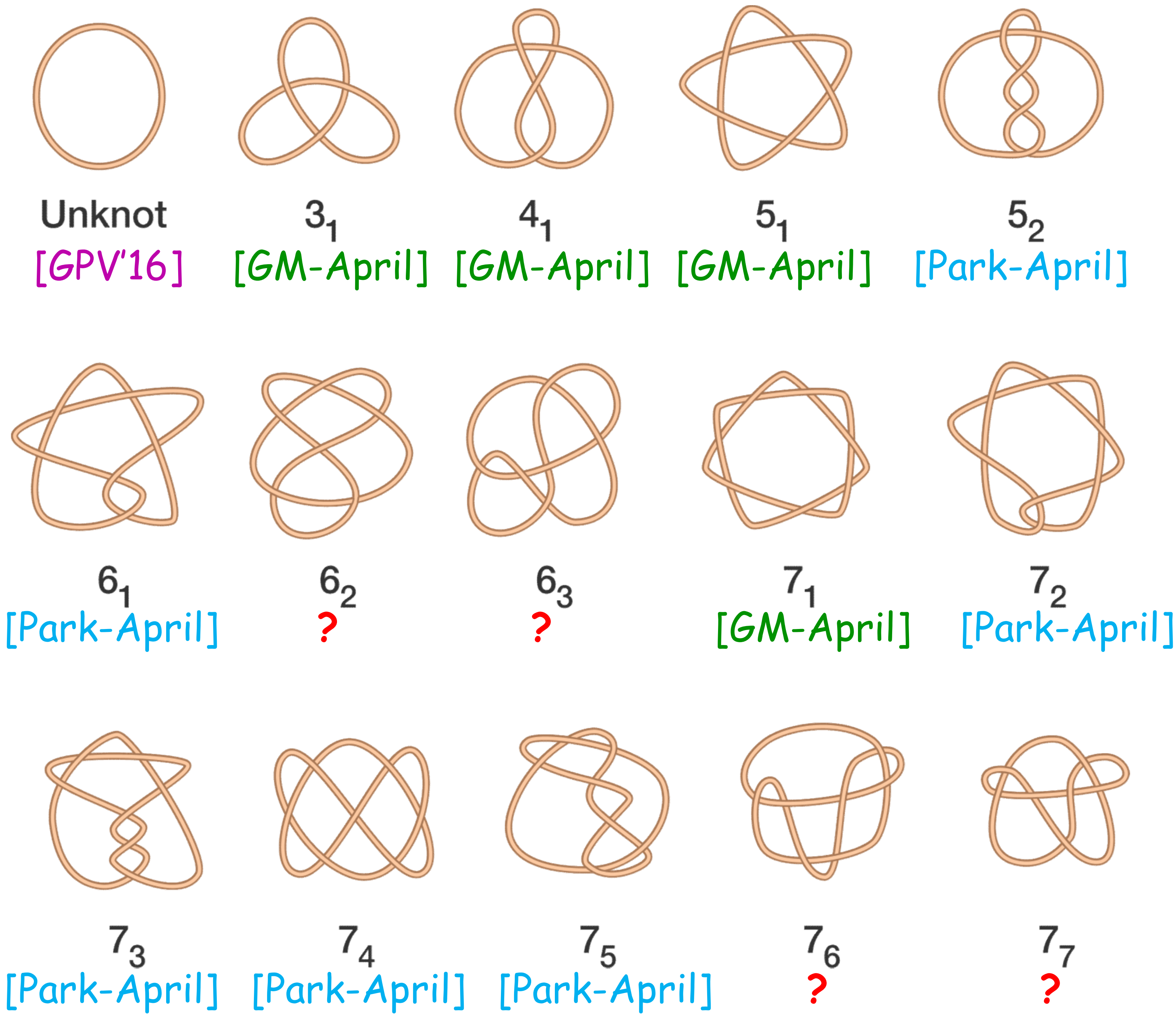
$|m,n\rangle$ or $|x,n\rangle$

$$\widehat{A}(\widehat{x},\widehat{y})\, Z_{\text{CS}}(M) \;=\; 0$$

Prospect #5: Knot complements

$$\widehat{A}(\widehat{x},\widehat{y})\ \widehat{Z}\left(S^3 \setminus K\right) \ = \ 0$$

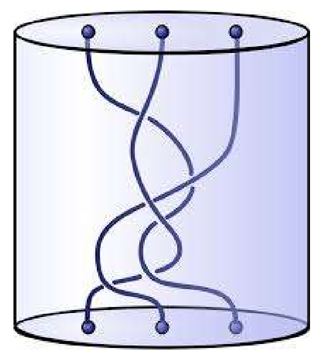
$S^3_{-1/r}(\mathbf{4_1})$	$\widehat{Z}_a(q)$
$r=2$	$-q^{-1/2}(1-q+2q^3-2q^6+q^9+3q^{10}+q^{11}-q^{14}-3q^{15}-3q^{15}-q^{16}+2q^{19}$ $+2q^{20}+5q^{21}+2q^{22}+2q^{23}-2q^{26}-2q^{27}-5q^{28}-2q^{29}-2q^{30}+\cdots)$
$r=3$	$-q^{-1/2}(1-q+2q^5-2q^8+q^{15}+3q^{16}+q^{17}-q^{20}-3q^{21}-q^{22}+2q^{31}$ $+2q^{32}+5q^{33}+2q^{34}+2q^{35}-2q^{38}-2q^{39}-5q^{40}-2q^{41}-2q^{42}+\cdots)$
$r=4$	$-q^{-1/2}(1-q+2q^7-2q^{10}+q^{21}+3q^{22}+q^{23}-q^{26}-3q^{27}-q^{28}+2q^{43}$ $+2q^{44}+5q^{45}+2q^{46}+2q^{47}-2q^{50}-2q^{51}-5q^{52}-2q^{53}-2q^{54}+\cdots)$


Unknot
[GPV'16]
3_1
[GM-April]
4_1
[GM-April]
5_1
[GM-April]
5_2
[Park-April]
6_1
[Park-April]
6_2
?
6_3
?
7_1
[GM-April]
7_2
[Park-April]
7_3
[Park-April]
7_4
[Park-April]
7_5
[Park-April]
7_6
?
7_7
?

LARGE COLOR R-MATRIX FOR KNOT COMPLEMENTS AND STRANGE IDENTITIES

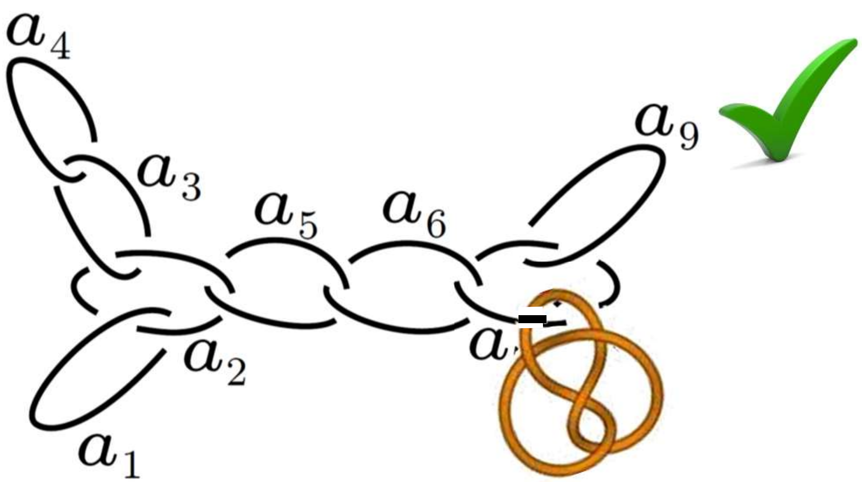
SUNGHYUK PARK

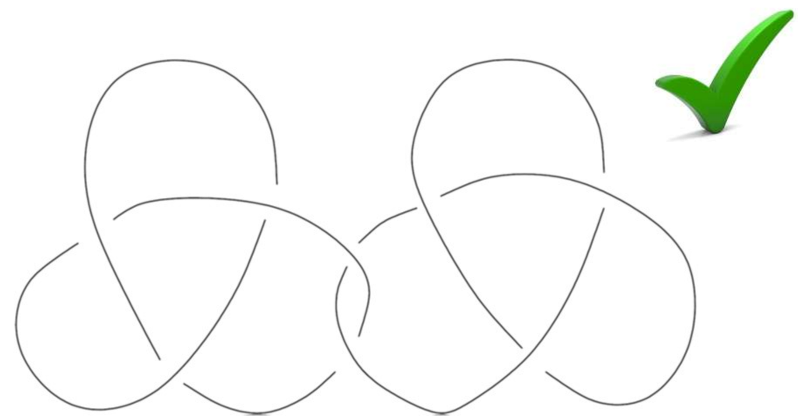
1. Introduction 1
2. Review of $U_q(\mathfrak{sl}_2)$ and R-matrix 4
2.1. $U_q(\mathfrak{sl}_2)$ 4
2.2. R-matrix for $V_n \otimes V_m$ 5
3. Large color R-matrix 7
3.1. Highest weight and lowest weight Verma modules V_∞^h and V_∞^l 7
3.2. R-matrices on Verma modules 8
3.3. Semi-classical limit of the R-matrix and Burau representation 9
4. F_K from large color R-matrix 10
4.1. Closing up the braid 10
4.2. Positive braid knots 12
4.3. Fibered strongly quasi-positive braid knots 14
4.4. Double twist knots 15
4.5. Other examples 18
5. Some surgeries 19
5.1. Some consistency checks 19
5.2. Reverse-engineering F_K for some simple links 21
6. Strange identity for positive braid knots 25
7. Open questions and future directions 26
References 26

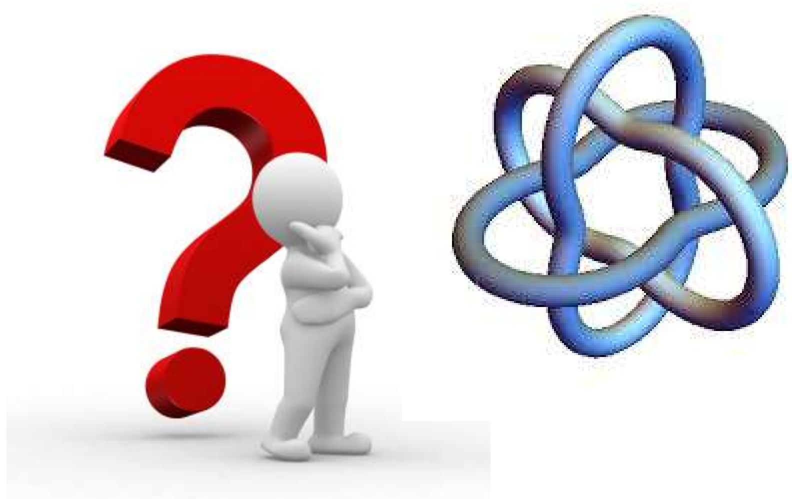


$$KEK^{-1} = q^2 E\,, \qquad KFK^{-1} = q^{-2} F\,, \qquad [E,F] = \frac{K - K^{-1}}{q - q^{-1}}$$

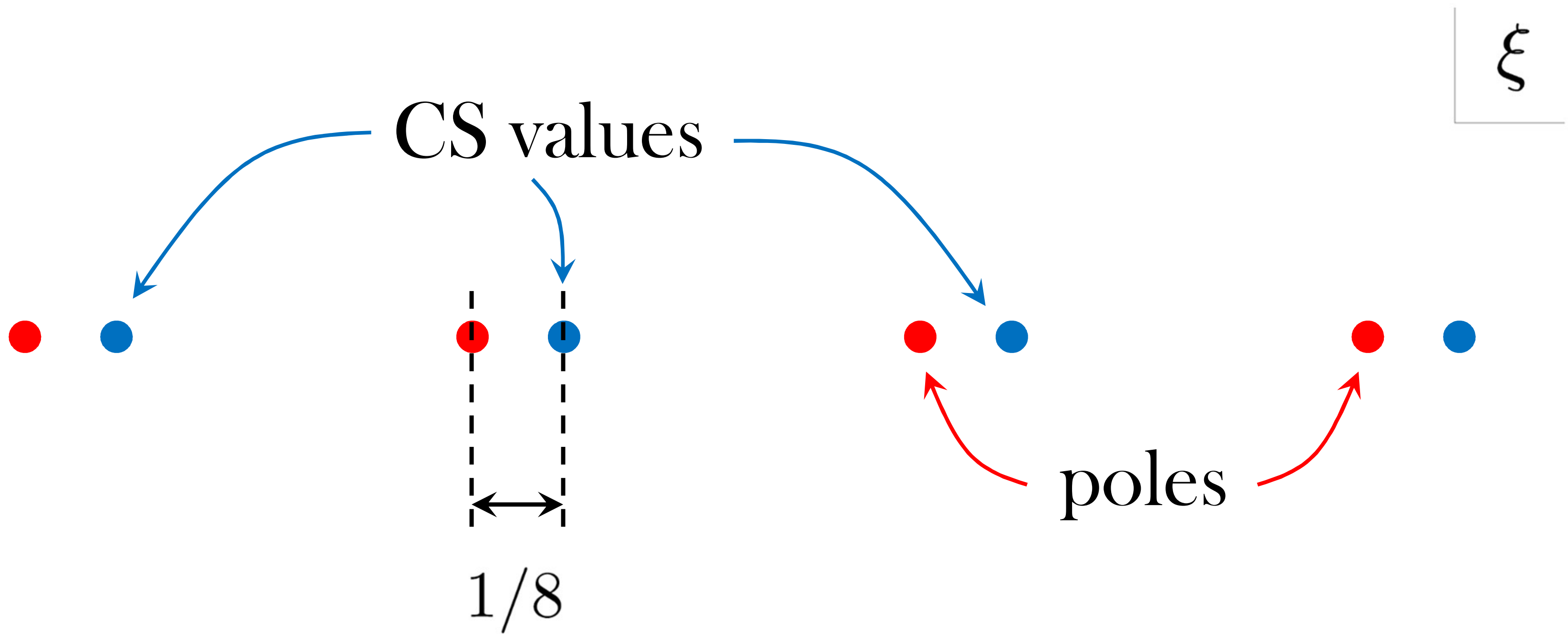
Question #3: Link complements?







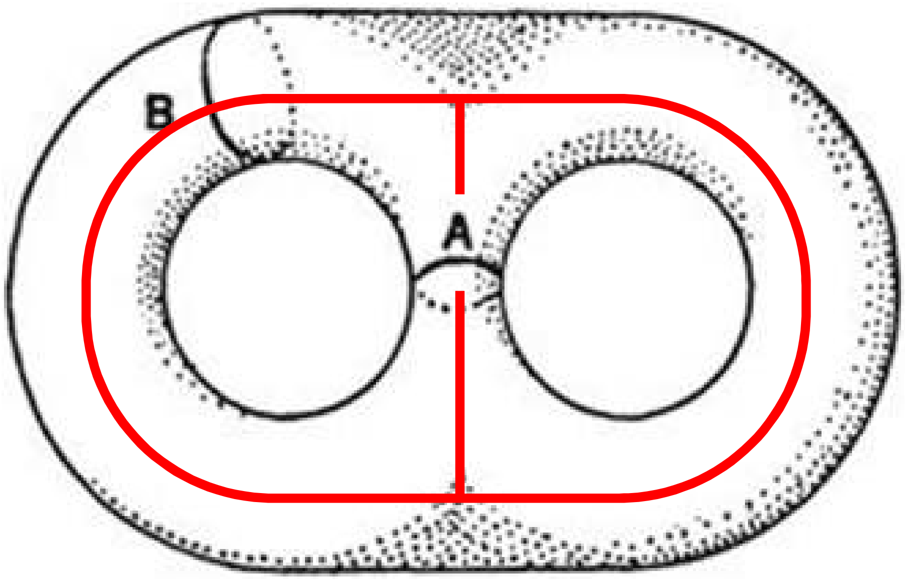
Question #4: "Renormalons"?



$$M_3 = S_0^3(\mathbf{5_2})$$

$$BZ_\alpha^{\text{pert}}(\xi) \overset{?}{\sim} \sum_\alpha \frac{n_{\alpha\beta}}{\xi - \text{CS}(\beta)}$$

Question #5: Heegaard decompositions and framing dependence?



$$1 + \left(n + \frac{3}{2}n^2 + \frac{1}{3}n^3 \right) \hbar^2 + \ldots$$