String-net condensation and topological phases in quantum spin systems

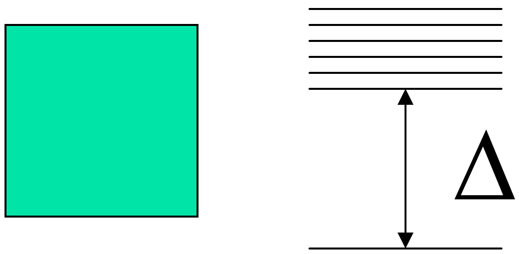
Michael Levin, Xiao-Gang Wen

MIT

Topological phases

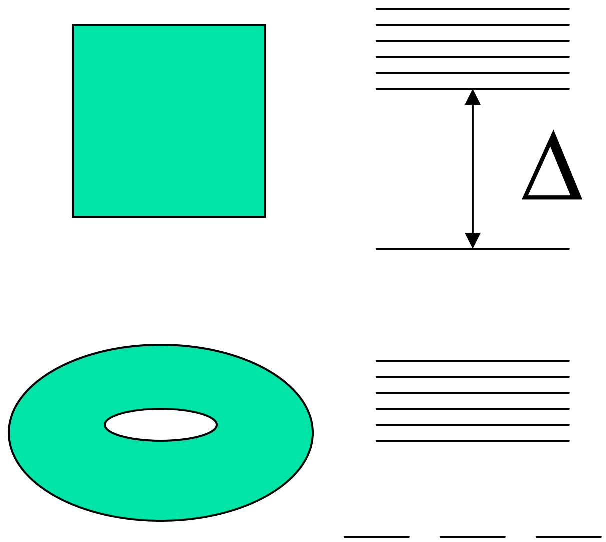
Topological phases

- Gapped



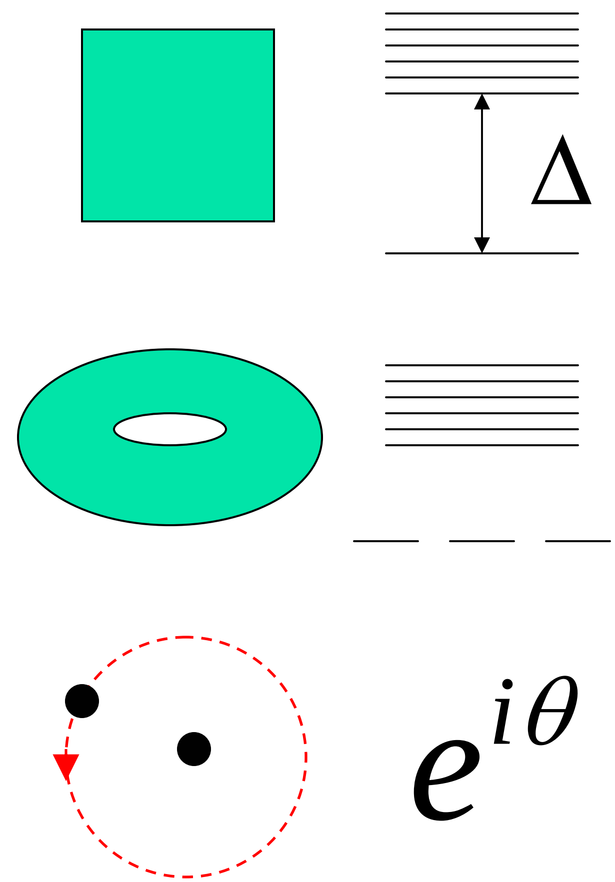
Topological phases

- Gapped
- Degenerate ground state on torus



Topological phases

- Gapped
- Degenerate ground state on torus
- Fractional statistics



Real life examples

- FQH liquids.

Real life examples

- FQH liquids.

- Hope: Frustrated magnets

  - Many theoretical models

  - A few candidate materials

    - Cs_2CuCl_4

    - κ-$(BEDT\text{-}TTF)_2Cu_2(CN)_3$

Theory of topological phases

Theory of topological phases

- We understand:

  - Low energy/Long distance physics

- We're missing:

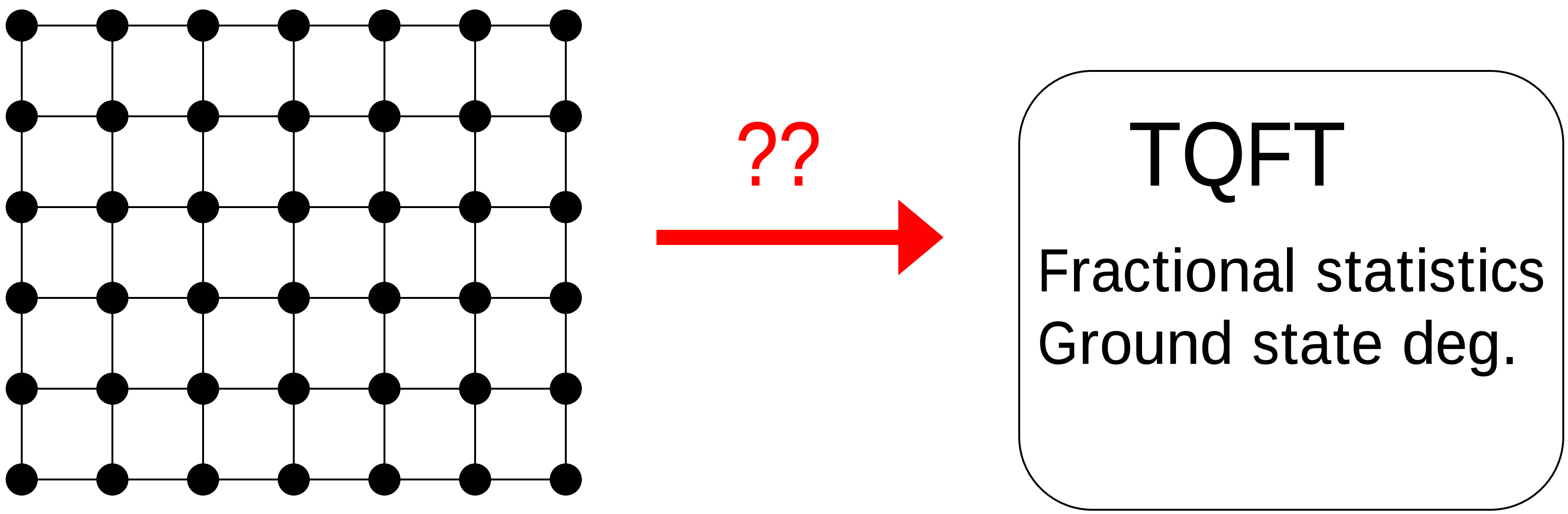
  - Connection with microscopics!

How do topological phases emerge from microscopic spins?

- How can we realize them? What interactions favor them?

How do topological phases emerge from microscopic spins?

- How can we realize them? What interactions favor them?



Outline

I. Physical picture

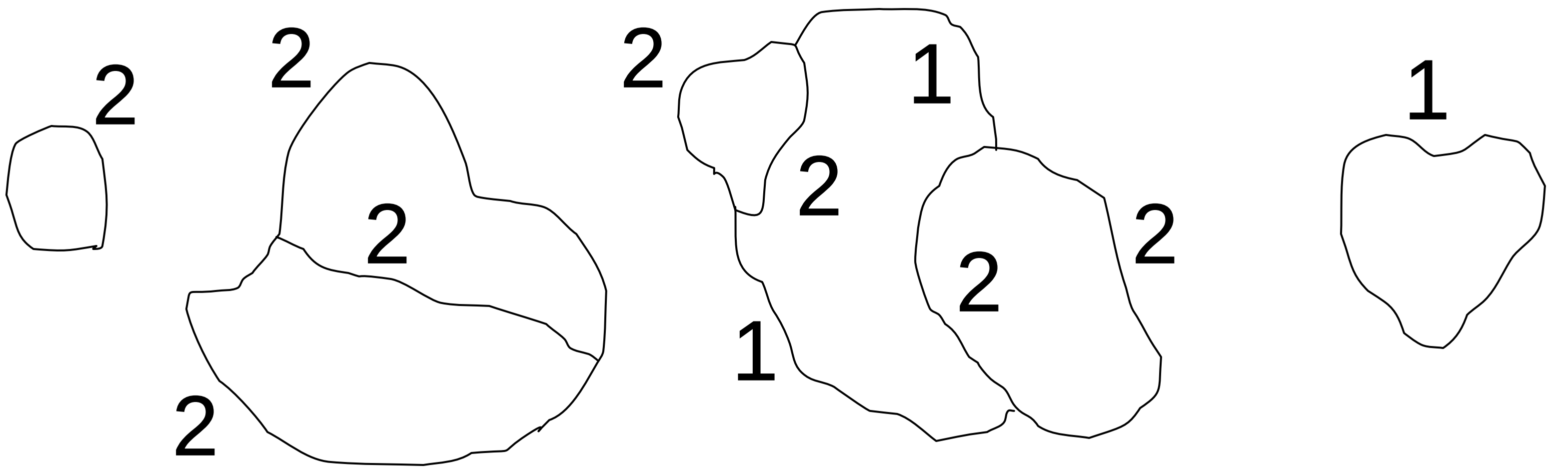
II. Quantitative results

A. Explicit ground state wave functions

B. Exactly soluble Hamiltonians

III. Examples

String-net models

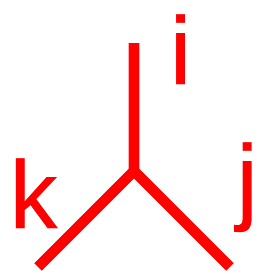


Data

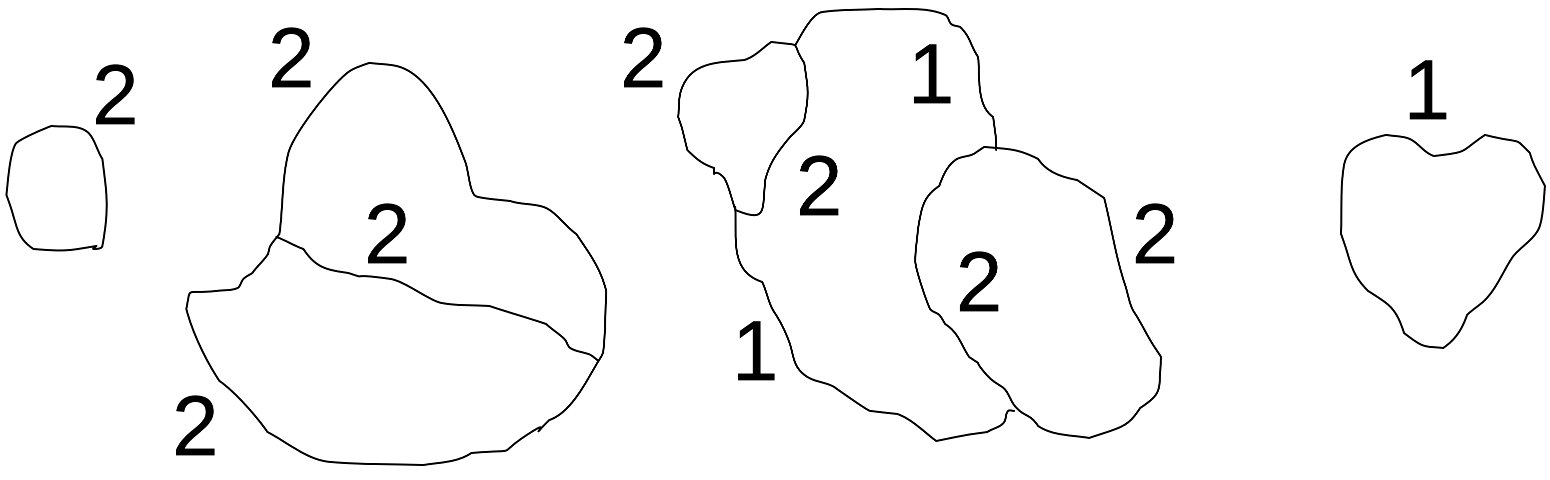
1. String types: Number of string types N.

(i = 1,...,N)

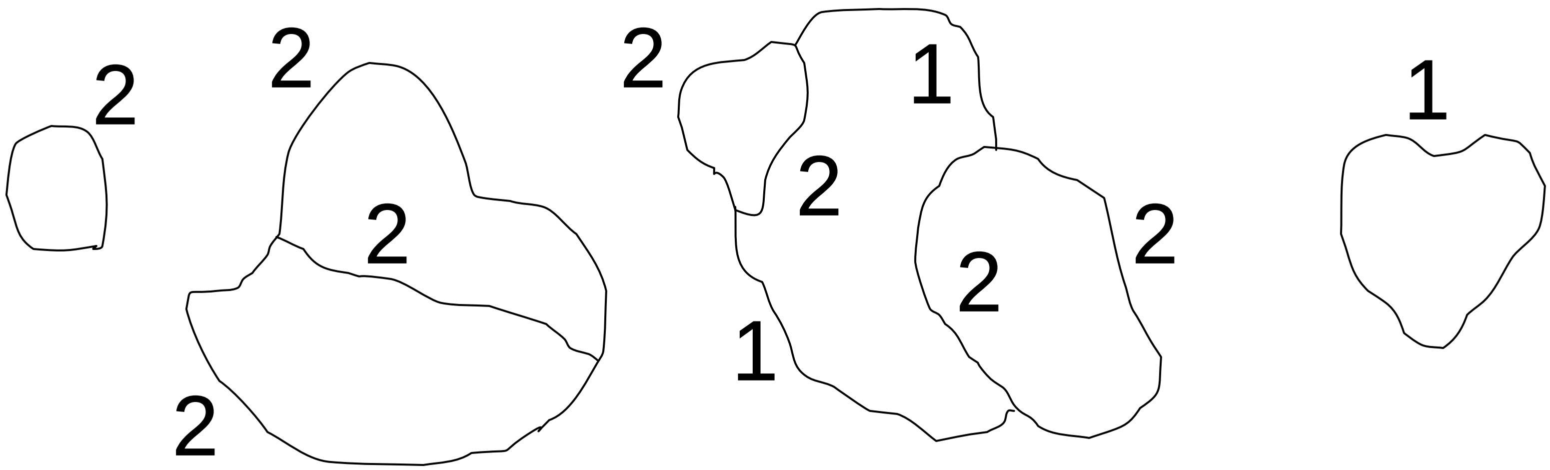
2. Branching rules: Triplets {i, j, k} allowed to meet at a point.



Data



Data



1. Number of string types: N = 2.
2. Branching rules: {2, 2, 2}, {1, 2, 2}.

String-net Hamiltonian

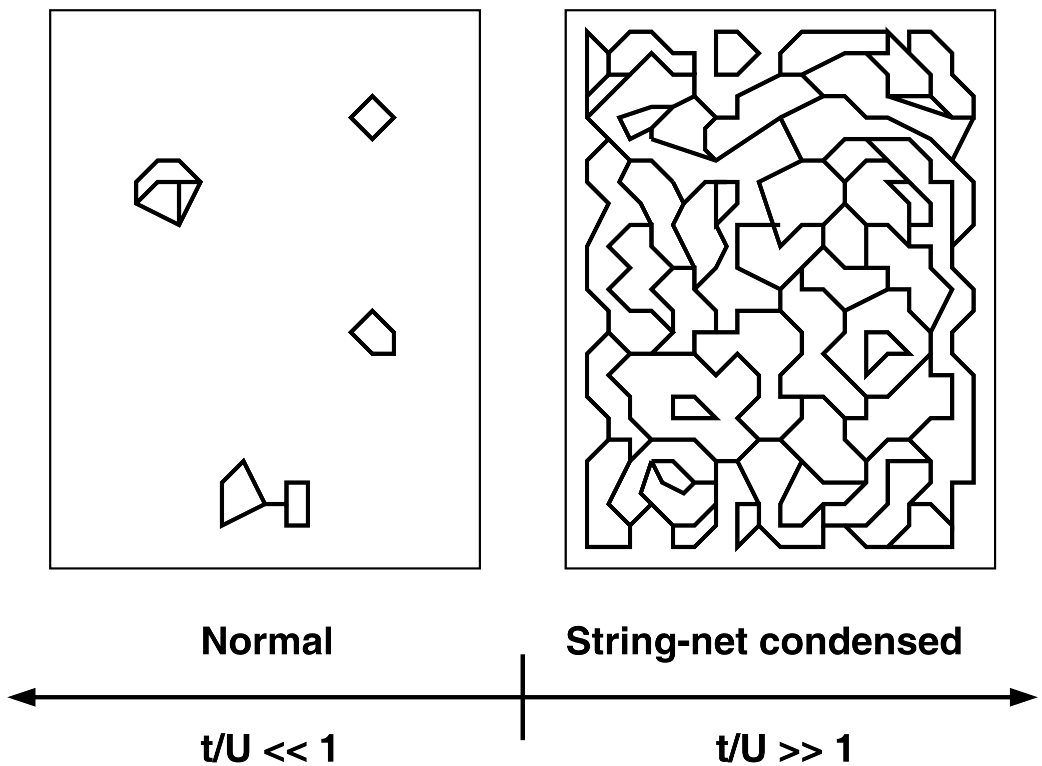
$$H = t\,H_t + U\,H_U$$

$t\,H_t$: String kinetic energy

$U\,H_U$: String tension

String-net Hamiltonian

$$H = t\,H_t + U\,H_U$$



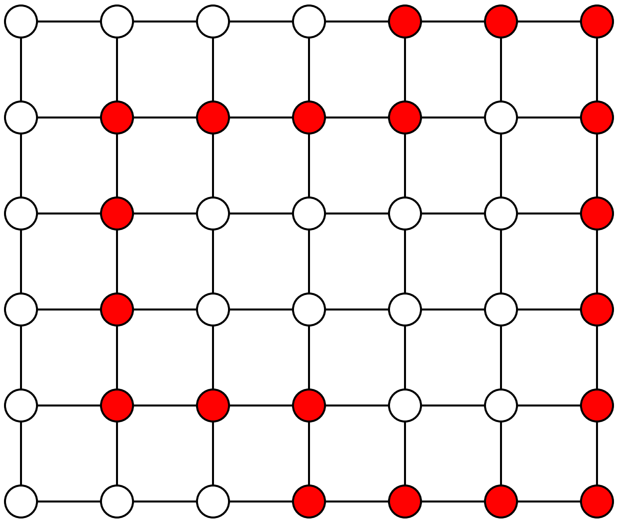
So what?

- String-net condensed phases ARE topological phases!
- Mechanism for topological phases
- Very general: all non-chiral topological phases can be realized

What does this have to do with spin systems?

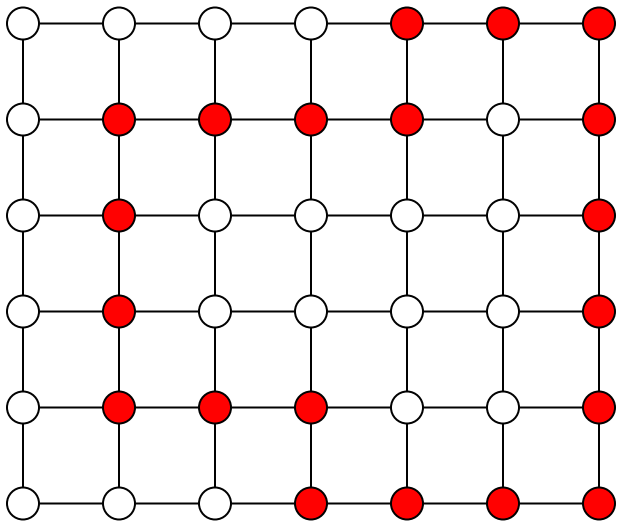
What does this have to do with spin systems?

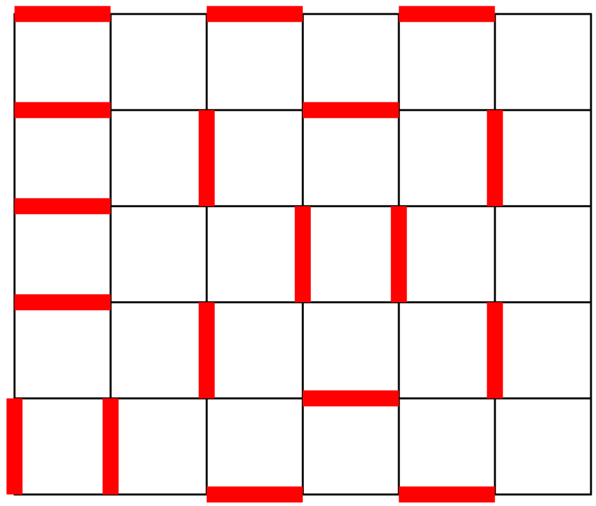
Low energy degrees of freedom can be string-like:



What does this have to do with spin systems?

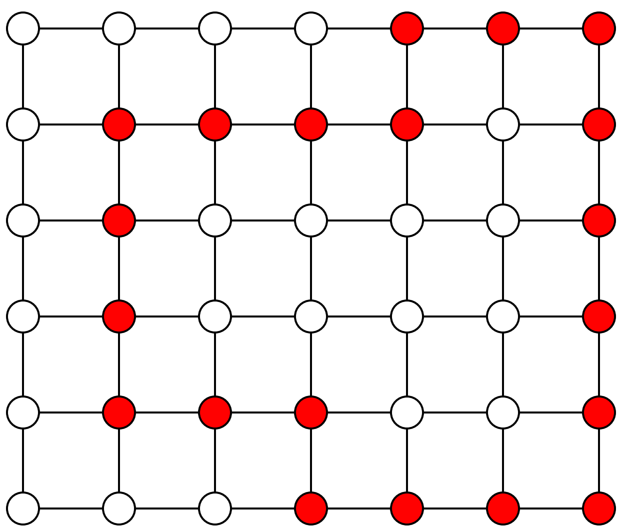
Low energy degrees of freedom can be string-like:

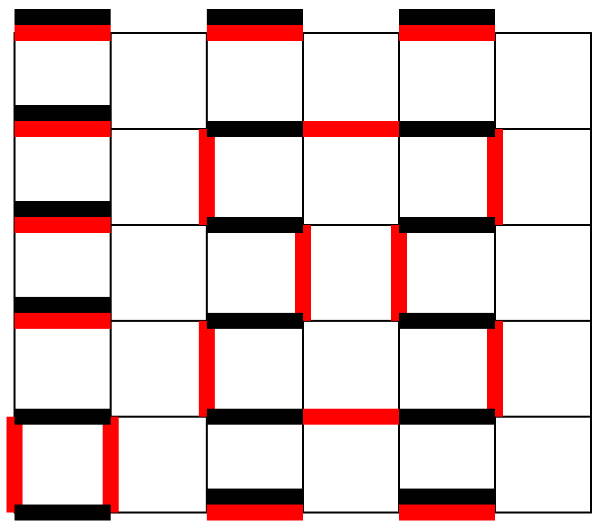




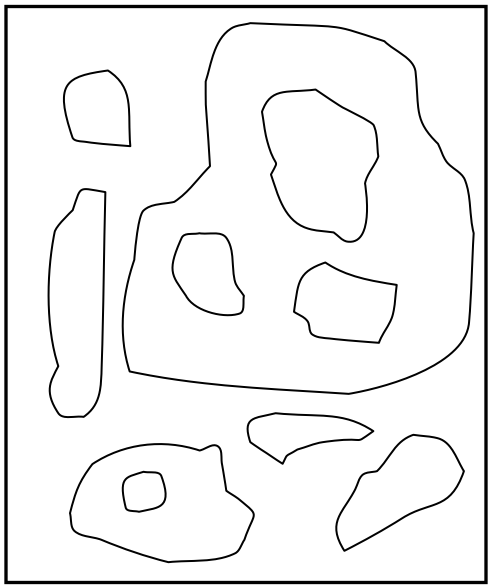
What does this have to do with spin systems?

Low energy degrees of freedom can be string-like:

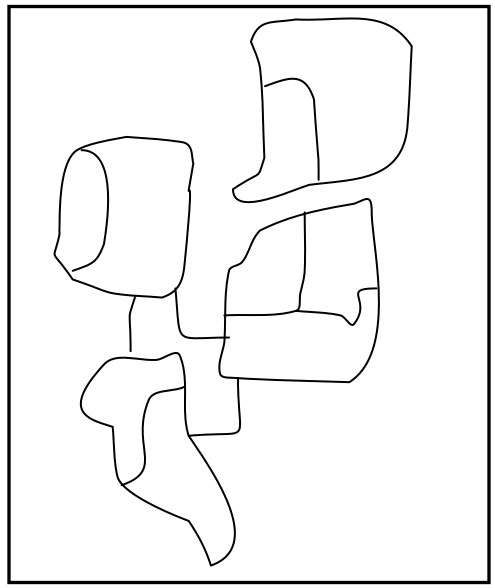




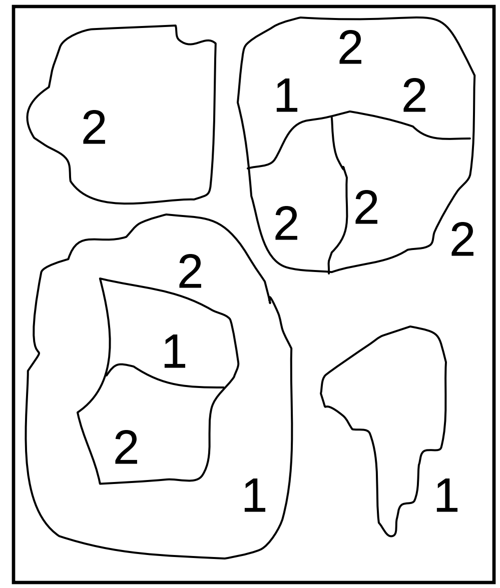
Examples



Z_2 gauge theory

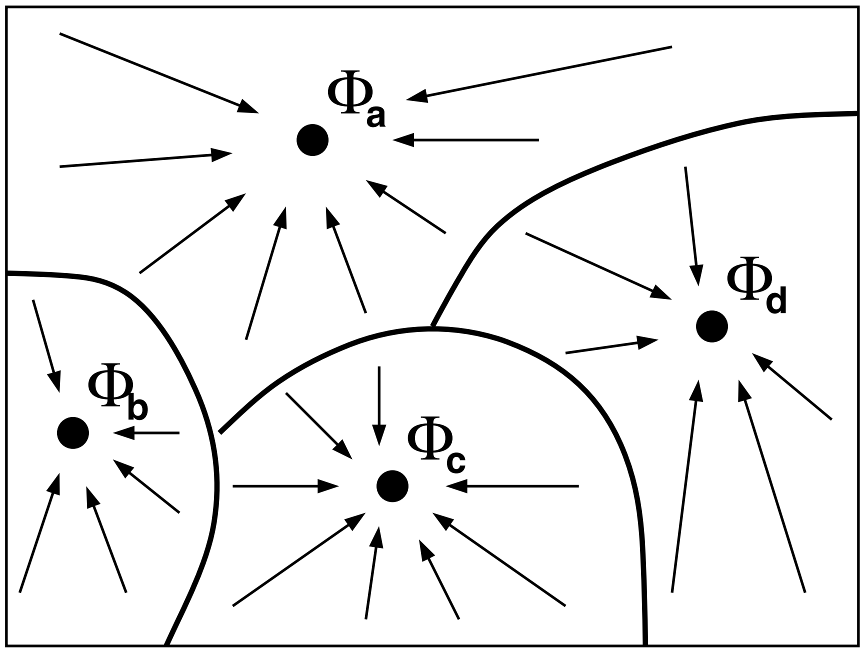


$SO_3(3)$ ¤ $SO_3(3)$
Chern-Simons

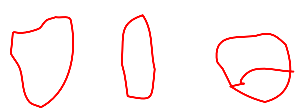


S_3 gauge theory

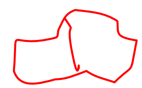
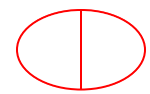
Representative wave functions



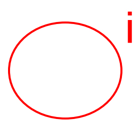
Want “fixed-point” wave functions:

$\Phi($  $) = \ldots$

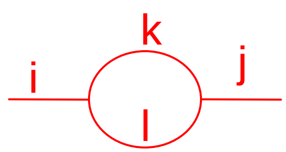
Ansatz

1. Amplitude of Φ only depends on topology of string-net: e.g., $\Phi($  $) = \Phi($  $)$

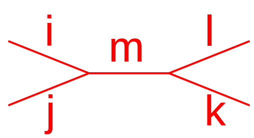
2. Φ satisfies local constraint equations:

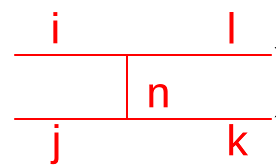
$\Phi($ 

$) = d_i\ \Phi(\qquad)$

$\Phi($ 

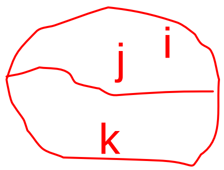
$) = 0 \quad \text{if } i \neq j$

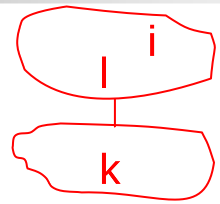
$\Phi($ 

$) = \sum_n F^{ijm}_{kln}\ \Phi($ 

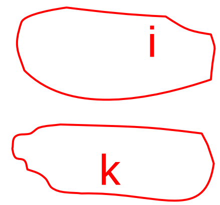
$)$

Local constraints specify Φ completely

$\Phi($ 

$) = \sum_l F^{ikj}_{kil} \Phi($ 

$)$

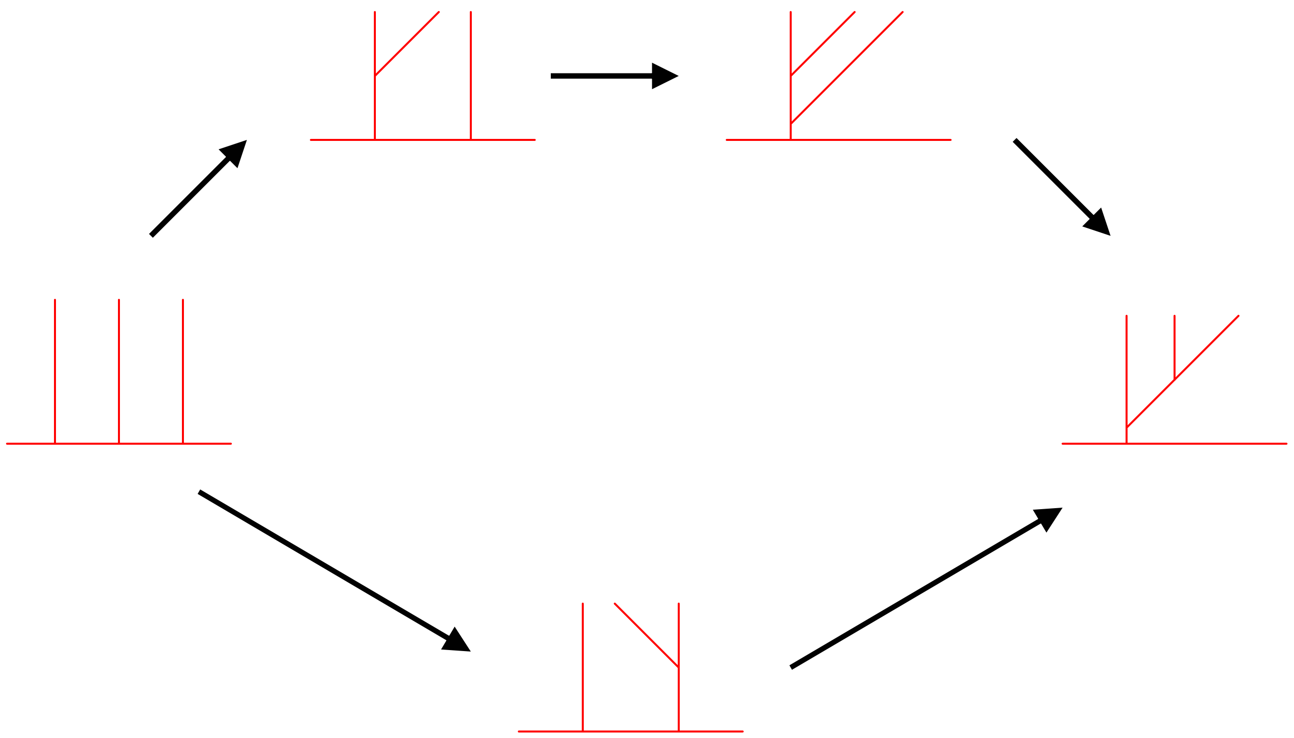
$= F^{ikj}_{ki0} \Phi($ 

$)$

$= F^{ikj}_{ki0} d_i d_k \Phi($vacuum$)$

$= F^{ikj}_{ki0} d_i d_k$

But rules are not usually self-consistent!



Self-consistency conditions

$$\sum_n F^{mlq}_{kpn}\, F^{jip}_{mns}\, F^{jsn}_{lkr} = F^{jip}_{qkr} F^{riq}_{mls} \qquad \text{(a)}$$

$$F^{ijm}_{kln} = F^{lkm}_{jin} = F^{jim}_{lkn} = F^{imj}_{knl}\,(d_m d_n / d_j d_l)^{1/2} \qquad \text{(b)}$$

$$F^{ijk}_{ji0} = (d_k/d_i d_j)^{1/2}\,\delta_{ijk} \qquad \text{(c)}$$

(where $\delta_{ijk} = 1$ if $\{i,j,k\}$ allowed, 0 otherwise).

Self-consistency conditions

$$\sum_n F^{mlq}_{kpn} F^{jip}_{mns} F^{jsn}_{lkr} = F^{jip}_{qkr} F^{riq}_{mls} \qquad \text{(a)}$$

$$F^{ijm}_{kln} = F^{lkm}_{jin} = F^{jim}_{lkn} = F^{imj}_{knl} \, (d_m d_n / d_j d_l)^{1/2} \qquad \text{(b)}$$

$$F^{ijk}_{ji0} = (d_k / d_i d_j)^{1/2} \, \delta_{ijk} \qquad \text{(c)}$$

(where $\delta_{ijk} = 1$ if $\{i,j,k\}$ allowed, 0 otherwise).

Solutions $\Leftrightarrow$ fixed point wave functions Φ

Classification of non-chiral topological phases

Solutions (F^{ijm}_{kln}, d_i, δ_{ijk}) of (a)-(c) ⟷ String-net condensates/ non-chiral topological phases

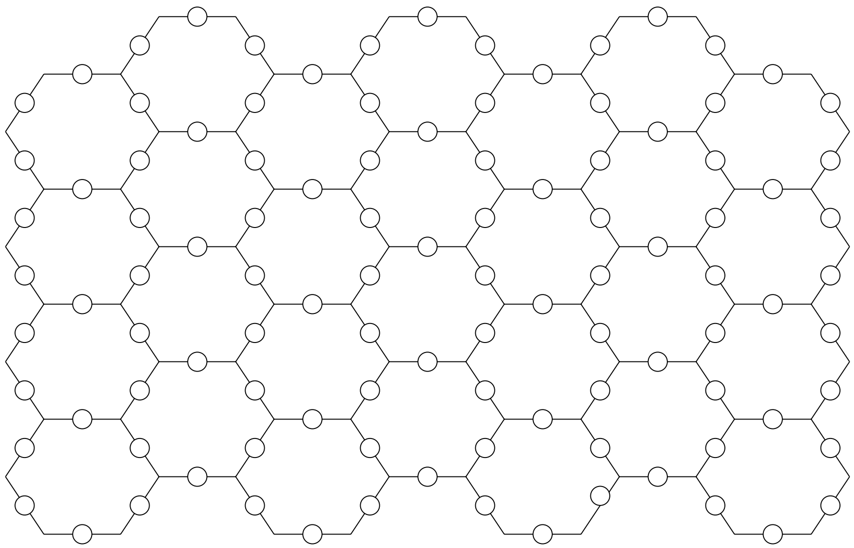
Classification of non-chiral topological phases

Solutions $(F^{ijm}_{kln}, d_i, \delta_{ijk})$ of (a)-(c)

String-net condensates/ non-chiral topological phases

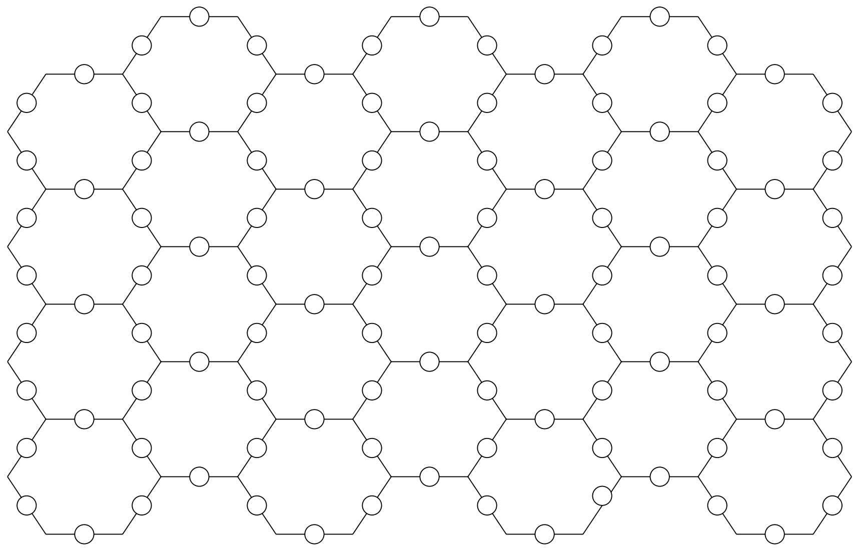
“Tensor categories”

Exactly soluble lattice models

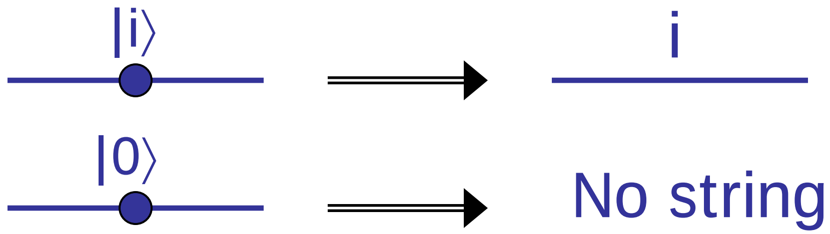


Each "spin" can be in N+1 states: $|0\rangle, |1\rangle, \ldots, |N\rangle$

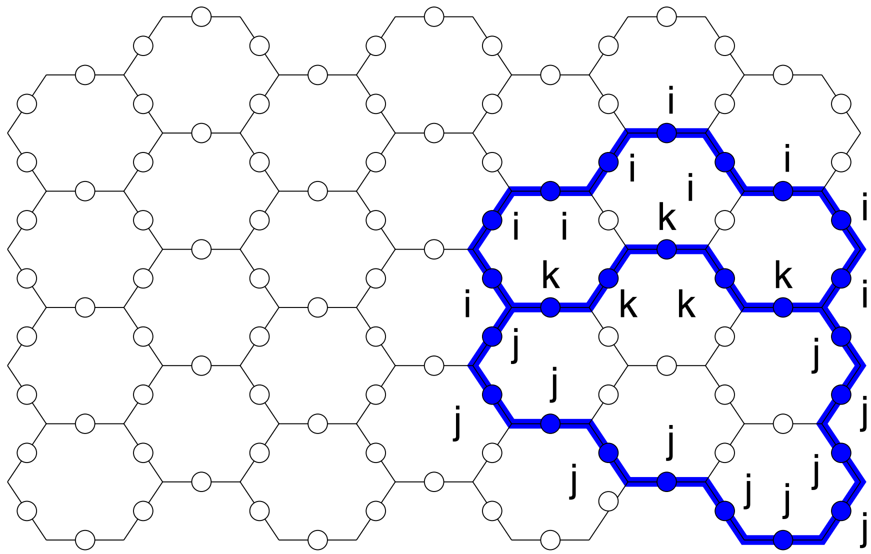
Exactly soluble lattice models



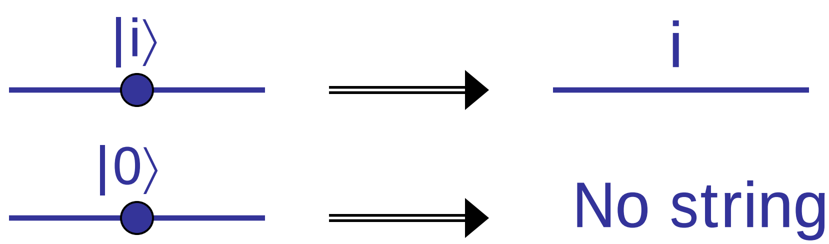
Each "spin" can be in N+1 states: $|0\rangle, |1\rangle, \ldots, |N\rangle$



Exactly soluble lattice models

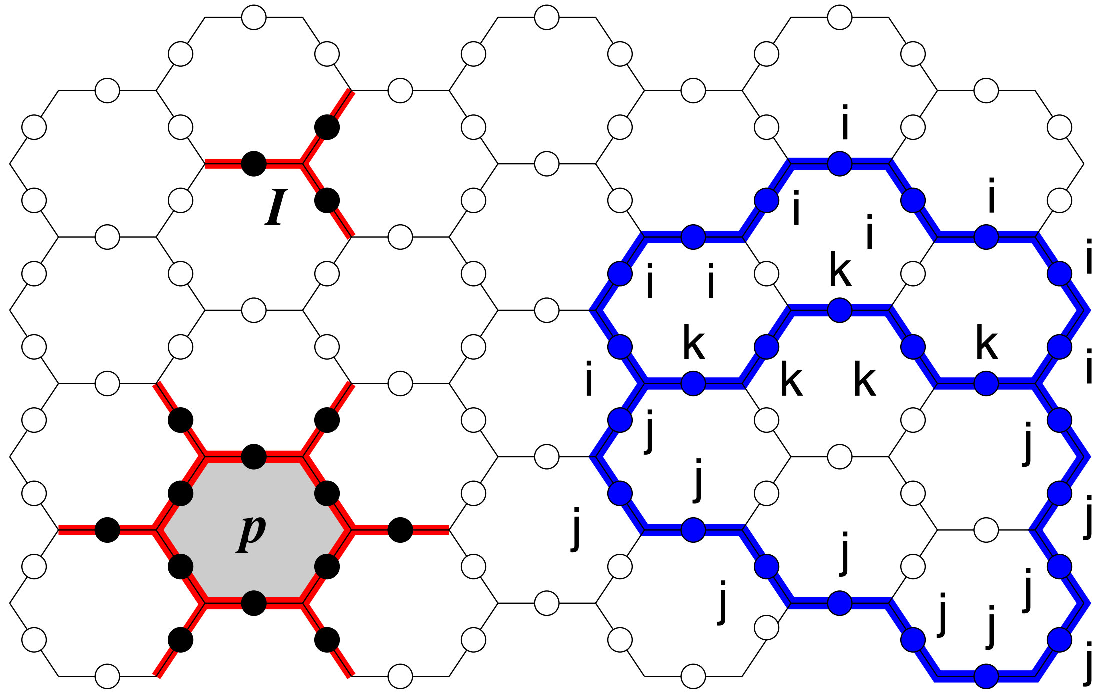


Each “spin” can be in N+1 states: $|0\rangle, |1\rangle, \ldots, |N\rangle$



Hamiltonians

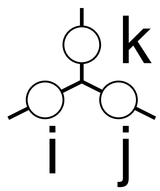
$$H = -\sum_{I} Q_{I} - \sum_{p} B_{p}$$

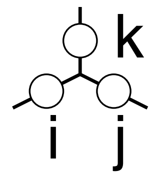


Generalization of Kitaev’s toric code

First term: Q_I

Defined by:

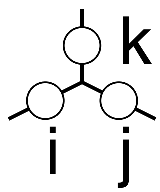
Q_I | 

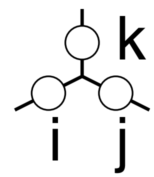
$\rangle = \delta_{ijk}$ | 

$\rangle$

First term: Q_I

Defined by:

Q_I | 

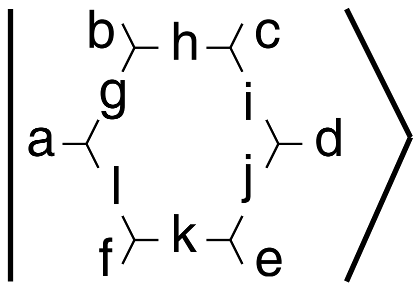
$\rangle = \delta_{ijk}$ | 

$\rangle$

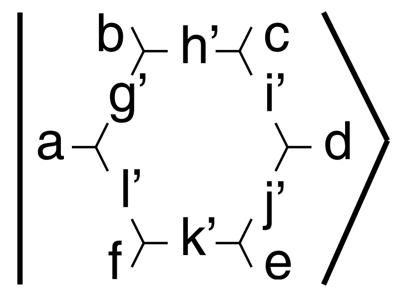
"Electric charge"

Second term: B_p

Defined by: $B_p = \sum_s d_s B^s_p$ where

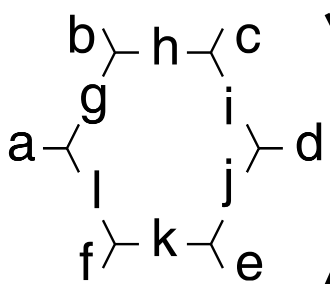
B^s_p 

$=$

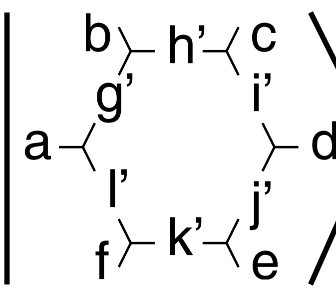
$\sum_{g'h' \cdots l'} F^{alg}_{sg'l'} F^{bgh}_{sh'g'} \cdots F^{fkl}_{sl'k'}$ 

Second term: B_p

Defined by: $B_p = \sum_s d_s B^s_p$ where:

B^s_p $\Big|$ 

$\Big\rangle$ =

$\sum_{g'h'\cdots\, l'} F^{alg}_{sg'l'} F^{bgh}_{sh'g'} \cdots F^{fkl}_{sl'k'}$ $\Big|$ 

$\Big\rangle$

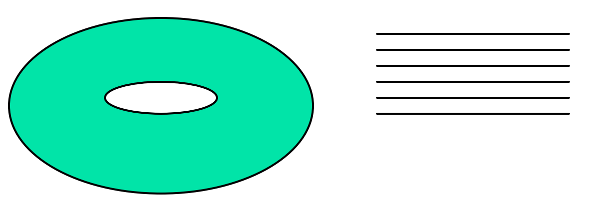
"Magnetic flux"

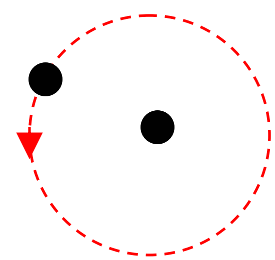
Properties of Hamiltonian

1. $\{B_p\}$, $\{Q_l\}$ commuting projectors $\Rightarrow$
   H is exactly soluble.

2. Ground state wave function is Φ.

3. Model describes a topological phase.





$$e^{i\theta}$$

Properties of Hamiltonian

4. Fixed points: Correlation length $\xi = 0$

~ zero coupling gauge theory

"Right way" to put topological theories on lattice.

Properties of Hamiltonian

4. Fixed points: Correlation length $\xi = 0$

~ zero coupling gauge theory

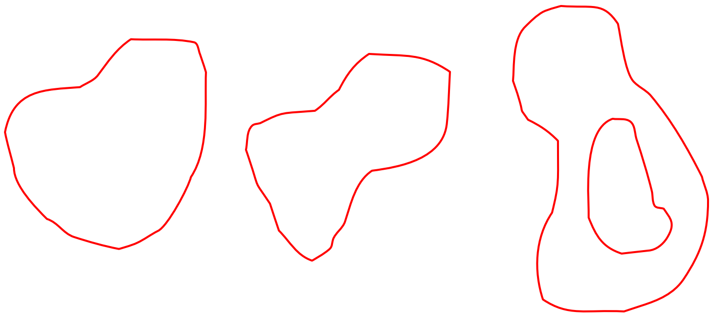
"Right way" to put topological theories on lattice.

Turaev/Viro (1992)
Ooguri (1992)

Loop quantum gravity:
"spin networks"

Example #1

1. String types: N = 1
2. Branching rules: No branching



What phase occurs when strings condense?

Example #1

Two solutions to self-consistency equations:

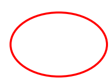
$d_0 = 1$

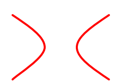
$d_1 = F^{110}_{110} = \pm 1$

$F^{000}_{000} = F^{101}_{101} = F^{011}_{011} = 1$

$F^{000}_{111} = F^{110}_{001} = F^{101}_{010} = F^{011}_{100} = 1$

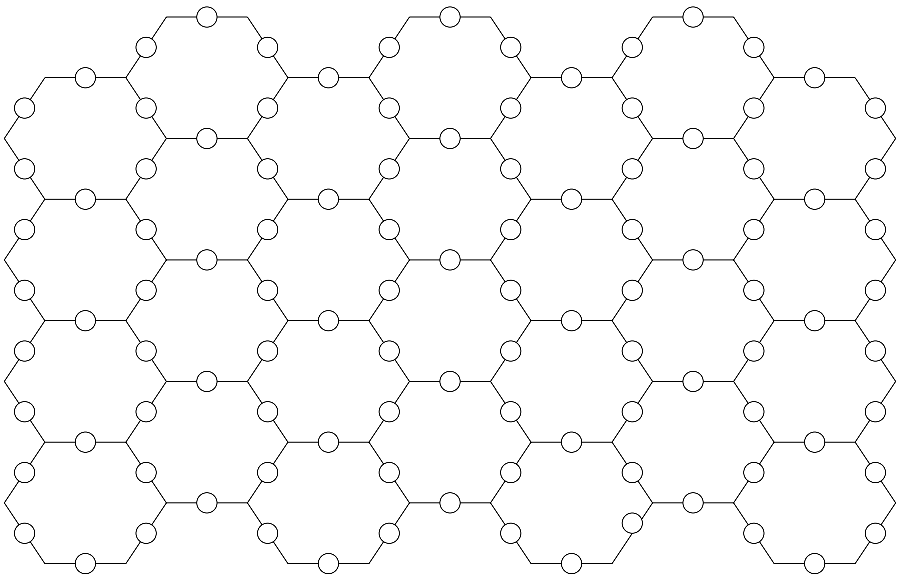
Two sets of local rules:

$\Phi($  $) = \pm\, \Phi(\quad)$

$\Phi($  $) = \pm\, \Phi($ $)$

Two solutions: $\Phi_{\pm}(X) = (\pm 1)^{N_{loops}(X)}$

Lattice realization

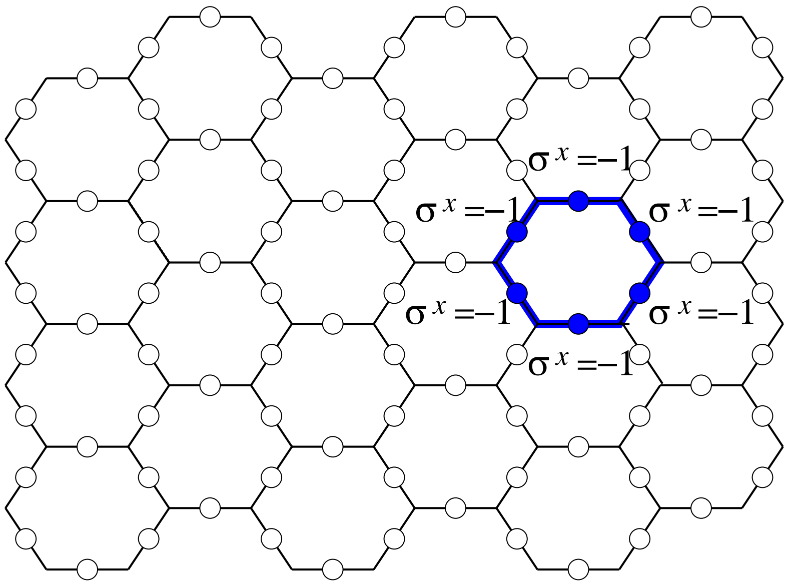


Each "spin" can be in 2 states: $|0\rangle$, $|1\rangle$

Convenient to use spin-1/2 notation:

$$|0\rangle = |\sigma^x = +1\rangle$$
$$|1\rangle = |\sigma^x = -1\rangle$$

Lattice realization



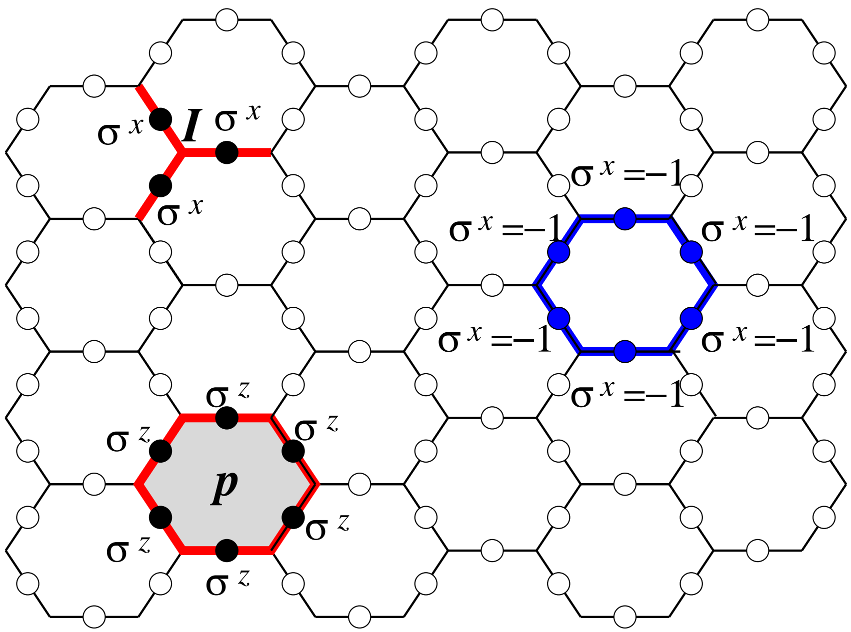
Each "spin" can be in 2 states: $|0\rangle$, $|1\rangle$

Convenient to use spin-1/2 notation:

$$|0\rangle = |\sigma^x = +1\rangle$$
$$|1\rangle = |\sigma^x = -1\rangle$$

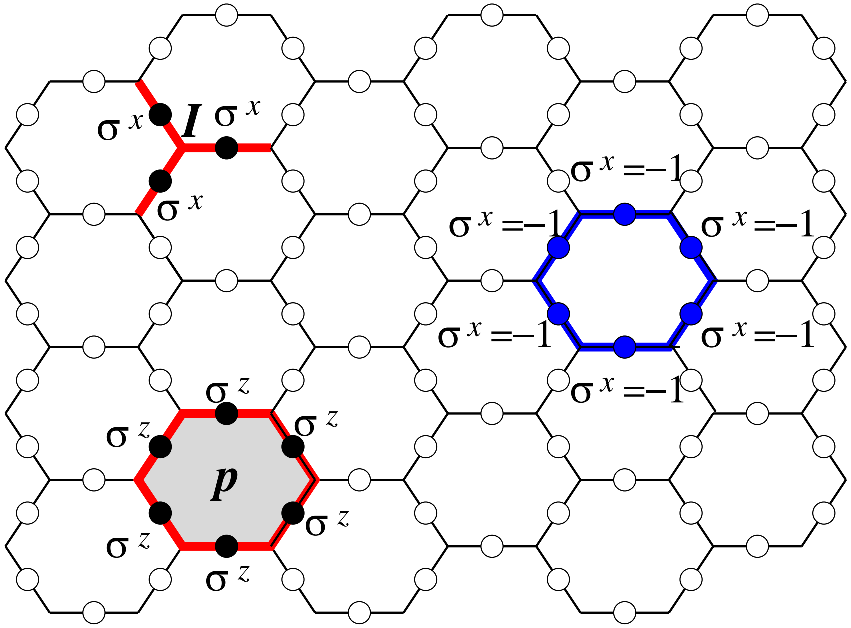
Hamiltonian: Φ_+

$$H_+ = -\sum_l \prod_a \sigma^x{}_a - \sum_p \prod_b \sigma^z{}_b$$



Hamiltonian: Φ_+

$$H_+ = -\sum_l \prod_a \sigma^x{}_a - \sum_p \prod_b \sigma^z{}_b$$

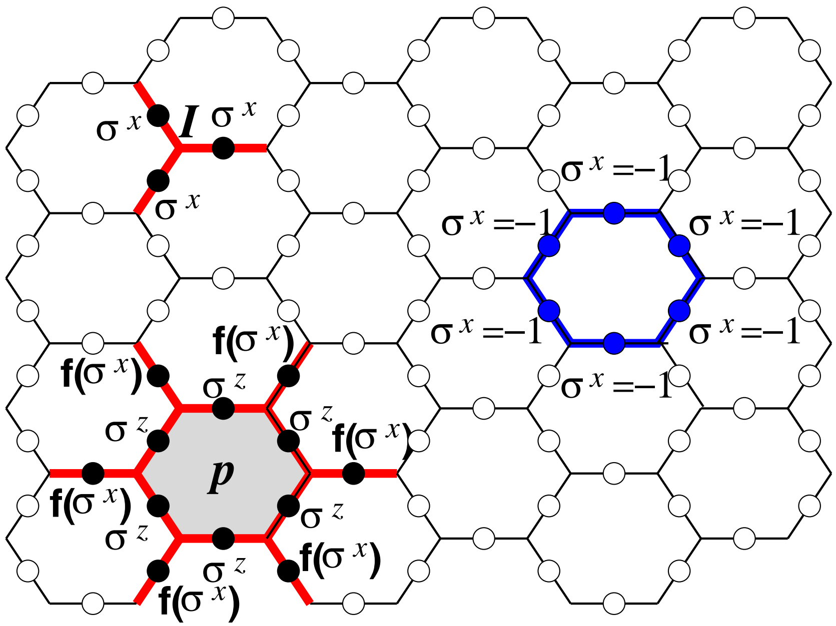


Toric code: Lattice model for Z_2 gauge theory!

Hamiltonian: Φ_-

$$H_- = -\sum_l \prod_a \sigma^x_a - \sum_p \prod_b \sigma^z_b \cdot \prod_c i^{(1-\sigma^x_c)/2}$$

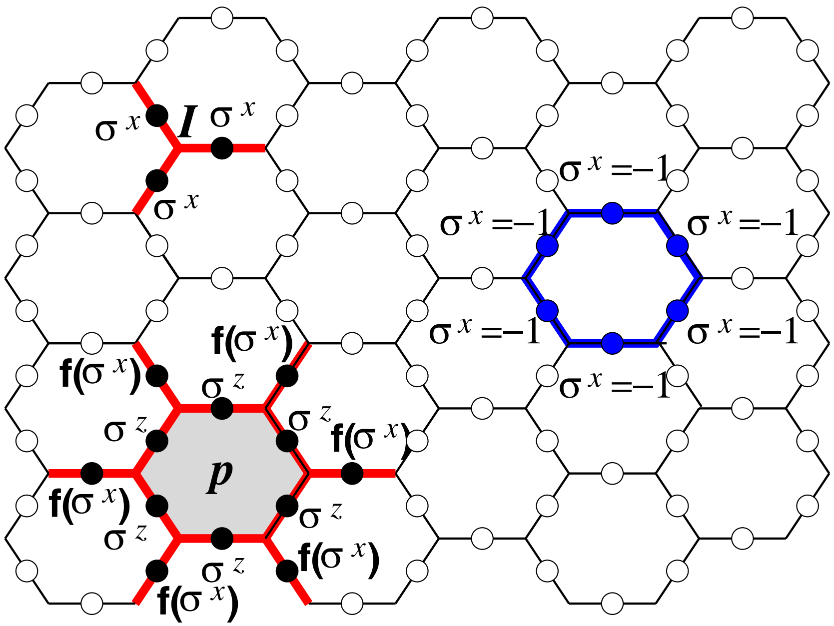
$f(\sigma^x) = i^{(1-\sigma^x)/2}$



Hamiltonian: Φ_-

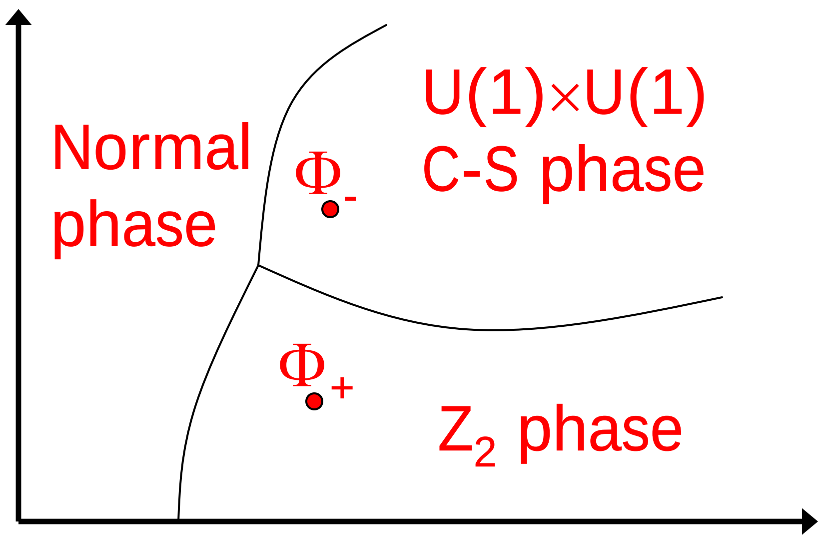
$$H_- = -\sum_l \prod_a \sigma^x_a - \sum_p \prod_b \sigma^z_b \cdot \prod_c i^{(1-\sigma^x_c)/2}$$

$f(\sigma^x) = i^{(1-\sigma^x)/2}$



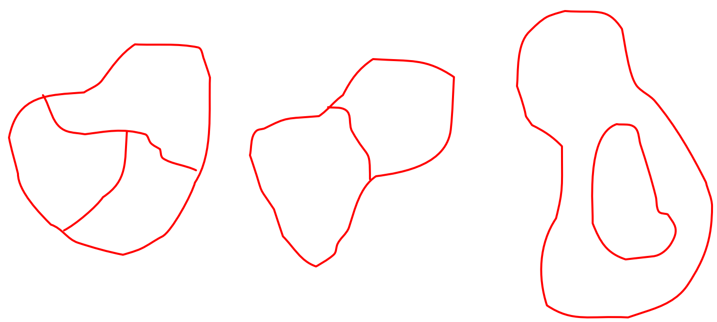
U(1)×U(1) Chern-Simons theory with semions!

Two string condensed phases



Example #2

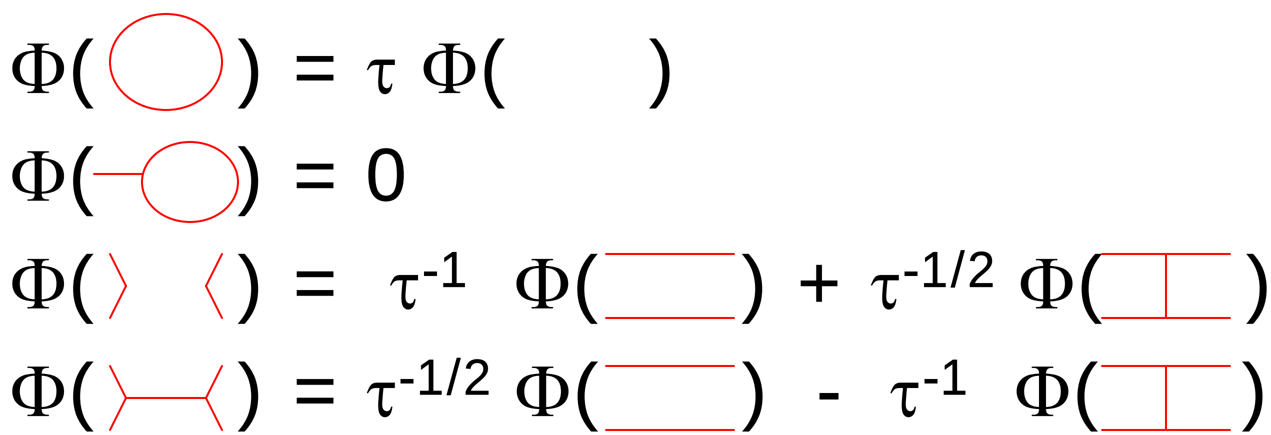
1. String types: N = 1
2. Branching rules: {1,1,1}



What phase occurs when string-nets condense?

Example #2

Only one set of self-consistent local rules:



$$\tau = (1+5^{1/2})/2$$

Example #2

Wave function: No closed form!

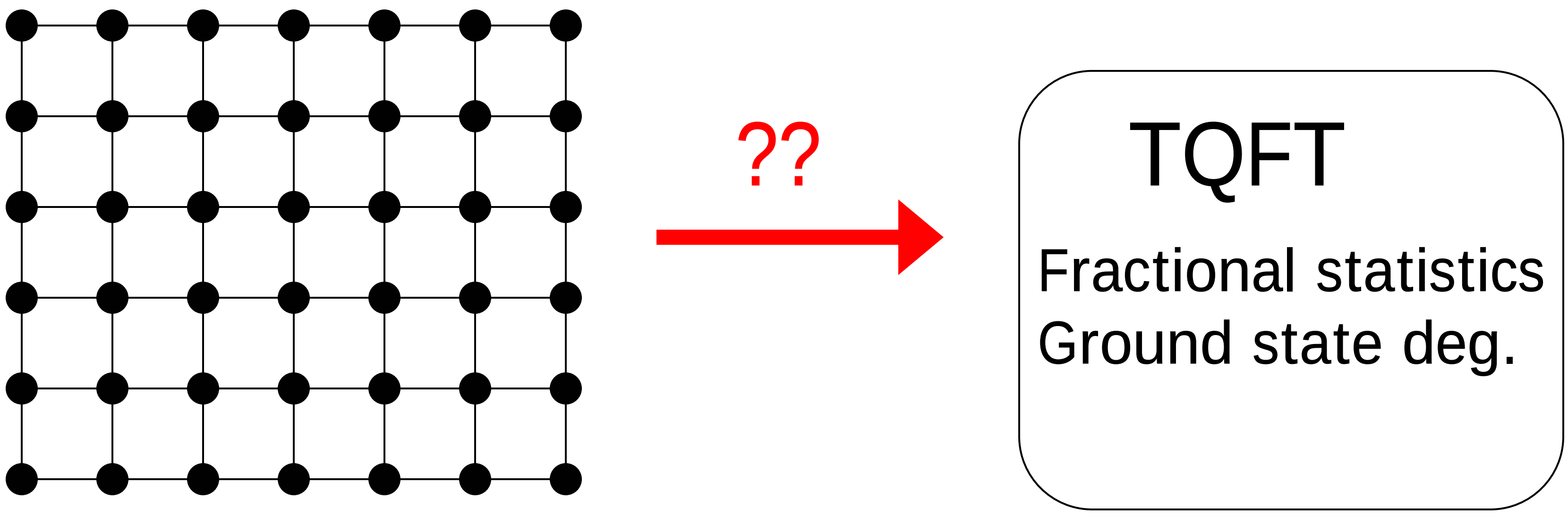
Hamiltonian: Spin-1/2 model (complicated)

Topological phase: $SO_3(3) \times SO_3(3)$ Chern-Simons theory

- "Fibonacci theory"
- Non-abelian anyons

How do topological phases emerge from microscopic spins?

- How can we realize them? What interactions favor them?



How do topological phases emerge from microscopic spins?

- How can we realize them? What interactions favor them?