Topological Quantum Compiling

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Fibonacci Anyons may actually exist!

\[ \nu = \frac{12}{5} \]
Possibly a Read-Rezayi \( k = 3 \) “Parafermion” state.

Read and Rezyai, ‘99

Charge \( \frac{e}{5} \) quasiparticles with braiding properties described by \( SU(2)_3 \) Chern-Simons Theory.

Slingerland and Bais, ‘01

Non-Abelian content is that of Fibonacci anyons.

J.S. Xia et al., PRL (2004).
Maybe, one day, **Fibonacci Anyons** will be everywhere!

Bosonic Read-Rezayi states (including $k=3$ at $\nu = 3/2$) may be realizable in rotating Bose condensates.  

Rezayi, Read, Cooper ‘05

Doubled Fibonacci “string-nets” may be found / realized.

Levin and Wen ‘04  
Fendley and Fradkin ‘05  
Freedman, Nayak, Shtengel, Walker and Wang ‘03

Think Golden!
\[ \tau \times \tau = 1 + \tau \]
Topological Quantum Computation

When quasiparticles are present there is an exponentially large Hilbert space whose states cannot be distinguished by local measurements. Quasiparticle world-lines forming braids carry out unitary transformations on this Hilbert space.

\[ |\Psi_f\rangle = \begin{pmatrix} a_{11} & \cdots & a_{1M} \\ \vdots & \ddots & \vdots \\ a_{M1} & \cdots & a_{MM} \end{pmatrix} |\Psi_i\rangle \]
Unitary transformation depends only on the topology of the braid swept out by anyon world lines!

Robust quantum computation?

(Kitaev ’97; Freedman, Larsen and Wang ‘01)
Quantum Circuit

What braid corresponds to this circuit?
The laws of Fibonacci anyons:

1. Fibonacci anyons have a quantum attribute I’ll call q-spin:
   \[ q\text{-spin} = 1 \]

2. A collection of Fibonacci anyons can have a total q-spin of either 0 or 1:

   Notation: Ovals are labeled by total q-spin of enclosed particles.
Fibonacci Anyon Basics

3. The “fusion” rule for combining q-spin is: \( 1 \times 1 = 0 + 1 \)

This means that two Fibonacci anyons can have total q-spin 0 or 1, or be in any quantum superposition of these two states.

\[ \alpha_0 + \beta_1 \]

Two dimensional Hilbert space

\[ \alpha_0 + \beta_1 + \gamma_1 \]

Three dimensional Hilbert space

For \( N \) Fibonacci anyons Hilbert space dimension is \( \text{Fib}(N-1) \)
The F Matrix

Changing fusion bases:

$$\sum_{a} F_{ab}^c = \begin{pmatrix} -\tau & \sqrt{\tau} & 0 \\ \sqrt{\tau} & \tau & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\tau = \frac{\sqrt{5} - 1}{2}$$
The R Matrix

Exchanging Particles:

\[ R = \begin{pmatrix} e^{-i4\pi/5} & 0 \\ 0 & e^{i3\pi/5} \end{pmatrix} \]

\[ F \text{ and } R \text{ must satisfy certain consistency conditions (the “pentagon” and “hexagon” equations). For Fibonacci anyons these equations uniquely determine } F \text{ and } R. \]
Encoding a Qubit

<table>
<thead>
<tr>
<th>Qubit States</th>
<th>Non-Computational State</th>
</tr>
</thead>
<tbody>
<tr>
<td>(</td>
<td>0\rangle = )</td>
</tr>
<tr>
<td>(</td>
<td>1\rangle = )</td>
</tr>
</tbody>
</table>

(Freedman, Larsen, and Wang, 2001)
Initializing a Qubit

Pull two quasiparticle-quasihole pairs out of the “vacuum”.
Initializing a Qubit

Pull two quasiparticle-quasihole pairs out of the “vacuum”.

These three particles have total q-spin 1
Initializing a Qubit

Pull two quasiparticle-quasihole pairs out of the “vacuum”.

\[ |0\rangle \]
Measuring a Qubit

Try to fuse the leftmost quasiparticle-quasiholes pair.

\[ \alpha |0\rangle + \beta |1\rangle \]
Measuring a Qubit

If they fuse back into the “vacuum” the result of the measurement is 0.
Measuring a Qubit

If they cannot fuse back into the “vacuum” the result of the measurement is 1.
Braiding Matrices for 3 Fibonacci Anyons

\[ \sigma_1 = \begin{bmatrix} e^{-i4\pi/5} & 0 & 0 \\ 0 & -e^{-i2\pi/5} & 0 \\ 0 & 0 & -e^{-i2\pi/5} \end{bmatrix} \]

\[ \sigma_2 = \begin{bmatrix} -\tau e^{-i\pi/5} & -i\sqrt{\tau} e^{-i\pi/10} & 0 \\ -i\sqrt{\tau} e^{-i\pi/10} & -\tau & 0 \\ 0 & 0 & -e^{-i2\pi/5} \end{bmatrix} \]

\[ \tau = \frac{\sqrt{5} - 1}{2} \]

\[ |\Psi_i\rangle \quad \sigma_1^{-1} \quad \sigma_2 \quad \sigma_1^{-1} \quad \sigma_2 = M \]

\[ |\Psi_f\rangle = M^{-1} |\Psi_i\rangle \]
Single Qubit Operations

**General rule:** Braiding inside an oval does not change the total q-spin of the enclosed particles.

**Important consequence:** As long as we braid *within* a qubit, there is **no leakage error**.

Can we do arbitrary single qubit rotations this way?
Single Qubit Operations are Rotations

The set of all single qubit rotations lives in a solid sphere of radius $2\pi$.

\[ |\psi\rangle - U_{\alpha} - U_{\alpha} |\psi\rangle \]

\[ U_{\alpha} = \exp \left( \frac{i \alpha \cdot \vec{\sigma}}{2} \right) \]
$N = 1$
$N = 2$
$N = 3$
$N = 4$
N = 5
\[ N = 6 \]
$N = 9$
$N = 10$
$N = 11$
Brute Force Search

\[
\begin{align*}
&\sigma_1^{-2} \sigma_2^{-4} \sigma_1^4 \sigma_2^{-2} \sigma_1^2 \sigma_2^{-2} \sigma_1^2 \sigma_2^{-2} \sigma_1^2 \sigma_2^{-2} \\
&= \left( \begin{array}{ccc}
0 & i & 0 \\
i & 0 & 0 \\
0 & 0 & 1 
\end{array} \right) + O(10^{-3})
\end{align*}
\]

For brute force search:

Braid Length \sim |\ln \varepsilon|
Brute Force Search

\[ \sigma_1^{-2} \sigma_2^{-4} \sigma_1^4 \sigma_2^{-2} \sigma_1^2 \sigma_2 \sigma_1^{-2} \sigma_2 \sigma_1^2 \sigma_2^{-4} \sigma_1^4 \sigma_2 \sigma_1^{-2} \sigma_2 \sigma_1^2 \sigma_2 \sigma_1^{-2} \sigma_2 \sigma_1^{-2} = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + O(10^{-3}) \]

Brute force searching rapidly becomes infeasible as braids get longer.

Fortunately, a clever algorithm due to Solovay and Kitaev allows for systematic improvement of the braid given a sufficiently dense covering of \( SU(2) \).
Solovay-Kitaev Construction

(Actual calculation)

\[ \begin{bmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + O(10^{-4}) \]

Braid Length \( \sim |\ln \varepsilon|^c \), \( c \approx 4 \)
What About Two Qubit Gates?

Problems:

1. We are pulling quasiparticles out of qubits: **Leakage error!**

2. **87 dimensional** search space (as opposed to 3 for three-particle braids). Straightforward “brute force” search is problematic.
Two Qubit Controlled Gates

Goal: Find a braid in which some rotation is performed on the target qubit only if the control qubit is in the state 1. ($b=1$)
Constructing Two Qubit Gates by “Weaving”

Weave a pair of anyons from the control qubit between anyons in the target qubit.

**Important Rule:** Braiding a q-spin 0 object does not induce transitions.

Target qubit is only affected if control qubit is in state $|1\rangle$ 

$(b = 1)$
Only nontrivial case is when the control pair has q-spin 1.

We’ve reduced the problem to weaving one anyon around three others. Still too hard for brute force approach!
OK, Try Weaving Through Only Two Particles

We’re back to $SU(2)$, so this is numerically feasible.

**Question:** Can we find a weave which does not lead to leakage errors?
A Trick: Effective Braiding

Actual Weaving $\approx$ Effective Braiding

The effect of weaving the blue anyon through the two green anyons has approximately the same effect as braiding the two green anyons twice.
Controlled–“Knot” Gate

Effective braiding is all within the target qubit. No leakage!

Not a CNOT, but sufficient for universal quantum computation.
Solovay-Kitaev Improved Controlled–“Knot” Gate
Another Trick: Injection Weaving

Step 1: Inject the control pair into the target qubit.

\[ \sigma^3 \sigma^{-2} \sigma^{-4} \sigma^2 \sigma^4 \sigma^2 \sigma^{-2} \sigma^{-4} \sigma^2 \sigma^{-2} \sigma^{-4} \sigma^2 \sigma^{-2} \sigma^2 \sigma^2 \sigma^{-2} \sigma^{-3} \approx \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]
Step 2: Weave the control pair inside the injected target qubit.

\[
\begin{pmatrix}
\sigma_1^{-2} \sigma_2^{-4} \sigma_1^4 \sigma_2 \sigma_1^{-2} \sigma_2^2 \sigma_1 \sigma_2^{-2} \sigma_1^4 \sigma_2 \sigma_1^{-2} \sigma_2^4 \sigma_1 \sigma_2^{-2} \sigma_1^2 \sigma_2^{-2} \sigma_1^{-2} \sigma_2^2 \\
\sigma_1^{-2} \sigma_2^{-4} \sigma_1^4 \sigma_2 \sigma_1^{-2} \sigma_2^2 \sigma_1 \sigma_2^{-2} \sigma_1^4 \sigma_2 \sigma_1^{-2} \sigma_2^4 \sigma_1 \sigma_2^{-2} \sigma_1^2 \sigma_2^{-2} \sigma_1^{-2} \sigma_2^2 \\
\end{pmatrix}
\approx
\begin{pmatrix}
0 & i & 0 \\
i & 0 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}
\]
Step 3: Extract the control pair from the target using the inverse of the injection weave.

Putting it all together we have a CNOT gate:
Solovay-Kitaev Improved CNOT
Universal Set of Fault Tolerant Gates

Single qubit rotations:  \( |\psi\rangle \xrightarrow{U_\phi} U_\phi |\psi\rangle \)

Controlled NOT:
Quantum Circuit
Quantum Circuit
Braid
Topological Quantum Computing with Only One Mobile Quasiparticle

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We know it is possible to carry out universal quantum computation by moving only a single quasiparticle.

Can we find an efficient CNOT construction in which only a single particle is woven through the other particles?
Another Useful Braid: The F-braid

F-Matrix:

\[
\begin{pmatrix}
-\tau & \sqrt{\tau} & 0 \\
\sqrt{\tau} & \tau & 0 \\
0 & 0 & 1 \\
\end{pmatrix}
\times
\begin{pmatrix}
1 & 0 & 1 \\
1 & 1 & 1 \\
0 & 1 & 0 \\
\end{pmatrix}
= 
\begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 0 & 1 \\
\end{pmatrix}
\]

F-Braid:
Single Particle Weave Gate: Part 1
Single Particle Weave Gate: Part 1

F-Braid
Single Particle Weave Gate: Part 1

<table>
<thead>
<tr>
<th>a</th>
<th>b</th>
<th>b'</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
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<td>1</td>
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</tbody>
</table>

Intermediate State
Single Particle Weave Gate: Part 2

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>b'</th>
<th>Phase</th>
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<tr>
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<td>0</td>
<td>0</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>+1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
</tr>
</tbody>
</table>

\[
\begin{pmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

Phase Braid

\( b' = 1 \)  \( b' = 0 \)
Single Particle Weave Gate: Part 2

<table>
<thead>
<tr>
<th>a</th>
<th>b</th>
<th>b'</th>
<th>Phase</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>+1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
</tr>
</tbody>
</table>
Single Particle Weave Gate: Part 3

<table>
<thead>
<tr>
<th>a</th>
<th>b</th>
<th>b'</th>
<th>Phase</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
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<td>1</td>
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<tr>
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<td>0</td>
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<td>+1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
</tr>
</tbody>
</table>
Controlled-Phase Gate

\[ U = -\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + O(10^{-3}) \]
Solovay-Kitaev-Improved Controlled-Phase Gate