

Non-intersecting Brownian motions: from Yang-Mills theory to KPZ

Grégory Schehr

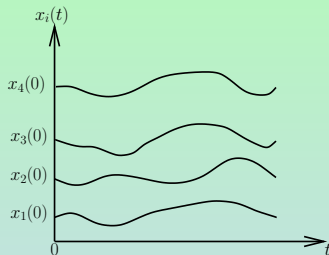
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Non-intersecting Brownian motions

- N non intersecting Brownian motions in one-dimension

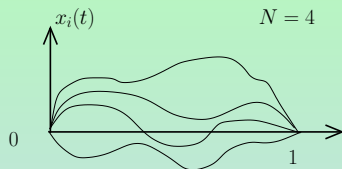
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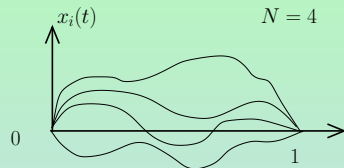
watermelons

Non-intersecting Brownian motions

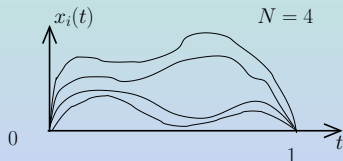
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Tracy & Widom '07

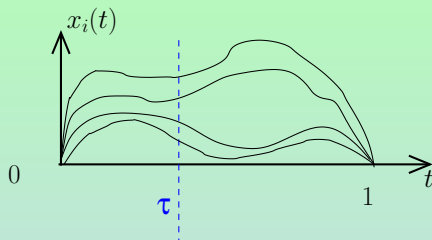


watermelons



watermelons "with a wall"

Non-intersecting excursions and RMT



- Joint probability of $x_1(\tau), x_2(\tau), \dots, x_N(\tau)$ at fixed time τ

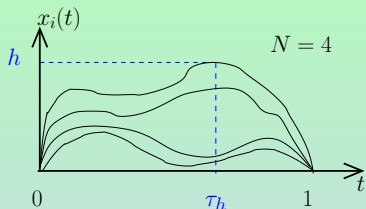
$$P_{\text{joint}}(\mathbf{x}, \tau) \propto \prod_{i=1}^N x_i^2 \prod_{1 \leq i < j \leq N} (x_i^2 - x_j^2)^2 e^{-\frac{1}{\sigma^2(\tau)} \sum_{i=1}^N x_i^2}$$
$$\sigma(\tau) = \sqrt{2\tau(1-\tau)}$$

- The rescaled positions $\frac{x_i}{\sigma(\tau)}$ are distributed like the **eigenvalues** of random matrices of the **Bogoliubov-de Gennes** type (class C)

A. Atland, M. R. Zirnbauer '96



Extremes of N non-intersecting excursions



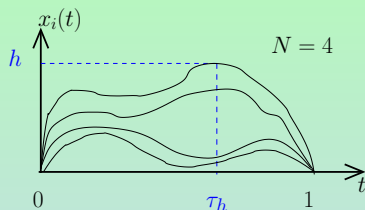
$$0 < x_1(t) < x_2(t) < \dots < x_N(t)$$

$$\mathcal{H}_N = \max_{t \in [0,1]} x_N(t)$$

$$\mathcal{T}_\mathcal{H} = \arg \max_{t \in [0,1]} x_N(t)$$

- 1 Distribution of the maximum $F_N(h) = \mathbb{P}(\mathcal{H}_N \leq h)$
- 2 Joint probability density function of $\mathcal{H}_N, \mathcal{T}_\mathcal{H}$: $P_N(h, \tau_h)$

Extremes of N non-intersecting excursions



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In this talk :

- Exact computation for $F_N(h)$ and $P_N(h, \tau_h)$ for all N
- Large N asymptotics (**typical** and **rare** fluctuations)

Outline

- 1 PDF of the maximum, Yang-Mills theory and 3rd order transition
- 2 Applications to directed polymers in random media
- 3 Relation with determinantal formulas from Airy_2 process
- 4 Conclusion

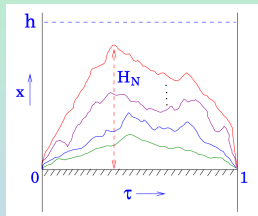
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An exact expression for $F_N(h)$

- Distribution of the maximal height \mathcal{H}_N

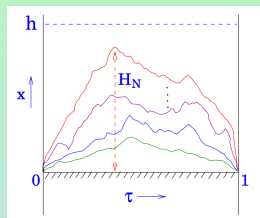
$$F_N(h) = \mathbb{P} [x_N(\tau) \leq h, \forall 0 \leq \tau \leq 1]$$



An exact expression for $F_N(h)$

- Distribution of the maximal height \mathcal{H}_N

$$F_N(h) = \mathbb{P} [X_N(\tau) \leq h, \forall 0 \leq \tau \leq 1]$$



- Exact result for finite N G. S., S. N. Majumdar, A. Comtet, J. Randon-Furling '08

$$F_N(h) = \frac{A_N}{h^{2N^2+N}} \sum_{n_1, \dots, n_N=0}^{+\infty} \prod_{i=1}^N n_i^2 \prod_{1 \leq j < k \leq N} (n_j^2 - n_k^2)^2 e^{-\frac{\pi^2}{2h^2} \sum_{i=1}^N n_i^2}$$

$$A_N = \frac{\pi^{2N^2+N}}{2^{N^2+\frac{N}{2}} \prod_{j=0}^{N-1} \Gamma(2+j) \Gamma(\frac{3}{2}+j)}$$

see also T. Feierl, M. Katori *et al.* '08

Connection to Yang-Mills theory in 2d

- Partition function of Yang-Mills theory on a $2d$ manifold \mathcal{M} with a gauge group G , described by a gauge field $A_\mu(x) \equiv A_\mu^a(x) T^a$, $\mu \in \{1, 2\}$

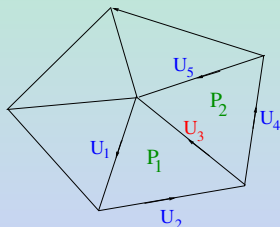
$$\mathcal{Z}_{\mathcal{M}} = \int [\mathcal{D}A_\mu] e^{-\frac{1}{4\lambda^2} \int \text{Tr}[F^{\mu\nu} F_{\mu\nu}] \sqrt{g} d^2x}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$$

- Regularization on the lattice Migdal '75

$$\mathcal{Z}_{\mathcal{M}} = \int \prod_L dU_L \prod_{\text{plaquettes}} Z_P[U_P]$$

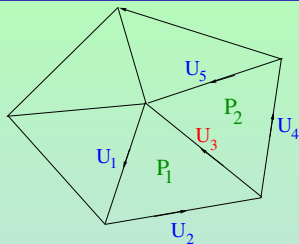
$$U_P = \prod_{L \in \text{plaquette } P} U_L$$



Heat-kernel action

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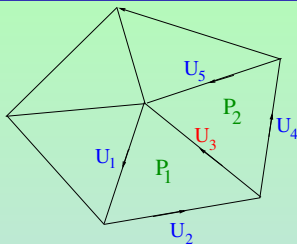
- A common choice : **Wilson's action** Wilson'74, Gross & Witten '80, Wadia '80

$$Z_P(U_P) = \exp \left[b N \text{Tr}(U_P + U_P^\dagger) \right]$$

Heat-kernel action

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$$Z_P(U_P) = \exp \left[b N \text{Tr}(U_P + U_P^\dagger) \right]$$

- Alternative choice : invariance under decimation \Rightarrow **Migdal's recursion relation**

$$\int dU_3 Z_{P_1}(U_1 U_2 U_3) Z_{P_2}(U_4 U_5 U_3^\dagger) = Z_{P_1+P_2}(U_1 U_2 U_4 U_5)$$

$$Z_P(U_P) = \sum_R d_R \chi_R(U_P) \exp \left[-\frac{A_P}{2N} C_2(R) \right]$$

Migdal'75, Rusakov'90

Partition function of Yang-Mills theory on the $2d$ -sphere

- Partition funct^o on \mathcal{M} , of genus g , computed with the heat-kernel action

$$\mathcal{Z}_{\mathcal{M}} = \sum_R d_R^{2-2g} \exp \left[-\frac{A}{2N} C_2(R) \right]$$

Partition function of Yang-Mills theory on the $2d$ -sphere

- Partition funct^o on the **sphere** computed with the heat-kernel action

$$Z_{\mathcal{M}} = \sum_R d_R^2 \exp \left[-\frac{A}{2N} C_2(R) \right]$$

Partition function of Yang-Mills theory on the $2d$ -sphere

- Partition function^o on the **sphere** computed with the heat-kernel action

$$\mathcal{Z}_{\mathcal{M}} = \sum_R d_R^2 \exp \left[-\frac{A}{2N} C_2(R) \right]$$

- Irreducible representations R of G are labelled by the lengths of the Young diagrams:

- If $G = U(N)$ Gross '93, Gross & Matytsin '94

$$\mathcal{Z}_{\mathcal{M}} = c_N e^{-A \frac{N^2-1}{24}} \sum_{n_1, \dots, n_N=0}^{\infty} \prod_{i < j} (n_i - n_j)^2 e^{-\frac{A}{2N} \sum_{j=1}^N n_j^2}$$

- If $G = Sp(2N)$ Crescimanno & Schnitzer '96

$$\mathcal{Z}_{\mathcal{M}} = \hat{c}_N e^{A(N+\frac{1}{2})\frac{N+1}{12}} \sum_{n_1, \dots, n_N=0}^{\infty} \left(\prod_{j=1}^N n_j^2 \right) \prod_{i < j} (n_i^2 - n_j^2)^2 e^{-\frac{A}{4N} \sum_{j=1}^N n_j^2}$$

Correspondence between YM_2 on the sphere and watermelons

- Partition function of YM_2 on the sphere with gauge group $Sp(2N)$

$$\mathcal{Z}_{\mathcal{M}} = \mathcal{Z}(A; Sp(2N))$$

$$\mathcal{Z}(A; Sp(2N)) = \hat{c}_N e^{A(N+\frac{1}{2})\frac{N+1}{12}} \sum_{n_1, \dots, n_N=0}^{\infty} \left(\prod_{j=1}^N n_j^2 \right) \prod_{i<j} (n_i^2 - n_j^2)^2 e^{-\frac{A}{4N} \sum_{j=1}^N n_j^2}$$

- Cumulative distribution of the maximal height of watermelons with a wall

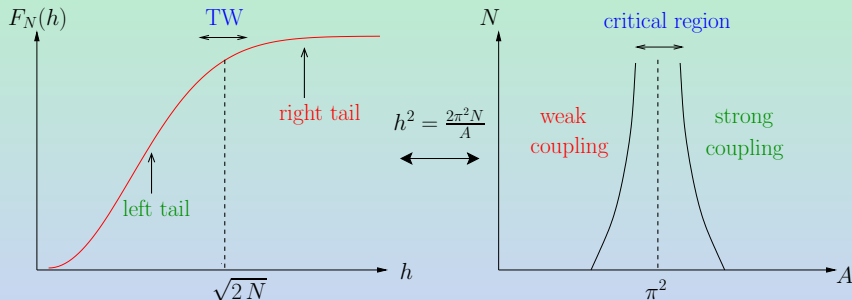
$$F_N(h) = \frac{A_N}{h^{2N^2+N}} \sum_{n_1, \dots, n_N=0}^{+\infty} \left(\prod_{j=1}^N n_j^2 \right) \prod_{i<j} (n_i^2 - n_j^2)^2 e^{-\frac{\pi^2}{2h^2} \sum_{j=1}^N n_j^2}$$

$$\propto \mathcal{Z} \left(A = \frac{2\pi^2 N}{h^2}; Sp(2N) \right)$$

P. J. Forrester, S. N. Majumdar, G. S. '11

Large N limit of YM_2 and consequences for $F_N(M)$

- Weak-strong coupling transition in YM_2 Douglas-Kazakov '93



Large N asymptotics: third order phase transition

- Third-order phase transition

$$\lim_{N \rightarrow \infty} -\frac{1}{N^2} \ln F_N(\tilde{h} \sqrt{2N}) = \begin{cases} \phi^-(\tilde{h}), & \tilde{h} < 1, \\ 0, & \tilde{h} \geq 1 \end{cases}, \quad \phi^-(\tilde{h}) \sim \frac{16}{3}(1 - \tilde{h})^3,$$

\implies the **third** derivative of the partition function is discontinuous at $\tilde{h} = 1$

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- A similar third order phase transition occurs for **Gaussian β -ensembles in presence of a wall** for a review see S. N. Majumdar, G. S. '14

$$Z_N(w) \propto \int_{-\infty}^w d\lambda_1 \cdots \int_{-\infty}^w d\lambda_N \exp \left[-\frac{\beta}{2} \left(N \sum_{i=1}^N \lambda_i^2 - \sum_{i \neq j} \ln |\lambda_i - \lambda_j| \right) \right]$$

exhibits a **third order phase transition** at the critical value $w_c = \sqrt{2}$

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- A similar third order transition occurs in the KPZ universality class
P. Le Doussal, S. N. Majumdar, G. S. '16

Discrete orthogonal polynomials

$$F_N(h) = \frac{A_N}{h^{2N^2+N}} \sum_{n_1, \dots, n_N=0}^{+\infty} \prod_{i=1}^N n_i^2 \prod_{1 \leq j < k \leq N} (n_j^2 - n_k^2)^2 e^{-\frac{\pi^2}{2h^2} \sum_{i=1}^N n_i^2}$$

Introduce discrete orthogonal polynomials

Gross & Matytsin '94

$$\sum_{n=-\infty}^{\infty} p_k(n) p_{k'}(n) e^{-\frac{\pi^2}{2h^2} n^2} = \delta_{k,k'} h_k, \quad F_N(h) = \frac{\tilde{A}_N}{h^{2N^2+N}} \prod_{j=1}^N h_{2j-1}$$

Large N analysis of the three terms recursion relation

$$x p_k(x) = p_{k+1}(x) + R_k p_{k-1}(x), \quad R_k = \frac{h_k}{h_{k-1}}$$

Large N asymptotics: main results

P. J. Forrester, S. N. Majumdar, G. S. '11

G. S., S. N. Majumdar, A. Comtet, P. J. Forrester '12

$$\left\{ F_N(h) \sim \exp \left[-N^2 \phi_- \left(h/\sqrt{2N} \right) \right], \quad h < \sqrt{2N} \text{ \& } |h - \sqrt{2N}| \sim \mathcal{O}(\sqrt{N}) \right.$$

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Large N asymptotics: main results

P. J. Forrester, S. N. Majumdar, G. S. '11

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\mathcal{F}_1 is the Tracy-Widom distribution for GOE

$$\mathcal{F}_1 = \exp \left(-\frac{1}{2} \int_t^\infty ((s-t)q^2(s) + q(s)) ds \right)$$

obtained here via a **double scaling** analysis of the discrete OP system

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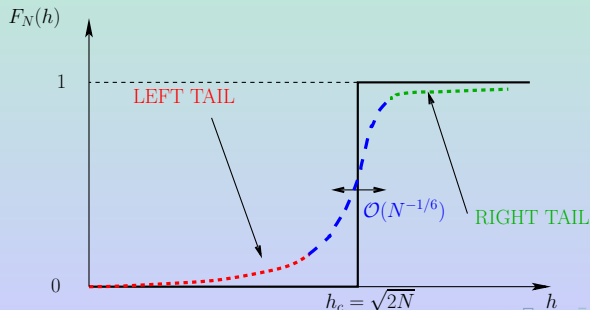
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see K. Liechty '12, K. Liechty & D. Wang '13 for a rigorous proof



Typical and rare fluctuations

$$\left\{ \begin{array}{l} F_N(h) \sim \exp \left[-N^2 \phi_- \left(h/\sqrt{2N} \right) \right], \quad h < \sqrt{2N} \text{ \& } |h - \sqrt{2N}| \sim \mathcal{O}(\sqrt{N}) \\ F_N(h) \sim \mathcal{F}_1 \left[2^{11/6} N^{1/6} (h - \sqrt{2N}) \right], \quad h \sim \sqrt{2N} \text{ \& } |h - \sqrt{2N}| \sim \mathcal{O}(N^{-1/6}) \\ 1 - F_N(h) \sim \exp \left[-N \phi_+ \left(h/\sqrt{2N} \right) \right], \quad h > \sqrt{2N} \text{ \& } |h - \sqrt{2N}| \sim \mathcal{O}(\sqrt{N}) \end{array} \right.$$

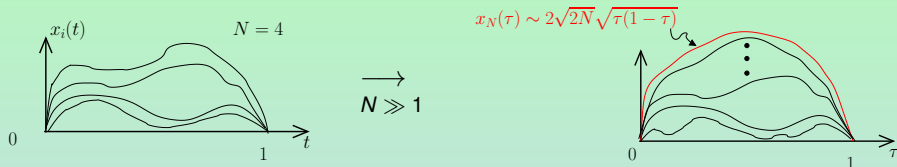


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Non-intersecting excursions and Airy_2 process

- N non-intersecting excursions for large N



- **Typical fluctuations** of the top path are related to **Airy_2 process** minus a parabola

$$\lim_{N \rightarrow \infty} \frac{\alpha \left[x_N\left(\frac{1}{2} + \beta u N^{-\frac{1}{3}}\right) - \sqrt{2N} \right]}{N^{-\frac{1}{6}}} \stackrel{d}{=} \mathcal{A}_2(u) - u^2$$

Prähofer & Spohn '02

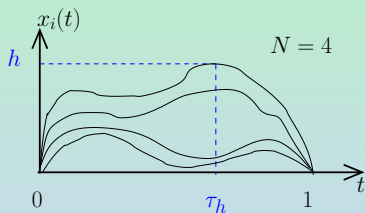
Tracy & Widom '07

$\mathcal{A}_2(u) \equiv \text{Airy}_2$ process

Extreme statistics of Airy_2 process minus a parabola

$$\mathcal{M} = \max_{u \in \mathbb{R}} \mathcal{A}_2(u) - u^2, \quad \mathcal{T} = \arg \max_{u \in \mathbb{R}} \mathcal{A}_2(u) - u^2$$

Johansson '03, Corwin & Hammond '11, Moreno-Flores, Quastel & Remenik '13



$$0 < x_1(t) < x_2(t) < \dots < x_N(t)$$

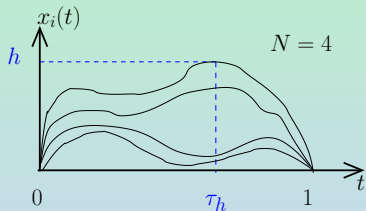
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$$0 < x_1(t) < x_2(t) < \dots < x_N(t)$$

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$$\mathcal{T}_{\mathcal{H}} = \arg \max_{t \in [0,1]} x_N(t)$$

$$\lim_{N \rightarrow \infty} 2^{2/3} N^{1/6} (\mathcal{H}_N - \sqrt{2N}) \stackrel{d}{=} \mathcal{M}$$

$$\lim_{N \rightarrow \infty} 2^{4/3} N^{1/3} (\mathcal{T}_{\mathcal{H}} - 1/2) \stackrel{d}{=} \mathcal{T}$$

Extreme statistics of Airy_2 process minus a parabola

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- Our result for the PDF of the maximum on non-intersecting excursion yields

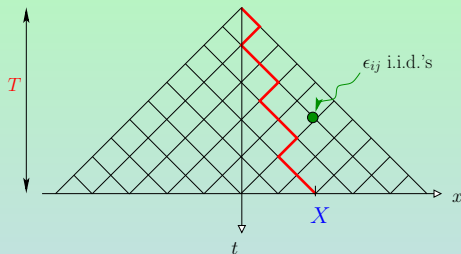
$$\mathbb{P}(\mathcal{M} \leq m) = \mathcal{F}_1(2^{2/3}m)$$

Johansson '03

Corwin, Quastel, Remenik '12

Relation to Directed Polymers in Random Media

- DPRM with one free end ("point to line")



$$\mathbb{E}(E_{\text{opt}}) \sim aT, \quad \mathbb{E}(X) = 0$$

$$E_{\text{opt}} - \mathbb{E}(E_{\text{opt}}) \sim \mathcal{O}(T^{1/3})$$

$$X \sim \mathcal{O}(T^{2/3})$$

Q: what is the joint pdf of E_{opt}, X ?

- E_{opt} \equiv Energy of the optimal polymer
- X \equiv Transverse coordinate of the optimal polymer

\implies Related to KPZ growth

DPRM and the Airy_2 process minus a parabola

- The " Airy_2 process minus a parabola"

$$Y(u) = \mathcal{A}_2(u) - u^2$$

where $\mathcal{A}_2(u)$ is the Airy_2 process Prähofer & Spohn '02

- Fluctuations in the DPRM

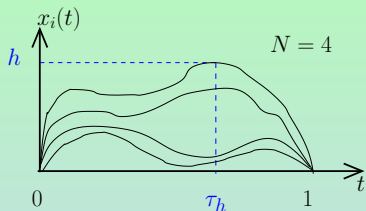
$$\lim_{T \rightarrow \infty} \frac{T^{-\frac{1}{3}}}{e_0} (E_{\text{opt}}(T) - \mathbb{E}(E_{\text{opt}}(T))) = \max_{u \in \mathbb{R}} Y(u) = \mathcal{M}$$

and

$$\lim_{T \rightarrow \infty} \frac{T^{-\frac{2}{3}}}{\xi} X = \arg \max_{u \in \mathbb{R}} Y(u) = \mathcal{T}$$

Johansson '03, Corwin & Hammond '11, Moreno-Flores, Quastel & Remenik '13

Extremes of N non-intersecting excursions



$$0 < x_1(t) < x_2(t) < \dots < x_N(t)$$

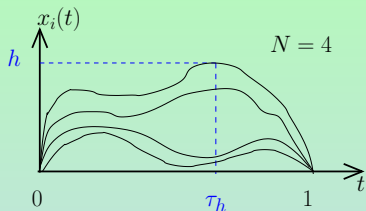
$$\mathcal{H}_N = \max_{t \in [0,1]} x_N(t)$$

$$\mathcal{T}_H = \arg \max_{t \in [0,1]} x_N(t)$$

$P_N(h, \tau_h) \equiv$ joint probability distribution function of $\mathcal{H}_N, \mathcal{T}_H$

HERE :

- Compute exactly $P_N(h, \tau_h)$ for any finite N
- Typical fluctuations of $\mathcal{H}_N, \mathcal{T}_H$ when $N \rightarrow \infty$ yield the statistics of the extremes of $\mathcal{A}_2(u) - u^2$

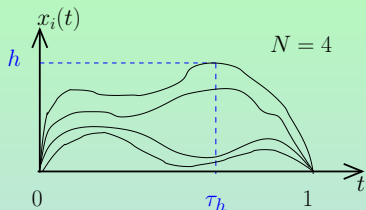


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$$\mathcal{T}_\mathcal{H} = \arg \max_{t \in [0,1]} x_N(t)$$

$$P_N(h, \tau_h) = \frac{B_N}{h^{N(2N+1)+3}} \sum_{(n_1, \dots, n_N, n'_N) \in \mathbb{Z}^{N+1}} \left[(-1)^{n_N+n'_N} n_N^2 n'_N{}^2 \prod_{i=1}^{N-1} n_i^2 \Delta_N(n_1^2, \dots, n_{N-1}^2, n_N^2) \right. \\ \left. \times \Delta_N(n_1^2, \dots, n_{N-1}^2, n'_N{}^2) e^{-\frac{\pi^2}{2h^2} \sum_{i=1}^{N-1} n_i^2 - \frac{\pi^2}{2h^2} [(1-\tau_h)n'_N{}^2 + \tau_h n_N^2]} \right]$$



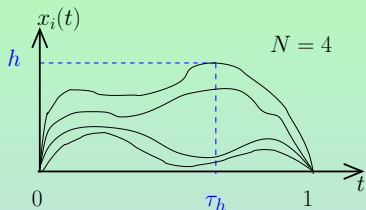
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$$B_N = \frac{N \pi^{2N^2+N+2}}{2^{N^2+N/2+1} \prod_{j=0}^{N-1} \Gamma(2+j) \Gamma\left(\frac{3}{2}+j\right)}$$



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Large N limit ?

Typical fluctuations for large N

$$P_N(h, \tau_h) \propto \sum_{(n_1, \dots, n_N, n'_N) \in \mathbb{Z}^{N+1}} \left[(-1)^{n_N + n'_N} n_N^2 n'_N{}^2 \prod_{i=1}^{N-1} n_i^2 \Delta_N(n_1^2, \dots, n_{N-1}^2, n_N^2) \right. \\ \left. \times \Delta_N(n_1^2, \dots, n_{N-1}^2, n'_N{}^2) e^{-\frac{\pi^2}{2h^2} \sum_{i=1}^{N-1} n_i^2 - \frac{\pi^2}{2h^2} [(1-\tau_h)n'_N{}^2 + \tau_h n_N^2]} \right]$$

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$$P_N(h, \tau h) \propto \sum_{(n_1, \dots, n_N, n'_N) \in \mathbb{Z}^{N+1}} \left[(-1)^{n_N + n'_N} n_N^2 n'^2_N \prod_{i=1}^{N-1} n_i^2 \Delta_N(n_1^2, \dots, n_{N-1}^2, n_N^2) \right. \\ \left. \times \Delta_N(n_1^2, \dots, n_{N-1}^2, n'^2_N) e^{-\frac{\pi^2}{2h^2} \sum_{i=1}^{N-1} n_i^2 - \frac{\pi^2}{2h^2} [(1-\tau h)n_N^2 + \tau h n'^2_N]} \right]$$

- Discrete orthogonal polynomials

$$\sum_{n=-\infty}^{\infty} p_k(n) p_{k'}(n) e^{-\frac{\pi^2}{2h^2} n^2} = \delta_{k,k'} h_k, \quad p_k(n) = n^k + \dots$$

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- After some manipulations G. S. '12

$$P_N(h, \tau_h) \propto \underbrace{\prod_{j=1}^N h_{2j-1}} \sum_{n,m} (-1)^{n+m} n m \sum_{k=1}^N \frac{p_{2k-1}(n) p_{2k-1}(m)}{h_{2k-1}} e^{-\frac{\pi^2}{2h^2} [(1-\tau_h)n^2 + \tau_h m^2]}$$

$$\mathbb{P}[\mathcal{H}_N \leq h] = F_N(h)$$

- Large N asymptotics for extremes of excursions

$$\lim_{N \rightarrow \infty} 2^{-\frac{9}{2}} N^{-\frac{1}{2}} P_N \left(\sqrt{2N} + 2^{-\frac{11}{6}} s N^{-\frac{1}{6}}, \frac{1}{2} + 2^{-\frac{8}{3}} w N^{-\frac{1}{3}} \right) = P(s, w)$$

$$P(s, w) = \frac{4}{\pi^2} \mathcal{F}_1(s) \int_s^\infty f(x, w) f(x, -w) dx$$

$$f(x, w) = \int_0^\infty \zeta \Phi_2(\zeta, x) e^{-w\zeta^2} d\zeta$$

$$\underbrace{\frac{\partial}{\partial \zeta} \Psi = A\Psi, \quad \frac{\partial}{\partial x} \Psi = B\Psi}_{\text{Lax Pair}}, \quad \Psi = \begin{pmatrix} \Phi_1(\zeta, x) \\ \Phi_2(\zeta, x) \end{pmatrix}, \quad \begin{cases} \Phi_1(\zeta, x) = \cos\left(\frac{4}{3}\zeta^3 + x\zeta\right) + \mathcal{O}(\zeta^{-1}) \\ \Phi_2(\zeta, x) = -\sin\left(\frac{4}{3}\zeta^3 + x\zeta\right) + \mathcal{O}(\zeta^{-1}) \end{cases}$$

Lax Pair

$$A(\zeta, x) = \begin{pmatrix} 4\zeta q & 4\zeta^2 + x + 2q^2 + 2q' \\ -4\zeta^2 - x - 2q^2 + 2q' & -4\zeta q \end{pmatrix}, \quad B(\zeta, x) = \begin{pmatrix} q & \zeta \\ -\zeta & -q \end{pmatrix}$$

Extreme statistics of Airy_2 process minus a parabola

- **This work:** joint pdf $\hat{P}(m, t)$ of \mathcal{M}, \mathcal{T} G. S. '12

$$\hat{P}(m, t) = \frac{8}{\pi^2} \mathcal{F}_1(2^{2/3}m) \int_{2^{2/3}m}^{\infty} f(x, 2^{4/3}t) f(x, -2^{4/3}t) dx$$

- Marginal distribution $\hat{P}(t)$ of \mathcal{T} G. S. '12

$$\log \hat{P}(t) = -ct^3 + o(t^3), \quad t \rightarrow \infty \text{ with } c = \frac{4}{3}$$

see also Quastel & Remenik '13

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A more refined asymptotic analysis Bothner & Liechty '13

$$\hat{P}(t) = C e^{-\frac{4}{3}\varphi(t)} t^{-\frac{81}{32}} (1 + \mathcal{O}(t^{-\frac{3}{4}})), \quad \varphi(t) = t^3 - 2t^{\frac{3}{2}} + 3t^{\frac{3}{4}}$$

Outline

- 1 PDF of the maximum, Yang-Mills theory and 3rd order transition
- 2 Applications to directed polymers in random media
- 3 Relation with determinantal formulas from Airy_2 process**
- 4 Conclusion

Extreme statistics of Airy_2 process minus a parabola

$$\mathcal{M} = \max_{u \in \mathbb{R}} \mathcal{A}_2(u) - u^2, \quad \mathcal{T} = \arg \max_{u \in \mathbb{R}} \mathcal{A}_2(u) - u^2$$

- Our results also yield the joint pdf of \mathcal{M}, \mathcal{T} as

$$\hat{P}(m, t) = 4P(2^{2/3}m, 2^{4/3}t)$$

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$$f(x, w) = \int_0^\infty \zeta \Phi_2(\zeta, x) e^{-w\zeta^2} d\zeta$$

Φ_2 is a ψ -function associated to Painlevé II

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- A **different** formula was obtained first by Moreno-Flores, Quastel, Remenik [arXiv:1106.2716](https://arxiv.org/abs/1106.2716), CMP '13

$$\hat{P}(m, t) = 2^{1/3} \mathcal{F}_1(2^{2/3} m) \int_0^\infty dx \int_0^\infty dy \Phi_{-t, m}(2^{1/3} x) \rho_{2^{2/3} m}(x, y) \Phi_{t, m}(2^{1/3} y)$$

$$\Phi_{t, m}(x) = 2e^{x^2} [t \text{Ai}(t^2 + m + x) + \text{Ai}'(t^2 + m + x)]$$

and

$$\rho_m(x, y) = (I - \Pi_0 \mathbf{B}_m \Pi_0)^{-1}(x, y), \quad \mathbf{B}_m(x, y) = \text{Ai}(x + y + m)$$

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Show that the two formulas coincide !!

Solution to the Lax pair

$$\underbrace{\frac{\partial}{\partial \zeta} \Psi = A\Psi, \frac{\partial}{\partial x} \Psi = B\Psi}_{\text{Lax Pair}}, \Psi = \begin{pmatrix} \Phi_1(\zeta, x) \\ \Phi_2(\zeta, x) \end{pmatrix}, \begin{cases} \Phi_1(\zeta, x) = \cos\left(\frac{4}{3}\zeta^3 + x\zeta\right) + \mathcal{O}(\zeta^{-1}) \\ \Phi_2(\zeta, x) = -\sin\left(\frac{4}{3}\zeta^3 + x\zeta\right) + \mathcal{O}(\zeta^{-1}) \end{cases}$$

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An explicit expression of Φ_1, Φ_2 in terms of $\mathbf{B}_m(x, y) = \text{Ai}(x + y + m)$

J. Baik, K. Liechty, G. S. '12

$$\begin{aligned} \mathbf{A}_s(x, y) &:= \mathbf{B}_s^2(x, y) = \int_0^\infty \text{Ai}(x + s + \xi) \text{Ai}(y + s + \xi) d\xi \\ &= \frac{\text{Ai}(x + s) \text{Ai}'(y + s) - \text{Ai}'(x + s) \text{Ai}(y + s)}{x - y} \end{aligned}$$

Define the functions Q and R as

$$Q := (\mathbf{1} - \mathbf{A}_s)^{-1} \mathbf{B}_s \delta_0, \quad R := (\mathbf{1} - \mathbf{A}_s)^{-1} \mathbf{A}_s \delta_0$$

Introduce the functions

$$\Theta_1(x) := \cos \left(\frac{4}{3} \zeta^3 + (s + 2x) \zeta \right), \quad \Theta_2(x) := -\sin \left(\frac{4}{3} \zeta^3 + (s + 2x) \zeta \right)$$

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Proposition

$$\Phi_1(\zeta, s) = \Theta_1(0) + \langle \Theta_1, R - Q \rangle_0, \quad \Phi_2(\zeta, s) = \Theta_2(0) + \langle \Theta_2, R + Q \rangle_0,$$

where $\langle \cdot, \cdot \rangle_0$ is the inner product on $L^2[0, \infty)$

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- An indirect proof of this proposition was obtained by Baik '06
- After further manipulations, this prop. allows to show that the two formulas for the jpdf $\hat{P}(m, t)$ do coincide

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Remark: yet another expression for $\hat{P}(m, t)$ was found by Dotsenko '12. The equivalence with our formula was established by Bothner & Liechty '12

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Conclusion

- Exact results for extreme statistics of N vicious walkers
- Connection with Yang-Mills theory
- Large N analysis: typical and large fluctuations
- Connection between extreme statistics of $\mathcal{A}_2(u) - u^2$ and Painlevé
- An explicit solution for the Lax pair associated to Painlevé II