Bethe Ansatz for a finite q-boson system with boundary interactions*

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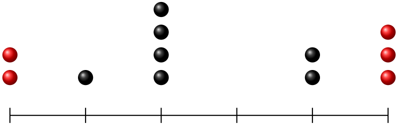
Non-equilibrium dynamics of stochastic and quantum integrable systems,

Kavli Institute for Theoretical Physics, Santa Barbara, February 2016



*Joint work with **Erdal Emsiz**, Pontificia Universidad Católica de Chile.

Periodic delta Bose gas with hyperoctahedral symmetry

Formal Schrödinger operator for n particles on the circle
(Gaudin '71)

$$H = -\Delta + \sum_{1 \leqslant j < k \leqslant n} c\Big(\delta(x_j - x_k) + \delta(x_j + x_k)\Big) + \sum_{1 \leqslant j \leqslant n} \Big(c_1 \delta(x_j) + c_2 \delta(2x_j)\Big)$$

- Δ is Laplacian in $\mathbb{R}^n$.
- $\delta(\cdot)$ is Dirac delta comb: $\delta(x + m) = \delta(x)$ $(m \in \mathbb{Z})$
- Configuration space: torus $(\mathbb{R}/\mathbb{Z})^n$

Laplacian on the hyperoctahedral Weyl alcove

Spectral problem for H equivalent to

$$-\Delta\psi = E\psi$$

on hyperoctahedral Weyl alcove

$$A = \{x \in \mathbb{R}^n \mid \tfrac{1}{2} > x_1 > x_2 > \cdots > x_n > 0\},$$

with Robin boundary conditions at the walls

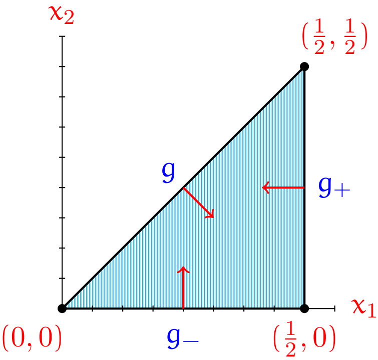
$$(\partial_{x_j} - \partial_{x_{j+1}} - g)\psi\Big|_{x_j - x_{j+1} = 0} = 0 \quad \text{for} \quad j = 1, \ldots, n-1$$

and

$$(\partial_{x_1} + g_+)\psi\Big|_{x_1 = \frac{1}{2}} = 0, \qquad (\partial_{x_n} - g_-)\psi\Big|_{x_n = 0} = 0$$

($c = 2g$, $c_1 = 2(g_- - g_+)$ and $c_2 = 4g_+$).

n=2



From now on: our walls are always repulsive ($g, g\pm > 0$).

Problem: verify orthogonality Bethe Ansatz eigenfunctions

— Idea: eigenfunctions of self-adjoint Laplacian

— Complication: possible degeneracies in spectrum of Laplacian

— Solution: employ q-boson lattice regularization + continuum limit

Similar previous results for delta bosons on the circle

—Bethe Ansatz eigenfunctions: Lieb and Liniger '63

—Orthogonality completeness: Dorlas '93

(via quantum lattice NLS and ideas of Yang and Yang)

—Completeness hyperoctahedral case: Emsiz '09

(by adapting Dorlas' methods)

II. Finite q-boson system with boundary interactions

q-Oscillator (Kulish '81)

Annihilation operator β, creation operator β^*, q-deformed number operator t^N ($t = q^2$) with relations:

$$t^N \beta^* = t\beta^* t^N, \quad \beta t^N = t t^N \beta, \quad t^N t^{-N} = t^{-N} t^N = 1$$

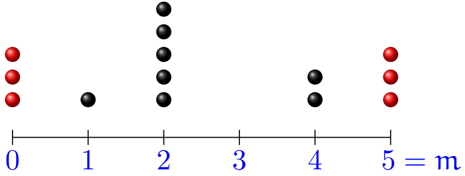
and

$$\beta\beta^* - \beta^*\beta = t^N, \qquad \beta\beta^* - t\beta^*\beta = 1.$$

q-Boson system (Bogoliubov-Bullough '92, Bogoliubov-Izergin-Kitanine '98)

$\mathfrak{m}$ q-oscillators placed on a finite lattice $\mathbb{N}_{\mathfrak{m}} := \{0, 1, 2, \ldots, \mathfrak{m}\}$:

$(\mathfrak{n}, \mathfrak{m}) = (14, 5)$



Particle configuration encoded by partition λ:

$$\lambda = (\underbrace{5,5,5}_{\substack{m_5(\lambda)\\=3}}, \underbrace{4,4}_{\substack{m_4(\lambda)\\=2}}, \underbrace{2,2,2,2,2}_{\substack{m_2(\lambda)\\=5}}, \underbrace{1}_{\substack{m_1(\lambda)\\=1}}, \underbrace{0,0,0}_{\substack{m_0(\lambda)\\=3}}) \qquad (m_3(\lambda) = 0)$$

Hamiltonian

We consider open-end boundary conditions (no periodicity)

(Li-Wang '12, vD-E '14)

$$\mathcal{H}_m = \underbrace{a_- \beta_0^* \beta_0 + a_+ \beta_m^* \beta_m}_{\text{Boundary Terms}} + \sum_{0 \leqslant l < m} \underbrace{\beta_l^* \beta_{l+1} + \beta_{l+1}^* \beta_l}_{\text{Hopping Terms}}$$

—Notice: $\beta_l^* \beta_l = (1 - t^{N_l})/(1 - t)$ (viz. q-number operator)

—$a_\pm$ are linear coupling parameters for the boundary interaction

Fock space representation

Let $f : \Lambda_{n,m} \to \mathbb{C}$ with

$$\Lambda_{n,m} = \{\lambda = (\lambda_1, \ldots, \lambda_n) \mid m \geqslant \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n \geqslant 0\}.$$

Action of the q-boson operators β_l, β_l^*, t^{N_l} ($l = 0, \ldots, m$) on the n-particle wave function f:

$$(\beta_l f)(\lambda) := f(\beta_l^* \lambda) \quad \text{if } n > 0$$

($\lambda \in \Lambda_{n-1,m}$) and $\beta_l f = 0$ if $n = 0$,

$$(\beta_l^* f)(\lambda) := \begin{cases} [m_l(\lambda)] f(\beta_l \lambda) & \text{if } m_l(\lambda) > 0 \\ 0 & \text{otherwise} \end{cases}$$

($\lambda \in \Lambda_{n+1,m}$),

$$(t^{\pm N_l} f)(\lambda) := t^{\pm m_l(\lambda)} f(\lambda)$$

($\lambda \in \Lambda_{n,m}$).

Here $\beta_l^* \lambda$ is obtained from λ by adding a part of size l while $\beta_l \lambda$ is obtained by deleting such a part (assuming $m_l(\lambda) > 0$), and

$$[k] := \frac{1 - t^k}{1 - t} \qquad \text{(q-integer)}.$$

n-Particle Hamiltonian

The action of the Hamiltonian on the n-particle wave functions reads:

$$(\mathcal{H}_m f)(\lambda) = \Big(a_-[m_0(\lambda)] + a_+[m_m(\lambda)] \Big) f(\lambda)$$

$$\sum_{\substack{1\leqslant j\leqslant n \\ \lambda+e_j\in\Lambda_{n,m}}} [m_j(\lambda)] f(\lambda + e_j) + \sum_{\substack{1\leqslant j\leqslant n \\ \lambda-e_j\in\Lambda_{n,m}}} [m_j(\lambda)] f(\lambda - e_j)$$

$\mathcal{H}_m$ is self-adjoint (because $(\beta_l)^* = \beta_l^*$ and $(t^{N_l})^* = t^{N_l}$) with respect to the inner product

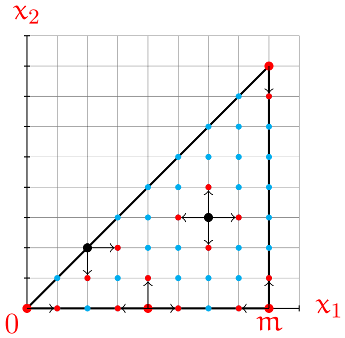
$$(f, g)_{n,m} := \sum_{\lambda\in\Lambda_{n,m}} f(\lambda)\overline{g(\lambda)}\delta_{n,m}(\lambda)$$

determined by the weight function

$$\delta_{n,m}(\lambda) := \frac{1}{\prod_{0\leqslant l\leqslant m}[m_l(\lambda)]!},$$

where $[k]! := [k][k-1]\cdots[2][1]$ (q-deformed factorial).

Action of $\mathcal{H}_m$ for $n = 2$:



III. Transfer operator

The transfer operator is obtained from Sklyanin's Quantum Inverse Scattering Formalism for systems with open-end boundary interactions (Sklyanin '88).

Monodromy matrix

Starting from the Lax matrix (Bogoliubov-Izergin-Kitanine '98)

$$L_l(u) = \begin{pmatrix} u^{-1} & (1-t)\beta_l^* \\ \beta_l & u \end{pmatrix} \qquad (l = 0, \ldots, m)$$

satisfying the Quantum Yang-Baxter Equation

$$R(u/v)L_l(u)_1 L_l(v)_2 = L_l(v)_1 L_l(u)_2 R(u/v)$$

associated with the R-matrix

$$R(u) = \begin{pmatrix} s(q^{-1}u) & 0 & 0 & 0 \\ 0 & s(q^{-1}) & q^{-1}s(u) & 0 \\ 0 & qs(u) & s(q^{-1}) & 0 \\ 0 & 0 & 0 & s(q^{-1}u) \end{pmatrix} \qquad s(u) := u - u^{-1},$$

and standard diagonal solutions (Cherednik '84)

$$K_-(u;a_-) = \begin{pmatrix} e(u;a_-) & 0 \\ 0 & f(u;a_-) \end{pmatrix}, \quad K_+(u;a_+) = \begin{pmatrix} f(u;a_+) & 0 \\ 0 & e(u;a_+) \end{pmatrix},$$

with $e(u;a) := au - u^{-1}$ and $f(u;a) := e(q^{-1}u^{-1};a)$, of the associated left– and right reflection equations

$$\begin{aligned} &R(u/v)K_-(u;a_-)_1R(quv)K_-(v;a_-)_1 \\ &\qquad = K_-(v;a_-)_1R(quv)K_-(u;a_-)_1R(u/v) \end{aligned}$$

and

$$\begin{aligned} &R(u/v)K_+(u;a_+)_2R(quv)K_+(v;a_+)_2 \\ &\qquad = K_+(v;a_+)_2R(quv)K_+(u;a_+)_2R(u/v)), \end{aligned}$$

one constructs the monodromy matrix

$$\mathcal{U}_m(u;a_-) := U_m(u)K_-(u;a_-)U_m^{-1}(q^{-1}u^{-1})$$
$$= \begin{pmatrix} \mathcal{A}_m(u;a_-) & \mathcal{B}_m(u;a_-) \\ \mathcal{C}_m(u;a_-) & \mathcal{D}_m(u;a_-) \end{pmatrix},$$

where

$$U_m(u) := L_m(u) \cdots L_1(u)L_0(u).$$

Transfer operator

The associated transfer operator

$$\mathcal{T}_m(u; a_+, a_-) := \mathrm{tr}\left(K_+(u^{-1}; a_+)\, \mathcal{U}_m(u; a_-)\right)$$
$$= f(u^{-1}; a_+)\mathcal{A}_m(u; a_-) + e(u^{-1}; a_+)\mathcal{D}_m(u; a_-)$$

satisfies the commutativity

$$[\mathcal{T}_m(u; a_+, a_-), \mathcal{T}_m(v; a_+, a_-)] = 0$$

and

$$[\mathcal{T}_m(u; a_+, a_-), \mathcal{H}_m] = 0$$

Explicit action in Fock space

For $\lambda \in \Lambda_{n,m}$ and $\mu \in \Lambda_{n,m} \cup \Lambda_{n-1,m}$, let $\mu \preceq \lambda$ if $\mu_j \leqslant \lambda_j$ for all j and the skew diagram λ/μ is a horizontal strip:

$$m \geqslant \lambda_1 \geqslant \mu_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \mu_{n-1} \geqslant \lambda_n \geqslant \begin{cases} \mu_n \geqslant 0 & \text{if } \mu \in \Lambda_{n,m}, \\ 0 & \text{if } \mu \in \Lambda_{n-1,m}. \end{cases}$$

Secondly, for $\lambda, \mu \in \Lambda_{n,m}$, let $\mu \sim_- \lambda$ if there exists a $\nu \in \Lambda_{n,m} \cup \Lambda_{n-1,m}$ such that $\nu \preceq \lambda$ and $\nu \preceq \mu$, and let $\mu \sim_+ \lambda$ if there exists a $\nu \in \Lambda_{n,m} \cup \Lambda_{n+1,m}$ such that $\lambda \preceq \nu$ and $\mu \preceq \nu$.

The action of the transfer operator on $f : \Lambda_{n,m} \to \mathbb{C}$ reads:

(proof by induction in m, cf. also Korff '13 for the action of the periodic transfer operator)

$$\begin{aligned}
&\big(\mathcal{T}_m(u; a_+, a_-)f\big)(\lambda) = \\
&q^{-m}t^{-n-1}(a_+ - tu^{-2}) \sum_{\substack{\mu \in \Lambda_{n,m} \\ \mu \sim_- \lambda}} A^{(n)}_{\lambda,\mu}(u^2; t, a_-) f(\mu) \\
&+ q^{-m}t^{-n-1}(a_+ - u^2) \sum_{\substack{\mu \in \Lambda_{n,m} \\ \mu \sim_+ \lambda}} D^{(n)}_{\lambda,\mu}(u^2; t, a_-) f(\mu),
\end{aligned}$$

with

$$A^{(n)}_{\lambda,\mu}(z;t,a) := z^{-m}(a - z^{-1}) \sum_{\substack{\nu\in\Lambda_{n,m} \\ \nu\preceq\lambda,\, \nu\preceq\mu}} \varphi_{\lambda/\nu}(t)\psi_{\mu/\nu}(t) z^{|\lambda|+|\mu|-2|\nu|}$$
$$+ z^{-m}(tz - a) \sum_{\substack{\nu\in\Lambda_{n-1,m} \\ \nu\preceq\lambda,\, \nu\preceq\mu}} \varphi_{\lambda/\nu}(t)\psi_{\mu/\nu}(t) z^{|\lambda|+|\mu|-2|\nu|},$$

$$D^{(n)}_{\lambda,\mu}(z;t,a) := z^{m}(z^{-1} - a) \sum_{\substack{\nu\in\Lambda_{n+1,m} \\ \lambda\preceq\nu,\, \mu\preceq\nu}} \psi_{\nu/\lambda}(t)\varphi_{\nu/\mu}(t) z^{|\lambda|+|\mu|-2|\nu|}$$
$$+ z^{m}(a - tz) \sum_{\substack{\nu\in\Lambda_{n,m} \\ \lambda\preceq\nu,\, \mu\preceq\nu}} \psi_{\nu/\lambda}(t)\varphi_{\nu/\mu}(t) z^{|\lambda|+|\mu|-2|\nu|},$$

and

$$\varphi_{\lambda/\mu}(t) := \prod_{\substack{0\leqslant l\leqslant m \\ m_l(\lambda)=m_l(\mu)+1}} (1 - t^{m_l(\lambda)}),$$

$$\psi_{\lambda/\mu}(t) := \prod_{\substack{0\leqslant l\leqslant m \\ m_l(\lambda)=m_l(\mu)-1}} (1 - t^{m_l(\mu)}).$$

IV. Algebraic Bethe Ansatz

The eigenfunctions of the transfer operator are constructed by means of Sklyanin's Algebraic Bethe Ansatz for systems with open-end boundary interactions (Sklyanin '88).

Vacuum state Let $\Lambda_{0,m} := \{\emptyset\}$ and let $|\emptyset\rangle : \Lambda_{0,m} \to \mathbb{C}$ be the constant function with value 1.

Bethe Ansatz wave function By acting iteratively with the Bethe Ansatz creation operator $\mathcal{B}_m(u; a_-)$ on the vacuum state $|\emptyset\rangle$, one arrives at the n-particle Bethe Ansatz wave function:

$$\Psi_{(v_1,\ldots,v_n)} := \mathcal{B}_m(v_1; a_-) \cdots \mathcal{B}_m(v_n; a_-)|\emptyset\rangle.$$

Theorem (Bethe Ansatz Eigenfunction)

The n*-particle Bethe Ansatz wave function* $\Psi_{(v_1,\ldots,v_n)}$ *solves the eigenvalue equation*

$$\mathcal{T}_m(u;a_+,a_-)\Psi_{(v_1,\ldots,v_n)} = E_{n,m}(u;v_1,\ldots,v_n)\Psi_{(v_1,\ldots,v_n)}$$

with eigenvalue

$$E_{n,m}(u;v_1,\ldots,v_n) :=$$
$$q^{-m-1}\Bigg(u^{-2m-2}\frac{s(q^{-1}u^2)}{s(u^2)}e(u;a_+)e(u;a_-)\prod_{1\leqslant j\leqslant n}\frac{s(quv_j)s(quv_j^{-1})}{s(uv_j)s(uv_j^{-1})}$$
$$+u^{2m+2}\frac{s(q^{-1}u^{-2})}{s(u^{-2})}e(u^{-1};a_+)e(u^{-1};a_-)\prod_{1\leqslant j\leqslant n}\frac{s(qu^{-1}v_j)s(qu^{-1}v_j^{-1})}{s(u^{-1}v_j)s(u^{-1}v_j^{-1})}\Bigg),$$

provided the spectral variables $v_1,\ldots,v_n$ *satisfy the following algebraic system of Bethe Ansatz equations:*

$$v_j^{4m+4} = \frac{e(v_j;a_+)e(v_j;a_-)}{e(v_j^{-1};a_+)e(v_j^{-1};a_-)}\prod_{\substack{1\leqslant k\leqslant n\\ k\neq j}}\frac{s(qv_jv_k)s(qv_jv_k^{-1})}{s(q^{-1}v_jv_k)s(q^{-1}v_jv_k^{-1})},$$

for $j=1,2,\ldots,n.$

V. Branching rule

From the explicit action of the Bethe Ansatz creation operator in Fock space, we compute a branching rule for the n-particle Bethe Ansatz wave function.

For $\lambda \in \Lambda_{n,m}$ and $\mu \in \Lambda_{n,m} \cup \Lambda_{n-1,m}$, let $\mu \leqslant \lambda$ if there exists a $\nu \in \Lambda_{n,m} \cup \Lambda_{n-1,m}$ such that $\mu \preceq \nu \preceq \lambda$.

Theorem (Branching Rule)

The value of the n-particle Bethe Ansatz wave function $\hat{\Psi}_{(\nu_1,\ldots,\nu_n)}$ at $\lambda \in \Lambda_{n,m}$ is determined by the following branching rule:

$$\hat{\Psi}_{(\nu_1,\ldots,\nu_n)}(\lambda) = \sum_{\substack{\mu \in \Lambda_{n-1,m} \\ \mu \leqslant \lambda}} B^{(n)}_{\lambda/\mu}(\nu_n^2; t, a_-)\hat{\Psi}_{(\nu_1,\ldots,\nu_{n-1})}(\mu),$$

with

$$B^{(n)}_{\lambda/\mu}(z; t, a) := \frac{(z^{-1} - a)}{(1-t)(z^{-1} - tz)} \sum_{\substack{\nu \in \Lambda_{n,m} \\ \mu \preceq \nu \preceq \lambda}} \varphi_{\lambda/\nu}(t)\varphi_{\nu/\mu}(t) z^{|\lambda|+|\mu|-2|\nu|}$$

$$+ \frac{(a - tz)}{(1-t)(z^{-1} - tz)} \sum_{\substack{\nu \in \Lambda_{n-1,m} \\ \mu \preceq \nu \preceq \lambda}} \varphi_{\lambda/\nu}(t)\varphi_{\nu/\mu}(t) z^{|\lambda|+|\mu|-2|\nu|}.$$

n-particle wave function

By iterating the branching rule, one ends up with a closed expression for the n-particle Bethe Ansatz wave function in terms one-particle wave functions.

Corollary (n-Particle Bethe Ansatz Wave Function)

One has that at $\lambda \in \Lambda_{n,m}$

$$\hat{\Psi}_{(\nu_1,\ldots,\nu_n)}(\lambda) = \sum_{\substack{\mu^{(j)} \in \Lambda_{j,m},\, j=1,\ldots,n \\ \mu^{(1)} \leqslant \mu^{(2)} \leqslant \cdots \leqslant \mu^{(n)} = \lambda}} \hat{\Psi}_{\nu_1}(\mu^{(1)}) \prod_{1<j\leqslant n} B^{(j)}_{\mu^{(j)}/\mu^{(j-1)}}(\nu_j^2; t, a_-).$$

One-Particle Bethe Ansatz Wave Function

By computing the action of the Bethe Ansatz creation operator on the vacuum state, we see that for $l \in \mathbb{N}_m$:

$$\hat{\Psi}_\nu(l) = s_l(\nu^2) - a_- s_{l-1}(\nu^2) \quad \text{with} \quad s_l(z) := \frac{z^{l+1} - z^{-l-1}}{z - z^{-1}}$$

(where $s_{-1}(z) = 0$ by convention).

Value at the origin

The value of the n-particle Bethe Ansatz wave function becomes particularly simple at the origin:

$$\hat{\Psi}_{(\nu_1,\ldots,\nu_n)}(0) = [n]!.$$

So $\hat{\Psi}_{(\nu_1,\ldots,\nu_n)} \not\equiv 0$.

Bethe Ansatz equations

Rewrite Bethe Ansatz equations by substituting $v_j = e^{i\xi_j/2}$:

$$e^{2i(m+1)\xi_j} = \left(\frac{1-a_+e^{i\xi_j}}{e^{i\xi_j}-a_+}\right)\left(\frac{1-a_-e^{i\xi_j}}{e^{i\xi_j}-a_-}\right)\prod_{\substack{1\leqslant k\leqslant n\\ k\neq j}}\left(\frac{1-te^{i(\xi_j+\xi_k)}}{e^{i(\xi_j+\xi_k)}-t}\right)\left(\frac{1-te^{i(\xi_j-\xi_k)}}{e^{i(\xi_j-\xi_k)}-t}\right),$$

$j = 1, \ldots, n$.

We seek solutions with ξ_j real, so v_j lies on the unit circle!

Spectral points

The solutions of the Bethe Ansatz equations are characterized via the minima of strictly convex Morse functions (Yang-Yang '69).

For any $\lambda \in \Lambda_{n,m}$, the (unique) global minimum $\xi_\lambda^{(n,m)} \in \mathbb{R}^n$ of the strictly convex Morse function $V_\lambda^{(n,m)} : \mathbb{R}^n \to \mathbb{R}$ given by

$$V_\lambda^{(n,m)}(\xi) := \sum_{1 \leqslant j \leqslant n} \left((m+1)\xi_j^2 - 2\pi(\rho_j + \lambda_j)\xi_j + \int_0^{\xi_j} \left(v_{a_+}(u) + v_{a_-}(u)\right) du \right) + \sum_{1 \leqslant j < k \leqslant n} \left(\int_0^{\xi_j + \xi_k} v_t(u) du + \int_0^{\xi_j - \xi_k} v_t(u) du \right),$$

with $\rho_j = n + 1 - j$ and

$$v_a(\theta) := \int_0^\theta \frac{(1-a^2)\, du}{1 - 2a\cos(u) + a^2} = i \log\left(\frac{1 - ae^{i\theta}}{e^{i\theta} - a} \right) \qquad (-1 < a < 1),$$

provides a solution of our Bethe Ansatz equations.

Theorem (Diagonalization and Orthogonality)

The n*-particle Bethe Ansatz wave functions* $\Psi^{(n,m)}(\xi_\lambda^{(n,m)})$, $\lambda \in \Lambda_{n,m}$*, which satisfy the eigenvalue equations*

$$\mathcal{T}_m(u; a_+, a_-)\Psi^{(n,m)}(\xi_\lambda^{(n,m)}) = E^{(n,m)}(u; \xi_\lambda^{(n,m)})\Psi^{(n,m)}(\xi_\lambda^{(n,m)})$$

with

$$E^{(n,m)}(u;\xi) := q^{-m}t^{-n} \times$$

$$\Big(u^{-2(m+2)}\frac{(1-t^{-1}u^4)}{(1-u^4)}(1-a_+u^2)(1-a_-u^2)\prod_{1\leqslant j\leqslant n}\frac{(1-tu^2e^{i\xi_j})(1-tu^2e^{-i\xi_j})}{(1-u^2e^{i\xi_j})(1-u^2e^{-i\xi_j})}+$$

$$u^{2(m+2)}\frac{(1-t^{-1}u^{-4})}{(1-u^{-4})}(1-a_+u^{-2})(1-a_-u^{-2})\prod_{1\leqslant j\leqslant n}\frac{(1-tu^{-2}e^{i\xi_j})(1-tu^{-2}e^{-i\xi_j})}{(1-u^{-2}e^{i\xi_j})(1-u^{-2}e^{-i\xi_j})}\Big),$$

and

$$\mathcal{H}_m\Psi^{(n,m)}(\xi_\lambda^{(n,m)}) = E^{(n)}(\xi_\lambda^{(n,m)})\Psi^{(n,m)}(\xi_\lambda^{(n,m)})$$

with

$$E^{(n)}(\xi) := \sum_{1\leqslant j\leqslant n} 2\cos(\xi_j),$$

constitute an orthogonal basis for $\ell^2(\Lambda_{n,m}, \delta_{n,m})$*:*

$$\Big(\Psi^{(n,m)}(\xi_\lambda^{(n,m)}), \Psi^{(n,m)}(\xi_\mu^{(n,m)})\Big)_{n,m} = 0 \quad \textit{iff } \lambda \neq \mu$$

($\lambda, \mu \in \Lambda_{n,m}$*).*

Proof Theorem follows from three observations:

— $\mathcal{T}_m(u; a_+, a_-)$ is self-adjoint in $\ell^2(\Lambda_{n,m}, \delta_{n,m})$.

—The Bethe Ansatz spectrum is nondegenerate:

$$E^{(n,m)}(u; \xi_\lambda^{(n,m)}) \neq E^{(n,m)}(u; \xi_\mu^{(n,m)}) \quad \text{if} \quad \lambda \neq \mu$$

(as Laurent polynomials in u).

— The Bethe Ansatz eigenfunctions are nontrivial:

$$\Psi^{(n,m)}(\xi_\lambda^{(n,m)}) \not\equiv 0.$$

Macdonald's formula

Macdonald's BC_n-type Hall-Littlewood polynomials are given by: (Macdonald '89)

$$P_\lambda(z_1,\dots,z_n;t,a,\hat{a}) = \sum_{\substack{\sigma\in S_n\\ \epsilon\in\{1,-1\}^n}} C(z_{\sigma_1}^{\epsilon_1},\dots,z_{\sigma_n}^{\epsilon_n};t,a,\hat{a})\, z_{\sigma_1}^{\epsilon_1\lambda_1}\cdots z_{\sigma_n}^{\epsilon_n\lambda_n},$$

with

$$C(z_1,\dots,z_n;t,a,\hat{a}) = \prod_{1\leqslant j\leqslant n}\frac{(z_j-a)(z_j-\hat{a})}{z_j^2-1}\prod_{1\leqslant j<k\leqslant n}\left(\frac{z_jz_k-t}{z_jz_k-1}\right)\left(\frac{z_jz_k^{-1}-t}{z_jz_k^{-1}-1}\right).$$

$P_\lambda(z_1,\dots,z_n;t,a,\hat{a})$ is a symmetric Laurent polynomial in $z_1,\dots,z_n$.

Coordinate representation of the n-particle wave function

By comparing our branching formula with a recent branching rule for Macdonald's hyperoctahedral Hall-Littlewood polynomials at $\hat{a} = 0$ (Wheeler–P.Zinn-Justin '15):

$$
\begin{aligned}
\hat{\Psi}_{(v_1,\ldots,v_n)}(\lambda) &= P_\lambda(v_1^2,\ldots,v_n^2;t,a_-,0) \\
&= \sum_{\substack{\sigma\in S_n \\ \epsilon\in\{1,-1\}^n}} C(v_{\sigma_1}^{2\epsilon_1},\ldots,v_{\sigma_n}^{2\epsilon_n};t,a_-,0)\, v_{\sigma_1}^{2\epsilon_1\lambda_1}\cdots v_{\sigma_n}^{2\epsilon_n\lambda_n}
\end{aligned}
$$

($\lambda \in \Lambda_{n,m}$).

cf. Tsilevich '05, vD '06, Korff '13 for periodic q-boson Bethe Ansatz wave functions in terms of Hall-Littlewood polynomials.

Affine Pieri formula and discrete orthogonality

For $\lambda, \mu \in \Lambda_{n,m}$, let $(z_1, \ldots, z_n)$ and $(y_1, \ldots, y_n)$ be equal to $(e^{i\xi_1}, \ldots, e^{i\xi_n})$ at $\xi = \xi_\lambda^{(n,m)}$ and $\xi = \xi_\mu^{(n,m)}$, respectively. Then:

$$
\begin{aligned}
P_\nu(z_1, \ldots, z_n; t, a_-, 0) \sum_{1 \leqslant j \leqslant n} \left(z_j + z_j^{-1} \right) = \\
& \left(a_-[m_0(\nu)] + a_+[m_m(\nu)] \right) P_\nu(z_1, \ldots, z_n; t, a_-, 0) \\
& + \sum_{\substack{1 \leqslant j \leqslant n \\ \nu \pm e_j \in \Lambda_{n,m}}} [m_{\nu_j}(\nu)] P_{\nu \pm e_j}(z_1, \ldots, z_n; t, a_-, 0)
\end{aligned}
$$

$(\nu \in \Lambda_{n,m})$, and

$$
\sum_{\nu \in \Lambda_{n,m}} P_\nu(z_1, \ldots, z_n; t, a_-, 0) \overline{P_\nu(y_1, \ldots, y_n; t, a_-, 0)} \delta_{n,m}(\nu) = 0 \quad \text{iff } \lambda \neq \mu.
$$

Important Feature: Orthogonality of the Bethe Ansatz for the Laplacian on the Weyl alcove follows from this orthogonality through a continuum limit!

System of affine Pieri formulas

From the explicit action of the transfer operator, we arrive at a system of affine Pieri formulas for Macdonald's BC_n Hall-Littlewood polynomials with $\hat{a} = 0$ and $\nu \in \Lambda_{n,m}$:

$$
\begin{aligned}
&P_\nu(z_1, \ldots, z_n; t, a_-, 0) E^{(n,m)}(u; z_1, \ldots, z_n) = \\
&q^{-m} t^{-n-1} (a_+ - t u^{-2}) \sum_{\substack{\mu \in \Lambda_{n,m} \\ \mu \sim_- \nu}} A^{(n)}_{\nu,\mu}(u^2; t, a_-) P_\mu(z_1, \ldots, z_n; t, a_-, 0) \\
&+ q^{-m} t^{-n-1} (a_+ - u^2) \sum_{\substack{\mu \in \Lambda_{n,m} \\ \mu \sim_+ \nu}} D^{(n)}_{\nu,\mu}(u^2; t, a_-) P_\mu(z_1, \ldots, z_n; t, a_-, 0)
\end{aligned}
$$

at $(z_1, \ldots, z_n) = (e^{i\xi_1}, \ldots, e^{i\xi_n})$ with $\xi = \xi^{(n,m)}_\lambda$ $(\lambda \in \Lambda_{n,m})$.

Thank You!