Realizing non-Abelian statistics in time-reversal-invariant condensed matter systems

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Outline

- Motivation
- Quantum Loop Gases, S-matrices and Braid Matrices
- Wave Function Engineering in $2+1$ Dimensions from Field Theories in $1+1$ Dimensions
- Quantum Loop Lattice Models with Non Abelian Statistics, $SU(2)$ and $SO(3)$ and Generalized Potts Models
- Topological Phases and Phase Transitions
- Entanglement Entropy of Scale Invariant Wave Functions
- Conclusions

Spin Liquids and Topological States of Matter

- Liquid phases of electron fluids and spin systems without long range order, with or without time reversal symmetry breaking
- Quasiparticles: vortices with fractional charge and fractional statistics (Abelian and non-Abelian)
- Hidden Topological Order and Topological Vacuum Degeneracy
- Finite-dimensional quasiparticle Hilbert spaces $\Rightarrow$ universal topological quantum computer

"Known" Topological Quantum Liquids

- 2DEG Fractional Quantum Hall Liquids
  - Abelian FQH states (Laughlin and Jain): fractional charge and Abelian fractional statistics
  - Non-Abelian FQH states: Is $\nu = 5/2$ a Pfaffian (Moore-Read) FQH state? (firm candidate) Is the plateau at $\nu = 12/5$ a parafermion state? (good possibility)
- Rapidly rotating Bose gases: possible non-Abelian (Pfaffian) FQH state of bosons at $\nu = 1$
- Time-Reversal Breaking Superconductors: Is Sr_2RuO_4 a $p + ip$ superconductor? (looks good)

Challenges

- To develop a consistent theory of topological phases (i.e. beyond FQH states) and to understand the underlying mechanisms
- What are the generic phases of models of topological liquids
- Is the gap necessary? Can a topological liquid be gapless?
- Concrete examples of lattice models with local interactions with topological phases
- Fractional Statistics: Abelian and non-Abelian
- There has been some progress in constructing models with Abelian statistics
- To find experimentally realizable models (looks promising, not quite there yet)

Time Reversal Invariant Spin Liquids: Quantum Dimer Models

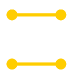
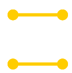
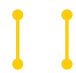
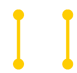
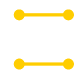
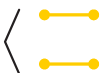
- Simple local models describing strongly frustrated and ring exchange quantum spin systems with a large spin gap and no long range spin order
- They typically exhibit spin gap phases with different types of valence bond crystal orders
- QDM have special solvable points, the Rokhsar-Kivelson (RK) point, where the exact ground state wave function has the short range RVB form

$$|\Psi_{\rm RVB}\rangle = \sum_{\{C\}} |C\rangle, \qquad \{C\} = \text{ all dimer coverings of the lattice}$$

- – Bipartite lattices: the RK points are quantum (multi) critical points, described by an effective field theory with $z = 2$ and massless deconfined spinons, or first order transitions
  – Non-bipartite lattices: QDMs have topological $\mathbb{Z}_2$ deconfined phases with massive spinons and a topological 4-fold ground state degeneracy on a torus (Moessner and Sondhi, 1998)

The Quantum Dimer Model

$$H_{\text{RK}} = \sum_i (vV_i - tF_i), \qquad \text{Rokhsar and Kivelson (1988)}$$

$V_i = \big|$  $\big\rangle\big\langle$  $\big| + \big|$  $\big\rangle\big\langle$  $\big|$ $\qquad$ $F_i = \big|$  $\big\rangle\big\langle$ $\big| + \big|$ $\big\rangle\big\langle$  $\big|$

Here each bar represents a spin singlet bond.

For $t = v$ $\qquad \Rightarrow H_{\text{RK}} = \sum_i Q_i^\dagger Q_i$, with $Q_i = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$.

- The ground state wave function $|\Psi_0\rangle$ has $E = 0$

$$|\Psi_0\rangle = \frac{1}{\sqrt{Z_{\text{cl}}}} \sum_C |C\rangle \ ,$$

  where Z_{cl} is the sum over all dimer configurations

- Equal-*time* correlators in the quantum dimer model at the RK point are given by correlators of the classical dimer model.

- This is actually a loop model: loops are the dimer moves from a reference state. This is the simplest loop model: the $SU(2)_1$ fully packed loop model.

Dimers, heights and effective field theory

Moessner, Sondhi and Fradkin; Ardonne, Fendley and Fradkin

- The QDM can be mapped to a height model
- Plaquette flip changes the height of that plaquette by ± 4, and the average height of the surrounding sites by ± 1.
- Equivalent configurations: $h \cong h + 4$.
- Continuum limit: $h \cong 4\varphi(x)$
  Compactification Radius: $\varphi(x) \cong \varphi(x) + 1$.
- The Quantum Lifshitz model
  Hamiltonian:
  $$H = \int d^2x \left[\frac{1}{2}\Pi^2 + \frac{\kappa^2}{2} \left(\nabla^2 \varphi \right)^2 \right]$$
  This is the Quantum Lifshitz Model. (Henley; Moessner, Sondhi and Fradkin)
- Action in imaginary time $\tau \Leftrightarrow$ smectic layers in 3D classical statistical mechanics at the Lifshitz transition.
  $$S = \int d^2x \int d\tau \left[\frac{1}{2} \left(\partial_\tau \varphi \right)^2 + \frac{\kappa^2}{2} \left(\nabla^2 \varphi \right)^2 \right]$$

Scale Invariant Ground State Wave Functions and 2D Classical Critical Phenomena

- $$\int d^2\vec{x}\left[-\frac{1}{2}\left(\frac{\delta}{\delta\varphi}\right)^2+\frac{\kappa^2}{2}(\nabla^2\varphi)^2\right]\Psi[\varphi]=E\Psi[\varphi]$$

$$Q(\mathbf{x})\equiv\frac{1}{\sqrt{2}}\left(\frac{\delta}{\delta\varphi}+\kappa\nabla^2\varphi\right)\qquad Q^\dagger(\mathbf{x})\equiv\frac{1}{\sqrt{2}}\left(-\frac{\delta}{\delta\varphi}+\kappa\nabla^2\varphi\right)$$

- Ground state wave-function, $\Psi_0[\varphi]$

$$Q(\vec{x})\Psi_0[\varphi]=0\qquad\Rightarrow\qquad\Psi_0[\varphi]\propto e^{-\frac{\kappa}{2}\int d^2x\ (\nabla\varphi(\mathbf{x}))^2}$$

$$\|\Psi_0\|^2=\int\mathcal{D}\varphi\ e^{-\kappa\int d^2x\ (\nabla\varphi(\mathbf{x}))^2}$$

- The ground state wave function is conformally invariant

Mapping to a 2D $c = 1$ Euclidean CFT

- The probability for a configuration $|\varphi\rangle$ is the Gibbs weight of a 2D classical Gaussian model, a Euclidean 2D free massless scalar field.

- At these quantum critical points the ground state wave function is scale invariant

- The equal-time expectation values of the observables are correlators in this $c = 1$ conformal field theory.

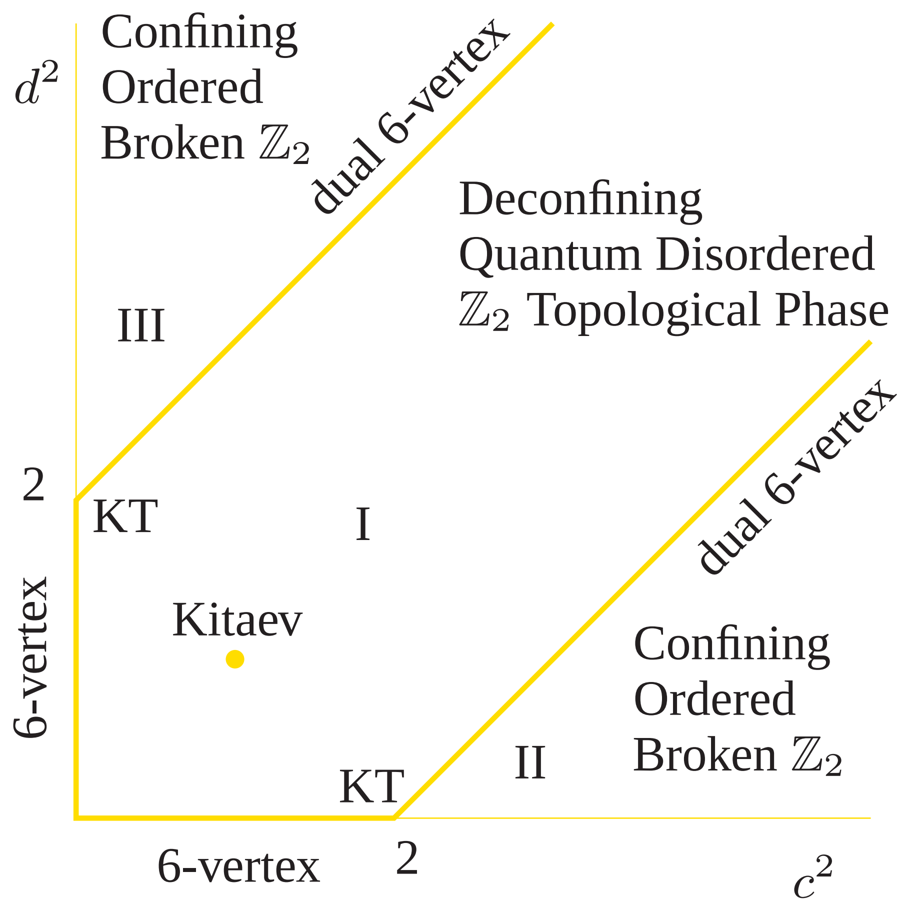
- The equal-time expectation value for operators in the quantum Lifshitz model are given by correlators of the massless free boson conformal field theory with central charge $c = 1$. Time-dependent correlators exhibit power-law behavior with dynamical exponent $z = 2$.

- Matching the correlation functions of the RK and Lifshitz models, one finds $\kappa = 1/2\pi$.

- This is a multicritical point with many relevant perturbations: e.g diagonal dimers drive the system into a $\mathbb{Z}_2$ topological phase

Phase diagram for a quantum eight vertex model

Ardonne, Fendley and Fradkin



Strategy for a Generalization

- Each basis state in the Hilbert space is a loop configuration in 2D
- We start with the statistics we wish to have, and work backward
- Algebraic characterization of braiding in both $SU(2)_k$ and $SO(3)_k$ Chern-Simons theories.
- Braid matrix of a 2+1-dimensional theory as a limit of the S-matrix of an associated relativistic 1+1 dimensional model
- We construct quantum 2D models with these braid relations by utilizing the structure of the factorizable S-matrices of integrable 1D models.
- We embed the 1D model in 2D Euclidean space, and find a (Rokhsar-Kivelson-type) quantum Hamiltonian whose ground state has the properties expected of a model with non-Abelian statistics.
- Loop gases:
  - $SU(2)_k$ case: $O(n)$ lattice model with $n = 2\cos(\pi/(k+2))$ (self-avoiding and mutually-avoiding loops)
  - $SO(3)_k$ case: domain walls of a Q-state Potts model with $Q = 4\cos^2(\pi/(k+2))$ (loops intersect and branch: nets)

Wave Function Engineering in $2+1$ Dimensions from Field Theories in $1+1$ Dimensions

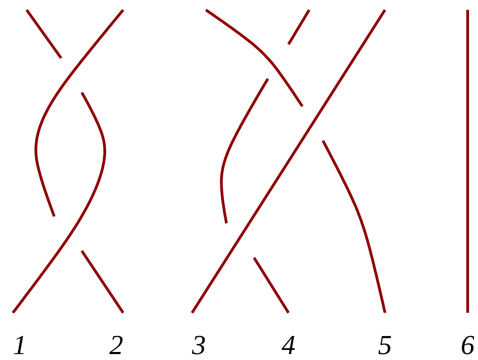
- Project the world lines of the particles down to the plane $\Rightarrow$ loops
- The basis states $|s\rangle$ of the Hilbert space are configurations of 2D loops
- This assumes that the $2+1$-dimensional theory is in some sense holographic
- The wave function Ψ of this ground state can be written as

$$\langle s|\Psi\rangle = \frac{e^{-\mathcal{S}(s)}}{\sqrt{Z}}$$

- $\mathcal{S}(s)$: action of the *classical* 2D loop model for the configuration s.
- Z is the 2D partition function with weight $|\langle s|\Psi\rangle|^2$, which is the functional integral over all configurations s with weight $e^{-\mathcal{S}(s)-\mathcal{S}^*(s)}$.

2D wave functions and 1D S-matrices

- The plane is a $1+1$-dimensional Euclidean space time
- A strand configuration is the "time" evolution of a system of particles in 1D



- A 2D wave function is given by an evolution in 1+1 dimensions
- it is the evolution of a vector in $V^{\otimes N}$, specified at the boundary
  a 1D wave function specified in terms of a set of coordinates and momenta of the particles, $x_1, p_1, \ldots, x_N, p_N$ at the boundary
- The evolution is specified by the 1D S-matrix which is a matching condition for $x_i < x_{i+1}$ and $x_i > x_{i+1}$

- $\psi_{V_i \otimes V_{i+1}}(x_i \gg x_{i+1}) = S_i(\theta_{i,i+1}) \psi_{V_i \otimes V_{i+1}}(x_i \ll x_{i+1})$
  where $\theta_{i,i+1}$ is the relative *rapidity* of the two particles

- For integrable systems the S-matrix obeys the Yang-Baxter Equation
  (same for Boltzmann weights in lattice models)

- Correspondence between the S-matrix and the braid generators

$$B = \lim_{\theta \to \infty} \widetilde{S}(\theta), \qquad B^{-1} = \lim_{\theta \to \infty} \widetilde{S}(-\theta)$$

- $\widetilde{S}$ obeys Yang-Baxter $\Rightarrow$ B obeys the Braid Group algebra

- There is a natural representation of the Braid Group associated to a given integrable theory with a factorizable S-matrix

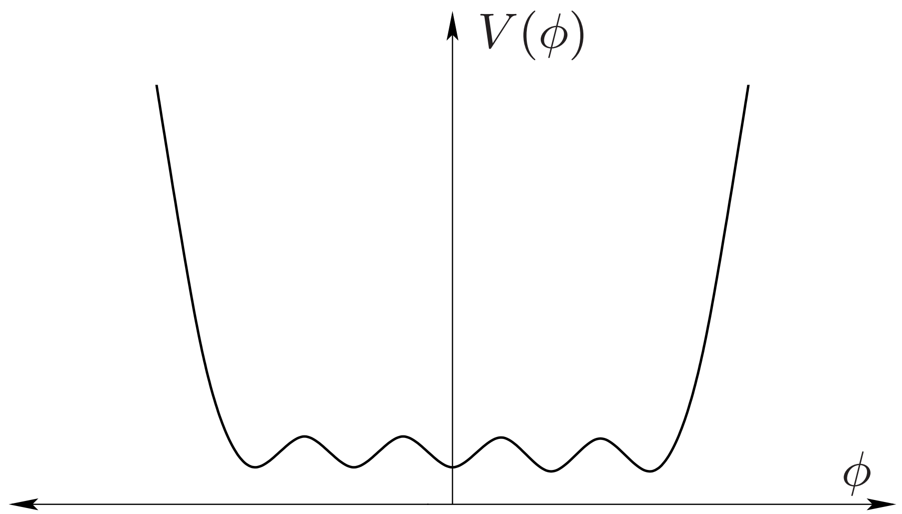
- This connection gives a prescription for constructing wave functions with excitations with non-Abelian braid statistics

- Topological theory: Unitarity requires that braiding be compatible with the Jones-Wenzl projection

- A 2D Hamiltonian can be found à la Rokhsar-Kivelson

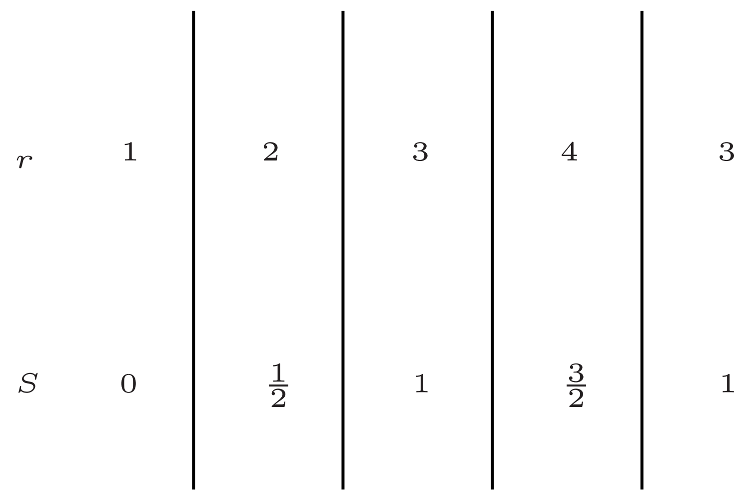
Quantum Loop Lattice Models with Non Abelian Statistics, and Generalized Potts Models

Consider a 2D classical problem with $k+1$ states; *e.g.* an RSOS model with dual spins (or heights) taking values $1, \ldots, k+1$ with a Landau-Ginzburg potential



Strands and Domain Walls

- The heights can only change by ± 1 across a domain wall
- We can regard the domain walls of this model as the strands carrying a $S = 1/2$ representation of $U_q(sl_2)$
- We can also associate a spin $S = \frac{h-1}{2}$ representation to a height $h = 1, \ldots, k+1$
- Crossing a strand $\Leftrightarrow$ tensor products of spin of the region on the left (S_L) and the spin (1/2) of the strand: $S_L \otimes 1/2 = (S_L + 1/2) \oplus (S_L - 1/2) \Leftrightarrow h_R = h_L \pm 1$
- The Jones-Wenzl projector is satisfied: $k+1$ consecutive strands cannot have spin $(k+1)/2$



Quantum Loop Models on a Honeycomb Lattice

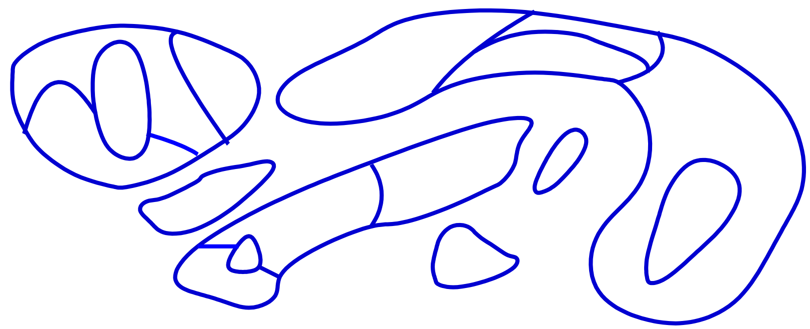
- We want to define local Hamiltonians on a honeycomb lattice whose ground state wave functions are given by the weights of loop models
- The Hamiltonians have the RK form: they are a sum of projection operators
- The ground state is annihilated by all projection operators and has zero energy
- The off-diagonal terms in the Hamiltonians are ergodic in the configuration space
  - Every link of the lattice is either occupied by a strand or empty
  - $SU(2)_k$: occupied links are assigned a spin $1/2$ representation of $U_q(sl_2)$
  - $SO(3)_k$: occupied links are assigned a spin 1 representation of $U_q(sl_2)$
  - An empty site corresponds to the identity
  - At each vertex the configurations that appear in the ground state obey the fusion rules of $U_q(sl_2)$

The $SU(2)_k$ lattice loop model

- The ground state must consist of a superposition of configurations where the strands form self- and mutually-avoiding loops which are not fully packed
- Each loop should have a weight d, and to be a purely topological ground state, there should be no weight per unit length.
- The strands form closed non-intersecting loops, *i.e.* each vertex has either 0 or 2 links with occupied links touching it.
- topologically identical configurations must have the same weight
- d-isotopy: If two configurations are identical except for one having a closed loop around a single plaquette then the weight of the configuration without the single-plaquette loop is d times that of the one with it.
- Freedman, Nayak and Shtengel constructed local Hamiltonians on a honeycomb lattice satisfying these rules for general d (without Jones-Wenzl projection)
- For $d \leq 2$ the ground states are critical and correspond to the CFT of the $O(n)$ loop models with $n = d = 2\cos(\pi/(k+2))$

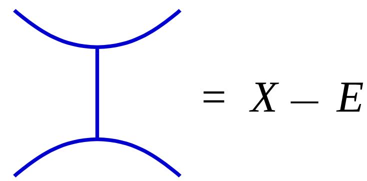
The $SO(3)_k$ loop model on the honeycomb lattice

- The strands are assigned a spin 1 representation of $U_q(sl_2)$
- The strand configurations form representations of a BMW algebra
- At a vertex we either have 0, 2 or 3 strands
- A typical configuration in the spin-1 loop model

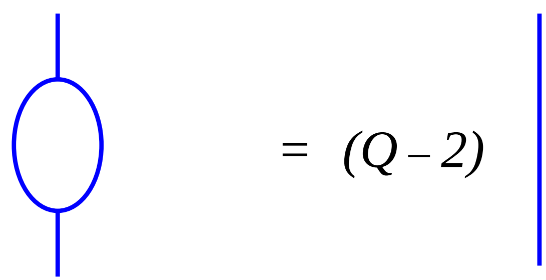


- The lines in this figure represent "spin-1" particles
- We must now allow for trivalent vertices, *i.e.* the loops are now allowed to branch and merge $\Rightarrow$ the spin-1 loop model has branching loops
- spin 1 appears in the tensor product of two spin-1 reps $\Rightarrow$ trivalent vertices
- The BMW relation $E_j^2 = (Q-1)E_j$ implies that isolated loops in the spin-1 model receive a weight of $Q - 1 = d^2 - 1$

- Because trivalent vertices occur here, however, all loops need not be isolated. The projector onto spin-1 is proportional to $X - E$, so we associate this with two neighboring trivalent vertices,



- The relation $(X_j - E_j)^2 = (Q-2)(X_j - E_j)$ means that a configuration with a loop with just two lines emanating from it has a weight $Q - 2$ times the configuration with the loop removed.



- Because $(X_j - E_j)E_j = 0$, no graph can contain any loop with just one external line attached to it $\Rightarrow$ no "tadpoles" are allowed.

The Chromatic Loop Model

- For the spin-1 loop model we will use the weights of the classical 2D Q state Potts model whose S-matrix yields the $SO(3)$ braid matrix
- The Potts spins reside on the dual (triangular) lattice and its domain walls occupy the links of the honeycomb lattice
- The low temperature expansion of a 2D Q state Potts model of coupling constant K can be written as a sum over configurations $\{\mathcal{L}\}$ of domain walls ("loops")

$$Z = \sum_{\mathcal{L}} e^{-KL(\mathcal{L})} \chi_Q(\mathcal{L})$$

  where $L(\mathcal{L})$ is the length of the domain walls in configuration $\mathcal{L}$, and $\chi_Q(\mathcal{L})$ is their multiplicity.
- The domain walls can intersect but do not have tadpoles
- If we shade each region of like dual spins with some color $\Rightarrow \chi_Q$ is the number of ways this shading can be done with Q colors so that no two adjacent regions have the same color

- The number of dual spin configurations which have the same loop configuration $\mathcal{L}$ is the number of Q-colorings $\chi_Q(\mathcal{L})$.
- $\chi_Q(\mathcal{L})$ is the *chromatic polynomial* of the graph dual to $\mathcal{L}$.
- The chromatic polynomial reduces to the number of colorings of the graph when Q is an integer, but can be defined for all Q by a recursion relation
- Consider two nodes connected by a line l (*i.e.* two loops sharing a boundary in the original picture). Then define $\mathcal{D}_l\mathcal{L}$ to be the graph with the line deleted, and $\mathcal{C}_l\mathcal{L}$ to be the graph with the two nodes joined into one. Then we have

$$\chi_Q(\mathcal{L}) = \chi_Q(\mathcal{D}_l\mathcal{L}) - \chi_Q(\mathcal{C}_l\mathcal{L}).$$

- $\chi_Q(\mathcal{L})$ vanishes for any configuration with a tadpole, or a strands with dangling end

The Chromatic Loop Model: the ground state of the $SO(3)$ quantum loop gas

- Strands form closed loops, but now we allow trivalent vertices
- topologically identical configurations have the same weight
- Each loop configuration $\mathcal{L}$ receives a weight $\chi_Q(\mathcal{L})$. For example, if two configurations are identical except for one having a closed loop around a single plaquette (a loop of length 6 on the honeycomb lattice, length 4 on the square), then the weight of the configuration without the single-plaquette loop is $Q-1$ times that of the one with it.
- We are only interested in the regime in which the domain walls proliferate: this is the disordered phase
- We have given an explicit construction on the honeycomb lattice

Topological Phases and Phase Transitions

- To determine the phase diagram, remember that a configuration s is weighted by $|\Psi(s)|^2$ in the quantum theory.
- Thus each weight is squared: each loop gets a weight $(Q-1)^2$.
- This suggests that the phase diagram is that of the $Q_{\rm eff}$-state Potts model, where

$$Q_{\rm eff} - 1 = (Q-1)^2 = (d^2-1)^2 = 1 + 2\cos[2\pi/(k+2)]$$

- Critical theory for $Q_{\rm eff} \leq 4$: $k = 1, 2, 3$. $k = 1$ is trivial, $k = 2$ is abelian.
- $k = 3$ is a CFT with the braiding rules of Fibonacci fractional statistics
- The critical point with $d = (1+\sqrt{5})/2$ and

$$Q_{\rm eff} = 1 + \left[\left(\frac{1+\sqrt{5}}{2}\right)^2 - 1\right]^2 = \frac{5+\sqrt{5}}{2}$$

  is a conformal field theory with $c = 14/15$ (before Jones-Wenzl projection)
- The consistency of this statement can be proven using Tutte's theorem (and a fugacity for trivalent vertices) (Freedman)
- After projection it presumably becomes a topological phase

Entanglement Entropy of 2D Quantum Critical States

ongoing project with Joel Moore and Matías Negrete

- Recent work by Kitaev and Preskill, and by Levin and Wen has shown that the entanglement (von Neumann) entropy S of a region of linear size L in 2D topological phases has the behavior

$$S = \alpha L + \gamma + O(1/L)$$

  where α is a non-universal coefficient (i.e. dependent on the short distance physics) and γ is a universal finite constant determined by the quantum dimensions of the excitations of the topological phase.

- This topological entropy plays a crucial role in single point contacts in non-Abelian FQH states (Fendley, Fisher and Nayak) and gives new meaning to the boundary entropy of quantum impurity problems and 1D boundary CFTs (Affleck and Ludwig)

- The proximity of the 2D conformal quantum critical points we discussed here to 2D topological phases suggest that they may hold clues on this behavior.

- Is there a universal signature in the von Neumann entropy of quantum critical systems?

Can you hear the shape of Schrödinger's Cat?

- In $1+1$-dimensional CFTs the entanglement entropy of a 1D interval of length L (in an otherwise infinite system) behaves as

$$S = \frac{c}{3} \log L + O(1)$$

where c is the central charge of the CFT (Calabrese and Cardy). This result has been verified in many 1D critical systems.

- For $d > 1$, it has been conjectured that S scales like L^{d-1}. This is known to hold for free fields (Srednicki) and it has been conjectured to generally hold at quantum criticality (Calabrese and Cardy), and it has been suggested (Verstraete and coworkers) that this behavior also holds for the systems we discussed here.

- We have found that in the case of a QCP with a conformally invariant ground state wave function, the entanglement entropy S obtained by observing a region A which is a subsystem of a region $A \cup B$ obeys the law

$$S = F_A + F_B - F_{A \cup B}$$

where $F = -\log Z = -\log ||\Psi_0||^2$, the normalization of the ground state wave function, with Dirichlet BC's in that region

- For the QCPs we are discussing here Z is the partition function of a CFT with Dirichlet BCs. For a bounded region of linear size L and smooth boundary, it obeys the 'Mark Kac law' ('Can you hear the shape of a drum?')

$$F = aL^2 + bL - \frac{c}{6}\chi \log L + O(1) \qquad \text{(Cardy and Peschel)}$$

where a and b are non-universal, and χ is the Euler characteristic of the region (manifold):

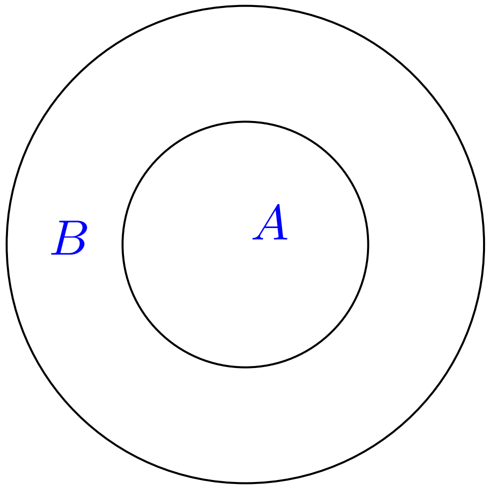
$$\chi = 2 - 2h - b, \qquad h = \#\ \text{handles},\ b = \#\ \text{boundaries}$$

- This result suggests the existence of a $\log L$ dependence with an universal coefficient associated with the central charge c of the associated CFT

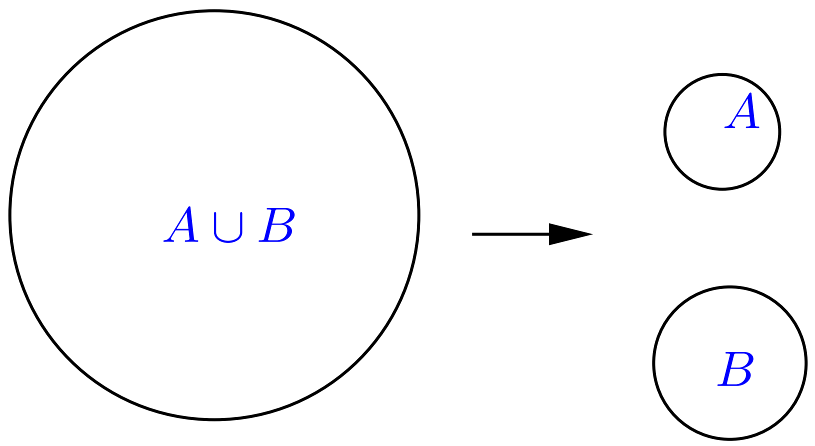
- This result implies that for a QCP described by a scale (and conformally) invariant ground state wave function, the entanglement entropy of regions A and B with a smooth common boundary has a universal logarithmic term of the form

$$\Delta S = -\frac{c}{6}\left(\chi_A + \chi_B - \chi_{A\cup B}\right) \log L$$

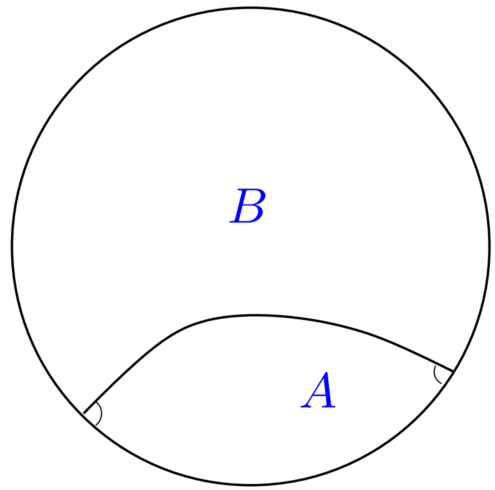
For regions $A \subseteq B$ the coefficient of the $\log L$ is zero since in this case $\chi_A + \chi_B = \chi_{A \cup B} \Rightarrow \Delta S = 0$



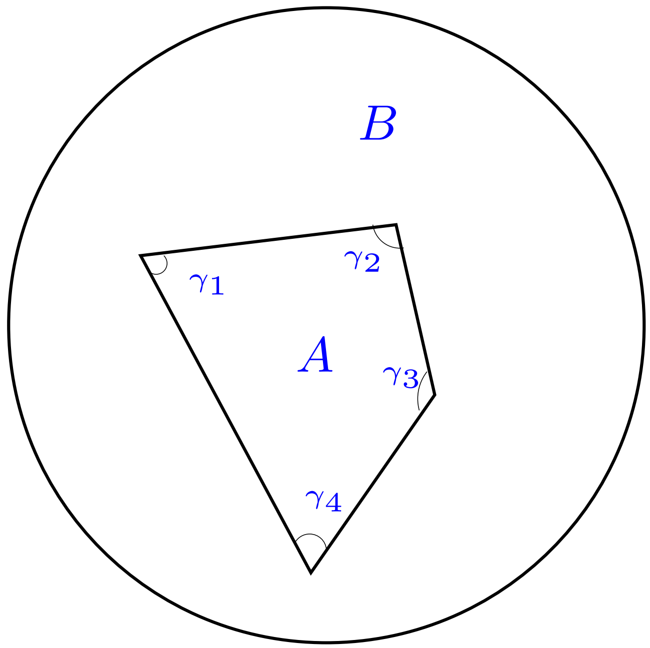
If the regions A and B are physically separate and have no common intersection, $\chi_A + \chi_B - \chi_{A \cup B} \neq 0$. In this case, which corresponds to a process in which the system physically splits in two disjoint parts, there is a universal $\log L$ term in the entanglement entropy at quantum criticality, proportional to the central charge c of the associated CFT!



If the A and B share a common boundary, there is a $\log L$ term whose coefficient is determined by the angles at the intersections



Or if the boundary of A is not smooth, in which case the coefficient depends on the angles γ_i for both regions



Finite terms in the entanglement entropy depend on scale-invariant aspect ratios

Conclusions

- There are lattice models and field theories which exhibit topological order and conformal quantum critical points. For $SO(3)_k$, Potts; for $SU(2)_k$, $O(n)$ model.
- The ground state wave function of the critical theory is a 2D CFT whose equal-time correlators at the critical points can be computed
- There is a gapped topological field theory describing the low energy physics.
- Do we always need a Chern-Simons description for these topological field theories?
- The excitations of this theory obey non-abelian statistics.
- Wanted: simpler and physically realizable models!
- There is a relation for systems with a scale-invariant ground state wave function between the entanglement entropy and the central charge of the associated CFT