String-net condensation and topological phases in quantum spin systems

Michael Levin, Xiao-Gang Wen

MIT
Topological phases
Topological phases

- Gapped
Topological phases

- Gapped
- Degenerate ground state on torus
Topological phases

- Gapped
- Degenerate ground state on torus
- Fractional statistics

\[ e^{i\theta} \]
Real life examples

- FQH liquids.
Real life examples

- FQH liquids.

- Hope: Frustrated magnets
  - Many theoretical models
  - A few candidate materials
    - $\text{Cs}_2\text{CuCl}_4$
    - $\kappa-(\text{BEDT-TTF})_2\text{Cu}_2(\text{CN})_3$
Theory of topological phases
Theory of topological phases

- We understand: 
  - Low energy/Long distance physics

- We’re missing: 
  - Connection with microscopics!
How do topological phases emerge from microscopic spins?

- How can we realize them? What interactions favor them?
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- How can we realize them? What interactions favor them?

TQFT
Fractional statistics
Ground state deg.
Outline

I. Physical picture

II. Quantitative results
   A. Explicit ground state wave functions
   B. Exactly soluble Hamiltonians

III. Examples
String-net models
1. **String types:** Number of string types $N$.

$$i \quad (i = 1, \ldots, N)$$

2. **Branching rules:** Triplets $\{i, j, k\}$ allowed to meet at a point.
Data
1. Number of string types: \( N = 2 \).

2. Branching rules: \{2, 2, 2\}, \{1, 2, 2\}.
String-net Hamiltonian

\[ H = t H_t + U H_U \]

String kinetic energy + String tension
String-net Hamiltonian

\[ H = t H_t + U H_U \]
So what?

- String-net condensed phases ARE topological phases!
- Mechanism for topological phases
- Very general: all non-chiral topological phases can be realized
What does this have to do with spin systems?
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Low energy degrees of freedom can be string-like:
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Low energy degrees of freedom can be string-like:
Examples

- $\mathbb{Z}_2$ gauge theory
- $SO_3(3) \rtimes SO_3(3)$ Chern-Simons
- $S_3$ gauge theory
Representative wave functions

Want “fixed-point” wave functions:
\[ \Phi(\emptyset \emptyset \emptyset) = \ldots \]
Ansatz

1. Amplitude of $\Phi$ only depends on topology of string-net: e.g., $\Phi(\text{\includegraphics[height=1cm]{string-net1}}) = \Phi(\text{\includegraphics[height=1cm]{string-net2}})$

2. $\Phi$ satisfies local constraint equations:
   
   $\Phi(\text{\includegraphics[height=1cm]{constraint1}}) = d_i \Phi(\text{\includegraphics[height=1cm]{constraint2}})$

   $\Phi(\text{\includegraphics[height=1cm]{constraint3}}) = 0 \quad \text{if } i \neq j$

   $\Phi(\text{\includegraphics[height=1cm]{constraint4}}) = \sum_n F_{ijm_{kn}} \Phi(\text{\includegraphics[height=1cm]{constraint5}})$
Local constraints specify $\Phi$ completely

\[
\Phi(\begin{array}{c} j \\ i \\ k \end{array}) = \sum_l F_{ki0}^{ij} \Phi(\begin{array}{c} i \\ k \end{array})
\]

\[
= F_{ki0}^{ij} \Phi(\begin{array}{c} i \\ k \end{array})
\]

\[
= F_{ki0}^{ij} d_i d_k \Phi(\text{vacuum})
\]

\[
= F_{ki0}^{ij} d_i d_k
\]
But rules are not usually self-consistent!
Self-consistency conditions

\[ \sum_n F_{mpl}^{kpn} F_{jip}^{mns} F_{jsn}^{lkr} = F_{jip}^{qkr} F_{riq}^{mls} \] (a)

\[ F_{ijm}^{kln} = F_{lkn}^{jim} = F_{jim}^{lkn} = F_{jm}^{lni} (d_m d_n / d_i d_j)^{1/2} \] (b)

\[ F_{ijk}^{ji0} = (d_k / d_i d_j)^{1/2} \delta_{ijk} \] (c)

(\text{where } \delta_{ijk} = 1 \text{ if } \{i,j,k\} \text{ allowed, } 0 \text{ otherwise}).
Self-consistency conditions

\[ \sum_n F^{mlq}_{kpn} F^{jip}_{mns} F^{jls}_{nkr} = F^{jip}_{qkr} F^{rlq}_{mls} \quad (a) \]

\[ F^{ijm}_{kln} = F^{lkm}_{jin} = F^{jim}_{lnk} = F^{imj}_{kln} \left( \frac{d_m d_n}{d_j d_i} \right)^{1/2} \quad (b) \]

\[ F^{ijk}_{ji0} = \left( \frac{d_k}{d_i d_j} \right)^{1/2} \delta_{ijk} \quad (c) \]

(where \( \delta_{ijk} = 1 \) if \( \{i,j,k\} \) allowed, 0 otherwise).

Solutions \( \Leftrightarrow \) fixed point wave functions \( \Phi \)
Classification of non-chiral topological phases

Solutions \((F_{ij}^{m_{kl}}, d_i, \delta_{ijk})\) of (a)-(c)  

String-net condensates/non-chiral topological phases
Classification of non-chiral topological phases

Solutions \((F_{ijk}^{m kl n}, d_{i}, \delta_{ijk})\) of (a)-(c)

String-net condensates/non-chiral topological phases

“Tensor categories”
Exactly soluble lattice models

Each “spin” can be in N+1 states: $|0\rangle, |1\rangle, \ldots, |N\rangle$
Exactly soluble lattice models

Each “spin” can be in $N+1$ states: $|0\rangle, |1\rangle, \ldots, |N\rangle$

$|i\rangle$  \rightarrow  i  \rightarrow  No string

$|0\rangle$
Exactly soluble lattice models

Each “spin” can be in $N+1$ states: $|0\rangle,|1\rangle,\ldots,|N\rangle$

|0⟩  | i  | No string
| i⟩  |
Hamiltonians

\[ H = -\sum_l Q_l - \sum_p B_p \]

Generalization of Kitaev’s toric code
First term: $Q_1$

Defined by:

$$Q_1 | ij \rangle = \delta_{ijk} | ij \rangle$$
First term: $Q_1$

Defined by:

$$Q_1 | i_j k \rangle = \delta_{ijk} | i_j k \rangle$$

“Electric charge”
Second term: $B_p$

Defined by: $B_p = \sum_s d_s B^s_p$ where

$$B^s_p = \sum_{g'h'\ldots l'} F_{g'li'}^\text{alg} F_{sh'g'}^\text{bgh} \ldots F_{sl'k'}^\text{fkl}$$
Second term: $B_p$

Defined by: $B_p = \sum_s d_s B_p^s$ where:

$$B_p^s = \sum_{g'h'...l'} F_{alg'g'...l'} F_{bgh'...l'} F_{fkl'...l'}$$

"Magnetic flux"
Properties of Hamiltonian

1. \( \{B_p\}, \{Q_i\} \) commuting projectors \( \Rightarrow \) H is exactly soluble.

2. Ground state wave function is \( \Phi \).

3. Model describes a topological phase.
Properties of Hamiltonian

4. Fixed points: Correlation length $\xi = 0$
   ~ zero coupling gauge theory

“Right way” to put topological theories on lattice.
Properties of Hamiltonian

4. Fixed points: Correlation length $\xi = 0$
   ~ zero coupling gauge theory

“Right way” to put topological theories on lattice.

Ooguri (1992)
Example #1

1. String types: $N = 1$
2. Branching rules: No branching

What phase occurs when strings condense?
Example #1

Two solutions to self-consistency equations:

\[ d_0 = 1 \]
\[ d_1 = F^{110}_{110} = \pm 1 \]
\[ F^{000}_{000} = F^{101}_{101} = F^{011}_{011} = 1 \]
\[ F^{000}_{111} = F^{110}_{001} = F^{101}_{010} = F^{011}_{100} = 1 \]

Two sets of local rules:

\[ \Phi(\bigcirc) = \pm \Phi(\quad) \]
\[ \Phi(\bigtriangledown\bigtriangleup) = \pm \Phi(\quad) \]

Two solutions: \( \Phi_{\pm}(X) = (\pm 1)^{N_{\text{loops}}(X)} \)
Lattice realization

Each “spin” can be in 2 states: \( |0\rangle \), \( |1\rangle \)

Convenient to use spin-1/2 notation:
\[
|0\rangle = |\sigma^x = +1\rangle \\
|1\rangle = |\sigma^x = -1\rangle
\]
Lattice realization

Each “spin” can be in 2 states: $|0\rangle$, $|1\rangle$

Convenient to use spin-1/2 notation:

$|0\rangle = |\sigma^x = +1\rangle$

$|1\rangle = |\sigma^x = -1\rangle$
Hamiltonian: $\Phi_+$

$$H_+ = -\sum_l \prod_a \sigma^x_a - \sum_p \prod_b \sigma^z_b$$
Hamiltonian: $\Phi_+$

$$H_+ = -\sum_l \prod_a \sigma^x_a - \sum_p \prod_b \sigma^z_b$$

Toric code: Lattice model for $\mathbb{Z}_2$ gauge theory!
Hamiltonian: $\Phi_-$

$$H_- = -\sum_l \prod_a \sigma^x_a - \sum_p \prod_b \sigma^z_b \cdot \prod_c i^{(1-\sigma^x_c)/2}$$

$$f(\sigma^x) = i^{(1-\sigma^x)/2}$$
Hamiltonian: $\Phi_-$

$$H_- = -\sum_l \prod_a \sigma^x_a - \sum_p \prod_b \sigma^z_b \cdot \prod_c i^{(1-\sigma^x_c)/2}$$

$$f(\sigma^x) = i^{(1-\sigma^x)/2}$$

$U(1) \times U(1)$ Chern-Simons theory with semions!
Two string condensed phases

![Diagram showing phases and points](image)

- Normal phase
- U(1)×U(1) phase
- C-S phase
- Z₂ phase

Points:
- \( \Phi^- \)
- \( \Phi^+ \)
Example #2

1. String types: \( N = 1 \)

2. Branching rules: \( \{1,1,1\} \)

What phase occurs when string-nets condense?
Example #2

Only one set of self-consistent local rules:

\[ \Phi(\bigcirc) = \tau \Phi(\bigcirc) \]
\[ \Phi(\overline{\bigcirc}) = 0 \]
\[ \Phi(\langle \rangle) = \tau^{-1} \Phi(\overline{\bigcirc}) + \tau^{-1/2} \Phi(\bigcirc) \]
\[ \Phi(\langle \rangle) = \tau^{-1/2} \Phi(\overline{\bigcirc}) - \tau^{-1} \Phi(\bigcirc) \]

\[ \tau = (1+5^{1/2})/2 \]
Example #2

Wave function: No closed form!

Hamiltonian: Spin-1/2 model (complicated)

Topological phase: $SO_3(3) \times SO_3(3)$ Chern-Simons theory
- “Fibonacci theory”
- Non-abelian anyons
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- How can we realize them? What interactions favor them?

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String-net condensation!

TQFT
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