Quantum Group Symmetry Breaking
and
Bose Condensation in Nonabelian Hall States

Joost Slingerland, Microsoft Station Q, May 2006

Joint work with Sander Bais

- Bais, Schroers, JKS. PRL 89:18601, 2002
- Bais, Schroers, JKS. JHEP 05:068, 2003
- Bais, Mathy, cond-mat/0602101, cond-mat/0602109, cond-mat/0602115

Topological Symmetry Breaking and Bose Condensation

- Can describe topological order by extended “symmetry” concepts: TQFTs, Tensor Categories, Hopf Algebras, Quantum Groups

Particle types	⟷	Irreducible representations
Fusion	⟷	Tensor Product
Braiding	⟷	R-matrix
Twist	⟷	Ribbon Element

- IDEA:

  Relate topological phases by “Symmetry Breaking”

- Mechanism? Bose Condensation!

  Break the Quantum Group to the “Stabiliser” of the condensate’s order parameter

Example: Z_N Gauge Theory/ Toric Code

Particles are:

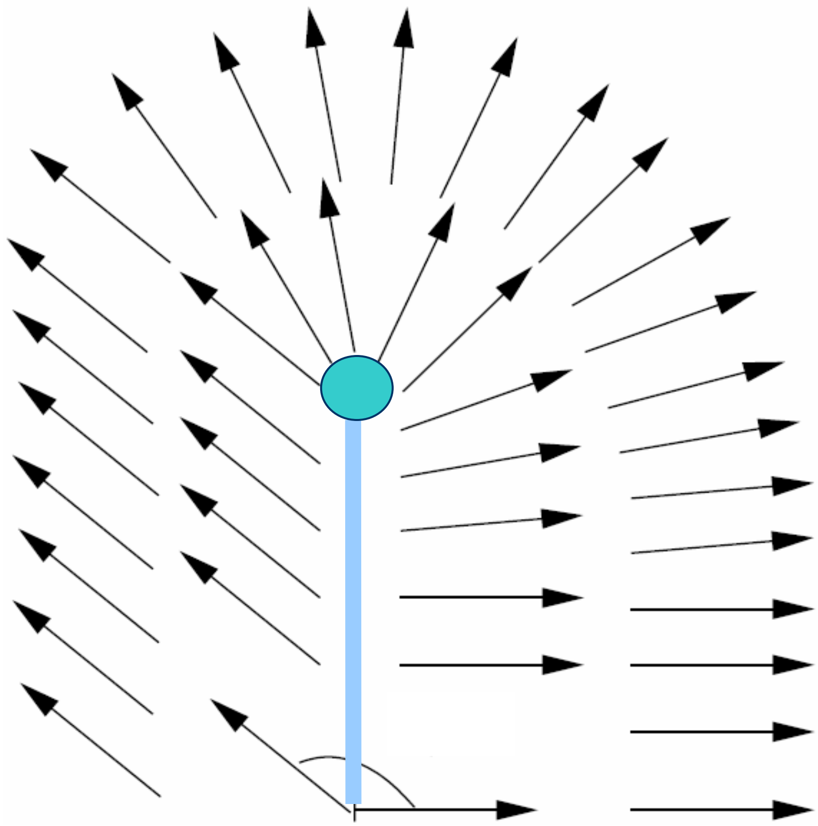
- Electric charges, labelled by representations of the gauge group.
  Under gauge transformations: $(\alpha \in Z_N) : |q\rangle \mapsto e^{\frac{2\pi i q \alpha}{N}} |q\rangle$
- Magnetic fluxes, labelled by monodromies (Wilson loop) 0,1,…N-1
  Can think of these as carriers of representations of a dual group (also Z_N)
- Dyons, with flux and charge (transform under Z_N x Z_N).

Topological interactions:

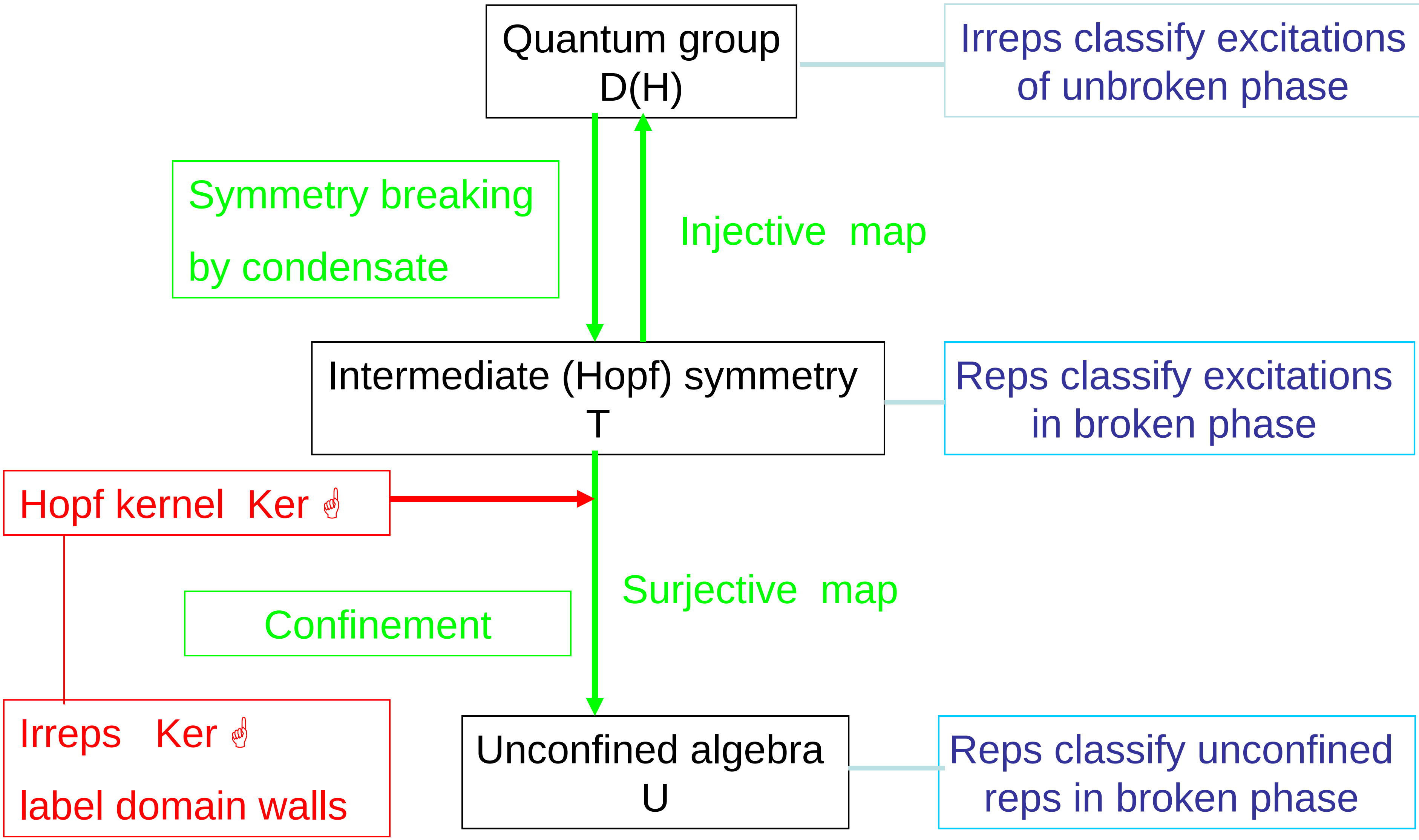
- Fusion (described by tensor product of Z_N x Z_N reps)
- Aharonov-Bohm effect, phase factors are $\exp(\frac{2\pi i(q_1 p_2 + p_1 q_2)}{N})$

Condensation described by breaking of the full symmetry group (incl dual).
On top of that, have confinement, from AB-interactions

Confined defect



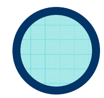
Symmetry breaking scheme



Bosons in the Read-Rezayi States

- What is a boson? A particle with
  - trivial twist factor/ integer conformal weight
  - trivial self braiding in at least on fusion channel, i.e. at least one of the fusion products also has trivial twist/integer weight

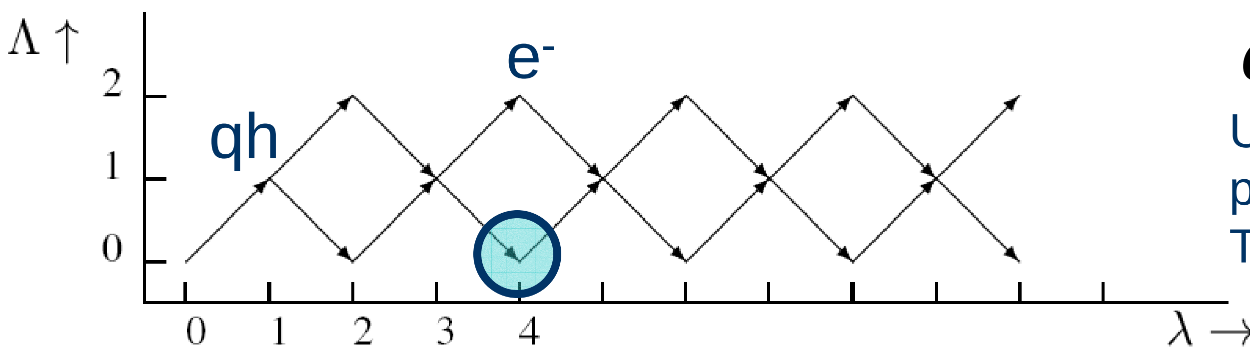
- Have a boson in the Pfaffian state (below) and lots of bosons in the higher RR-states (k=4 upwards)



Contains

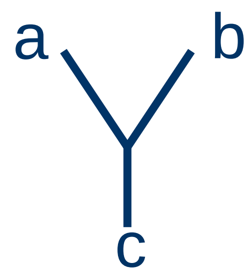
$$e^{i\sqrt{2}\varphi}$$

U(1) conformal primary with h_b=1, Trivial self-braiding



TQFT reminder

Fusion

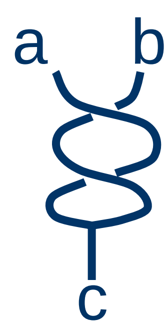


$$a \times b = \sum_c N_c^{ab} c$$

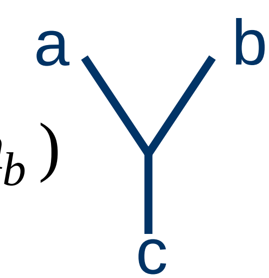
Twist

$$= e^{2\pi i h_a}$$

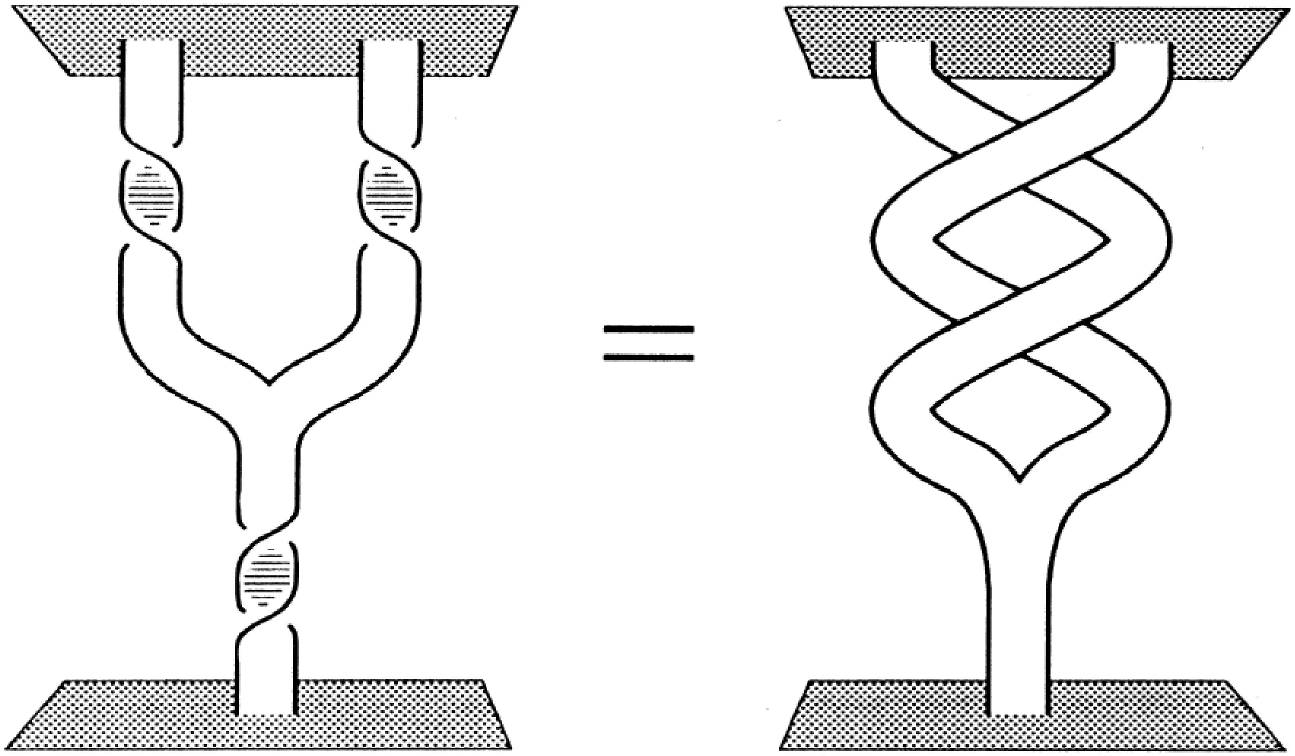
Monodromy



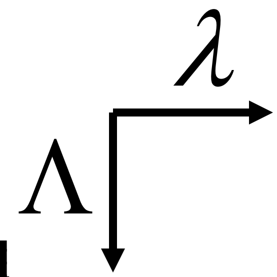
$$= e^{2\pi i (h_c - h_a - h_b)}$$



Braiding and Twisting



Another RR-boson



$$h_{\lambda}^{\Lambda} = \frac{\Lambda(\Lambda+2)-\lambda^2}{4(k+2)} + \text{integer}$$

k=2

0	X	$\frac{3}{4}$	X	0	X	$\frac{3}{4}$	X	0
X	$\frac{1}{8}$	X	$\frac{5}{8}$	X	$\frac{5}{8}$	X	$\frac{1}{8}$	X
$\frac{1}{2}$	X	$\frac{1}{4}$	X	$\frac{1}{2}$	X	$\frac{1}{4}$	X	$\frac{1}{2}$

k=3

0	X	$\frac{4}{5}$	X	$\frac{1}{5}$	X	$\frac{1}{5}$	X	$\frac{4}{5}$	X	0
X	$\frac{1}{10}$	X	$\frac{7}{10}$	X	$\frac{9}{10}$	X	$\frac{7}{10}$	X	$\frac{1}{10}$	X
$\frac{2}{5}$	X	$\frac{1}{5}$	X	$\frac{3}{5}$	X	$\frac{3}{5}$	X	$\frac{1}{5}$	X	$\frac{2}{5}$
X	$\frac{7}{10}$	X	$\frac{3}{10}$	X	$\frac{1}{2}$	X	$\frac{3}{10}$	X	$\frac{7}{10}$	X

k=4

0	X	$\frac{5}{6}$	X	$\frac{1}{3}$	X	$\frac{1}{2}$	X	$\frac{1}{3}$	X	$\frac{5}{6}$	X	0
X	$\frac{1}{12}$	X	$\frac{3}{4}$	X	$\frac{1}{12}$	X	$\frac{1}{12}$	X	$\frac{3}{4}$	X	$\frac{1}{12}$	X
$\frac{1}{3}$	X	$\frac{1}{6}$	X	$\frac{2}{3}$	X	$\frac{5}{6}$	X	$\frac{2}{3}$	X	$\frac{1}{6}$	X	$\frac{1}{3}$
X	$\frac{7}{12}$	X	$\frac{1}{4}$	X	$\frac{7}{12}$	X	$\frac{7}{12}$	X	$\frac{1}{4}$	X	$\frac{7}{12}$	X
0	X	$\frac{5}{6}$	X	$\frac{1}{3}$	X	$\frac{1}{2}$	X	$\frac{1}{3}$	X	$\frac{5}{6}$	X	0

Even more RR-bosons

k=5

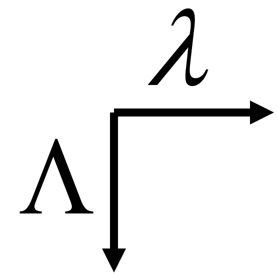
0	X	$\frac{6}{7}$	X	$\frac{3}{7}$	X	$\frac{5}{7}$	X
X	$\frac{1}{14}$	X	$\frac{11}{14}$	X	$\frac{3}{14}$	X	$\frac{5}{14}$
$\frac{2}{7}$	X	$\frac{1}{7}$	X	$\frac{5}{7}$	X	0	X
X	$\frac{1}{2}$	X	$\frac{3}{14}$	X	$\frac{9}{14}$	X	$\frac{11}{14}$
$\frac{6}{7}$	X	$\frac{5}{7}$	X	$\frac{2}{7}$	X	$\frac{4}{7}$	X
X	$\frac{3}{14}$	X	$\frac{13}{14}$	X	$\frac{5}{14}$	X	$\frac{1}{2}$

k=6

0	X	$\frac{7}{8}$	X	$\frac{1}{2}$	X	$\frac{7}{8}$	X	0
X	$\frac{1}{16}$	X	$\frac{13}{16}$	X	$\frac{5}{16}$	X	$\frac{9}{16}$	X
$\frac{1}{4}$	X	$\frac{1}{8}$	X	$\frac{3}{4}$	X	$\frac{1}{8}$	X	$\frac{1}{4}$
X	$\frac{7}{16}$	X	$\frac{3}{16}$	X	$\frac{11}{16}$	X	$\frac{15}{16}$	X
$\frac{3}{4}$	X	$\frac{5}{8}$	X	$\frac{1}{4}$	X	$\frac{5}{8}$	X	$\frac{3}{4}$
X	$\frac{1}{16}$	X	$\frac{13}{16}$	X	$\frac{5}{16}$	X	$\frac{9}{16}$	X
$\frac{1}{2}$	X	$\frac{3}{8}$	X	0	X	$\frac{3}{8}$	X	$\frac{1}{2}$

k=7

0	X	$\frac{8}{9}$	X	$\frac{5}{9}$	X	0	X	$\frac{2}{9}$	X
X	$\frac{1}{18}$	X	$\frac{5}{6}$	X	$\frac{7}{18}$	X	$\frac{13}{18}$	X	$\frac{5}{6}$
$\frac{2}{9}$	X	$\frac{1}{9}$	X	$\frac{7}{9}$	X	$\frac{2}{9}$	X	$\frac{4}{9}$	X
X	$\frac{7}{18}$	X	$\frac{1}{6}$	X	$\frac{13}{18}$	X	$\frac{1}{18}$	X	$\frac{1}{6}$
$\frac{2}{3}$	X	$\frac{5}{9}$	X	$\frac{2}{9}$	X	$\frac{2}{3}$	X	$\frac{8}{9}$	X
X	$\frac{17}{18}$	X	$\frac{13}{18}$	X	$\frac{5}{18}$	X	$\frac{11}{18}$	X	$\frac{13}{18}$
$\frac{1}{3}$	X	$\frac{2}{9}$	X	$\frac{8}{9}$	X	$\frac{1}{3}$	X	$\frac{5}{9}$	X
X	$\frac{13}{18}$	X	$\frac{1}{2}$	X	$\frac{1}{18}$	X	$\frac{7}{18}$	X	$\frac{1}{2}$



$$h_{\lambda}^{\Lambda} = \frac{\Lambda(\Lambda+2) - \lambda^2}{4(k+2)} + \text{integer}$$

λ

k=8

0	X	$\frac{9}{10}$	X	$\frac{3}{5}$	X	$\frac{1}{10}$	X	$\frac{2}{5}$	X	$\frac{1}{2}$
X	$\frac{1}{20}$	X	$\frac{17}{20}$	X	$\frac{9}{20}$	X	$\frac{17}{20}$	X	$\frac{1}{20}$	X
$\frac{1}{5}$	X	$\frac{1}{10}$	X	$\frac{4}{5}$	X	$\frac{3}{10}$	X	$\frac{3}{5}$	X	$\frac{7}{10}$
X	$\frac{7}{20}$	X	$\frac{3}{20}$	X	$\frac{3}{4}$	X	$\frac{3}{20}$	X	$\frac{7}{20}$	X
$\frac{3}{5}$	X	$\frac{1}{2}$	X	$\frac{1}{5}$	X	$\frac{7}{10}$	X	0	X	$\frac{1}{10}$
X	$\frac{17}{20}$	X	$\frac{13}{20}$	X	$\frac{1}{4}$	X	$\frac{13}{20}$	X	$\frac{17}{20}$	X
$\frac{1}{5}$	X	$\frac{1}{10}$	X	$\frac{4}{5}$	X	$\frac{3}{10}$	X	$\frac{3}{5}$	X	$\frac{7}{10}$
X	$\frac{11}{20}$	X	$\frac{7}{20}$	X	$\frac{19}{20}$	X	$\frac{7}{20}$	X	$\frac{11}{20}$	X
0	X	$\frac{9}{10}$	X	$\frac{3}{5}$	X	$\frac{1}{10}$	X	$\frac{2}{5}$	X	$\frac{1}{2}$

Problem with the old scheme

Problem is: need to select a state inside a representation as order parameter.
Problem because:

- Quantum Group Symmetry is often "hidden":
  internal labels of particles are not (fully) physical
  ⟶ what does the "order parameter" mean?

- Particles' "internal spaces" may have non-integer quantum dimensions…

- Tensor product in many theories (incl. the ones relevant to the Hall effect) is truncated. Product states as expected for Bose condensates usually do not survive the truncation

The New Symmetry Breaking: Forget the Algebra!

No algebra, No states, Just Labels and Branching:

$$a \rightarrow \sum_i n_{a,i} a_i$$

Three requirements:

1. The new labels themselves form a fusion algebra (need associativity, vacuum and charge conjugation)
2. Branching and fusion are compatible,

$$a \otimes b \rightarrow (\sum_i n_{a,i} a_i) \otimes (\sum_i n_{b,i} b_i)$$

3. Not more branching/identification than necessary for 1. and 2. (want the full “stabiliser”)

Breaking $U_q(su(4))$

$SU(2)_4$

0	$d_0 = 1$	$h_0 = 0$
1	$d_1 = \sqrt{3}$	$h_1 = 1/8$
2	$d_2 = 2$	$h_2 = 1/3$
3	$d_3 = \sqrt{3}$	$h_3 = 5/8$
4	$d_4 = 1$	$h_4 = 1$

$1 \times 1 = 0 + 2$

$1 \times 2 = 1 + 3 \qquad 2 \times 2 = 0 + 2 + 4$

$1 \times 3 = 2 + 4 \qquad 2 \times 3 = 1 + 3 \qquad 3 \times 3 = 0$

$1 \times 4 = 3 \qquad 2 \times 4 = 2 \qquad 3 \times 4 = 1 \qquad 4 \times 4 = 0$

Condensate, splitting and identification

Assume a bosonic condensate forms in the 4 rep of $SU(2)_4$:

$$2\times 2=0+2+4=0+2+0$$

$\Rightarrow 2\coloneqq 2_1+2_2$ possible because $d_2=2$

$$2_1\times 2_1+2_1\times 2_2+2_2\times 2_1+2_2\times 2_2=0+2_1+2_2+0$$

$$\Rightarrow 2_1\times 2_2=0$$

if $2_1\times 2_1=2_1$

then $2_2\times(2_1\times 2_1)=2_2\times 2_1=0$

$(2_2\times 2_1)\times 2_1=0\times 2_1=2_1$

$\Rightarrow 2_1\times 2_1=2_2$ and $2_2\times 2_2=2_1$

$$1\times 1=0+2_1+2_2$$

$$1\times 3=0+2_1+2_2$$

$$\Rightarrow 1\Leftrightarrow 3$$

0

1

➡ $2\coloneqq 2_1+2_2$

$3\Leftrightarrow 1$

$4\Leftrightarrow 0$

$$1\times 1=0+1$$

$1\times 2_1=1$ $\quad 2_1\times 2_1=2_2$

$1\times 2_2=1$ $\quad 2_1\times 2_2=1$ $\quad 2_2\times 2_2=2_2$

Confinement and Braiding

To see which of the particles in the broken theory are confined, look at braiding with the condensed particle.

How? For particle a_i, look in all channels of the old theory that cover $a_i x 1 = a_i$

Now notice: Fields that cover 1 have trivial twist factor (condensate is bosonic). Hence braiding with the vacuum is trivial and a_i is not confined precisely when all the fields that branch to a_i have equal twist factors (or conformal dimensions that differ by integers).

The non-confined particles all have well defined monodromies with each other, given by their twist factors (which are unambiguously defined from the branching).

Confinement

From fusion rules (of the original algebra)
with the condensate 4 and conformal weights one finds that
the 1 and 3 are confined.

The unconfined algebra becomes $SU(3)_1$:

$$\begin{aligned} 2_1 \times 2_1 &= 2_2 \\ 2_1 \times 2_2 &= 0 \\ 2_2 \times 2_2 &= 2_1 \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} 3 \times 3 &= \bar{3} \\ 3 \times \bar{3} &= 1 \\ \bar{3} \times \bar{3} &= 3 \end{aligned}$$

Relation to Conformal Embedding

Central charges satisfy c(G) = c(H) ==> c(G/H) = 0
Coset algebra is trivial.
==> Finite branching of inf. Dim. KM representations

Example: $SU(2)_4$ ==> $SU(3)_1$ (c=2)

$SU(3)_1$ Irreps:

1	$d_1 = 1$	$h_1 = 0$
3	$d_3 = 1$	$h_3 = 1/3$
$\bar{3}$	$d_{\bar{3}} = 1$	$h_{\bar{3}} = 1/3$

$$3 \times 3 = \bar{3}$$
$$3 \times \bar{3} = 1$$
$$\bar{3} \times \bar{3} = 3$$

→

branching

$$1 \to 0 + 4$$
$$3 \to 2$$
$$\bar{3} \to 2$$

Some results for the Read-Rezayi states

- k=2: Ising x $U(1)_8$ ➔ $SO(3)_2$ x $U(1)_2$ or (Ising/Z_2) x $U(1)_2$
  Interpretation not obvious (superconductor?!).
  Possibly connected to Fradkin-Nayak-Schoutens '98?

- k=4: Pf_4 x $U(1)_{24}$ ➔ ($U(1)_6$ x $U(1)_6$)/Z_2
  Expect a Hall state at filling 2/3 (neutral condensate)
  Get Abelian topological order with right behavior for the quasiholes.
  More precise connection difficult (naïve ground state gives nu=1/3 Laughlin)

- k=6: Much like k=2, but stays nonabelian (contains $SO(3)_6$)

- k=7: Stays nonabelian (contains “twisted” version of Pf_7)

- k=8: Stays nonabelian and contains Fibonacci x Fibonacci (chiral version)

Conformal embedding of KM algebra's

The case $SO(5)_1$

Irreps:

1	$d_1 = 1$	$h_1 = 0$	$4 \times 4 = 1 + 5$
4	$d_4 = \sqrt{2}$	$h_4 = 5/16$	$5 \times 5 = 1$
5	$d_5 = 1$	$h_5 = 1/2$	$4 \times 5 = 4$

Conformal embedding: $SU(2)_{10}$ ➔ $SO(5)_1$ (c= 5/2)

Branching rules:
0 => 0+6
4 => 3+7
5 => 4+10

The condensate

Condensation of the 6 demands splitting $6 := 6_1 + 6_2$ (with dim $6_1 = 1$ and $6_1 \mathrm{x} 6_1 = 0$)
Because of condensate we get further splitting and identifications:

$$d_0 = d_{10} = 1$$
$$d_1 = d_9 = \sqrt{2+\sqrt{3}}$$
$$d_2 = d_8 = 1+\sqrt{3}$$
$$d_3 = d_7 = \sqrt{2} + \sqrt{2+\sqrt{3}}$$
$$d_4 = d_6 = 2+\sqrt{3}$$
$$d_5 = 2\sqrt{2+\sqrt{3}}$$

$$0$$
$$1$$
$$2$$
$$3 := 3_1 + 3_2$$
$$4 := 4_1 + 4_2 \Leftrightarrow 4_1 + 2$$
$$5 := 5_1 + 5_2 \Leftrightarrow 1 + 3_1$$
$$6 := 6_1 + 6_2 \Leftrightarrow 0 + 2$$
$$7 := 7_1 + 7_2 \Leftrightarrow 1 + 4_2$$
$$8 \Leftrightarrow 2$$
$$9 \Leftrightarrow 3_1$$
$$10 \Leftrightarrow 4_1$$

$$d_0 = 1$$
$$d_1 = \sqrt{2+\sqrt{3}}$$
$$d_2 = d_8 = 1+\sqrt{3}$$
$$d_{3_1} = \sqrt{2+\sqrt{3}}$$
$$d_{3_2} = \sqrt{2}$$
$$d_{4_1} = 1$$

Intermediate model

Fusion rules:

$$1 \times 1 = 0 + 2$$

$$1 \times 2 = 1 + 3_1 + 3_2 \qquad 2 \times 2 = 0 + 2 + 2 + 4$$

$$1 \times 3_1 = 2 + 4_1 \qquad 2 \times 3_1 = 1 + 3_1 + 1 + 3_2 \qquad 3_1 \times 3_1 = 0 + 2$$

$$1 \times 3_2 = 2 \qquad 2 \times 3_2 = 1 + 3_1 \qquad 3_1 \times 3_2 = 2 \qquad 3_2 \times 3_2 = 0 + 4_1$$

$$1 \times 4_1 = 3_1 \qquad 2 \times 4_1 = 2 \qquad 3_1 \times 4_1 = 1 \qquad 3_2 \times 4_1 = 3_2 \qquad 4_1 \times 4_1 = 0$$

Braiding and confinement

SO(5)$_1$

$h_1 = 0$
$h_4 = 5/16$
$h_5 = 1/2$

SU(2)$_{10}$

$h_0 = 0$
$h_1 = \frac{1}{16}$
$h_2 = \frac{1}{6}$
$h_3 = \frac{5}{16}$
$h_4 = \frac{1}{2}$
$h_5 = \frac{35}{48}$
$h_6 = 1$
$h_7 = \frac{21}{16}$
$h_8 = \frac{5}{3}$
$h_9 = \frac{33}{16}$
$h_{10} = \frac{5}{2}$

0
1
2
$3 := 3_1 + 3_2$
$4 := 4_1 + 4_2 \Leftrightarrow 4_1 + 2$
$5 := 5_1 + 5_2 \Leftrightarrow 1 + 3_1$
$6 := 6_1 + 6_2 \Leftrightarrow 0 + 2$
$7 := 7_1 + 7_2 \Leftrightarrow 1 + 3_2$
$8 \Leftrightarrow 2$
$9 \Leftrightarrow 3_1$
$10 \Leftrightarrow 4_1$

0
3_2
4_1

$3_2 \times 3_2 = 0 + 4_1$
$3_2 \times 4_1 = 3_2$
$4_1 \times 4_1 = 0$

?

$4 \times 4 = 1 + 5$
$4 \times 5 = 4$
$5 \times 5 = 1$

(Algebra of Ising or SU(2)$_2$
have different conformal weights!)

Summary and Outlook

Results

- Extended Topological symmetry breaking to TQFTs with non-integer quantum dimensions
- Found connection to conformal embeddings
- Had a first go at application to nonabelian FQH states

Questions/Future Work

- Found Fusion and twist factors.
  How to determine the rest of the TQFT (half-braidings, F-symbols…) ?
  Note: often fixed by consistency (always?)
- Work suggests conformal embeddings of coset chiral algebras.
  Interesting CFT problem…
- Further Physical applications….