Nonabelian Quantum Hall States:

Properties and Prospects

N. Read

Recent + forthcoming papers: N.R., cond-mat/0601678

Rezayi + Read (to appear)

Thanks to: G. Moore
E. Rezayi
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D. Green
K. Schoutens
E. Ardonne
N. Cooper

Overview

1) Nonabelian QH states
   Read-Rezayi series
2) Numerical results on $\nu = \frac{12}{5}, \frac{13}{5}$
3) Quasihole counting + wavefunctions for RR_k.

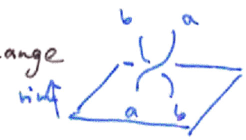
Quantum Hall phases

("universality classes") Moore + N.R. 1991

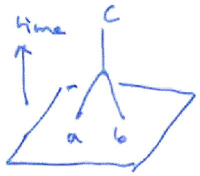
for fully-gapped (incompressible) QH phases, (in 2+1)
characterized by properties of elementary excitations (qptcles)

1) quantum numbers (e.g. charge) $a = 1, 2, \ldots$

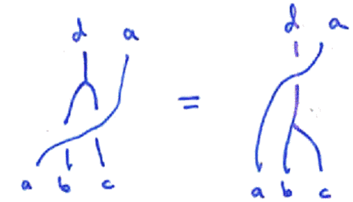
2) "statistics" under adiabatic exchange

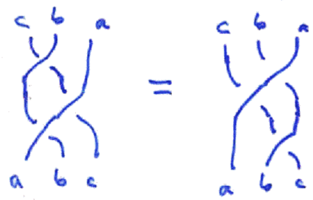


3) fusion



Consistency conditions: e.g.





Quasiparticles can be identified under addition of "invisible" ptcles, so labels form finite set.

(Rational) Conformal field theory (in 2 dims) provides a supply of consistent solutions! (Moore + Seiberg 1989)

For this, use correspondence of a QH phase with a RCFT

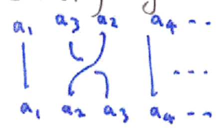
$$\boxed{\text{states with qptcles } a_1, a_2, \ldots \text{ in } 2+1 \longleftrightarrow \text{conformal blocks for fields } a_1, a_2, \ldots}$$

Conformal blocks are chiral parts of correlators:

in the RCFT $$\langle \Phi_{a_1}(z_1,\bar z_1)\, \Phi_{a_2}(z_2,\bar z_2) \ldots \rangle = \sum_{\substack{\text{blocks}\\ \alpha}} \left| \mathcal{F}^{(\alpha)}_{a_1 a_2 \ldots}(z_1, z_2, \ldots) \right|^2$$

[Later it emerged that all fully-gapped QH phases in pos. def. Hilb space corres to unitary RCFTs]

Braiding given by unitary matrices (nonabelian if dim > 1 !)



$$\mathcal{F}^{(\alpha)} \to \sum_\beta B_{\alpha\beta}\, \mathcal{F}^{(\beta)}$$

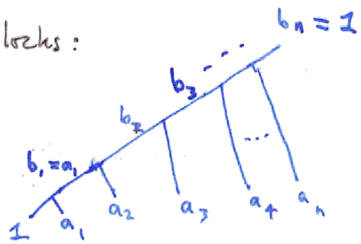
Fusion rules:

O.P.E.s: $$\Phi_a(z,\bar z)\, \Phi_b(0,0) \sim \sum_c \Phi_c(0,0)\, C^c_{ab}$$

Fusion rules are no of times Φ_c appears in $\Phi_a \Phi_b$:

$$\Phi_a \times \Phi_b = \sum_c N^c_{ab} \Phi_c \qquad (N^c_{ab} \geq 0, \text{ integer})$$

No of blocks:



The set $b_1, b_2, \ldots b_{n-2}$ (with multiplicities $N_{a_i a_{i+1}}^{b_i}$), equals number of possible choices for b_i

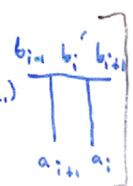
equals range of values for α in $\mathcal{F}^{(\alpha)}_{a_1 \ldots a_n}$.

Nonabelian statistics when no of blocks > 1.

[Braiding:

$\rightarrow$

$$= \sum_{b_i'} B^{(a_i, a_{i+1}, b_{i-1}, b_{i+1})}_{b_i, b_i'}$$



"Clustered" QH states

N.R. + Rezayi 1999

Look at bosons with $k+1$-body int Hamiltonian

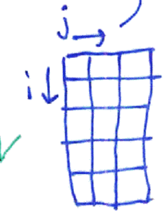
$$H_{int} = V \sum_{i_1 < i_2 < \cdots < i_{k+1}} \delta(z_{i_1} - z_{i_2}) \cdots \delta(z_{i_k} - z_{i_{k+1}})$$

acting within lowest Landau level (basis states $\propto z^m e^{-\frac{1}{4}|z|^2}$, $m = 0, 1, 2, \ldots$)

Ground state (lowest degree) wavefunction:

$$\Psi_{RR_k} = \mathcal{S}\left\{ \prod_{i<i'} \prod_j (z_{ij} - z_{i'j})(z_{i,j+1} - z_{i'j}) \right\}$$

We relabelled $i \rightarrow (ij)$



$j = 1, \ldots, k$
$i = 1, \ldots, N/k$

$\mathcal{S}$ = symmetrizer

— vanishes when any $k+1$ coincide: $z_1 = z_2 \cdots = z_{k+1}$ (N boxes)

Cases:

$k = 1$: $\Psi_{RR_1} = \prod_{i<j} (z_i - z_j)^2 \equiv \Psi_{LJ}^2$ Laughlin 1983 (Ham: Haldane 1983)

$k = 2$: $\Psi_{RR_2} = \mathrm{Pf}\left(\frac{1}{z_i - z_j}\right) \cdot \prod_{i<j} (z_i - z_j)$ Moore - N.R. 1991 (Ham: Greiter, Wen, Wilczek 1992)

$k > 2$: R.R.

More generally, multiply by $\prod_{i<j}(z_i - z_j)^M$, $M = 0, 1, 2 \ldots$

– symmetric (bosons) for M even,
antisymmetric (fermions) for M odd

Filling factor $\nu = \lim_{N\to\infty} \frac{N}{N_\phi}$

N = # ptcles, N_ϕ = # flux = degree in each z_i

$$\rightarrow \quad \boxed{\nu = \frac{k}{Mk+2}}$$

There are Hams for $M \geq 1$ also.

The spectra have a gap above ground state (at least for $M=0, 1$, k small) – on sphere

[Sphere $\dfrac{z^m}{\left(1+\frac{|z|^2}{4R^2}\right)^{1+N_\phi/2}}$, $m = 0, 1, \ldots, N_\phi$]
in place of $z^m e^{-\frac{1}{4}|z|^2}$ (plane)

<u>Conformal field theory connection</u> Moore + N.R. (1991)

"Nice" trial wavefunctions <u>are</u> "conformal blocks"
ie chiral parts of correlators of 2D conformal (ie critical) field theory

E.g. MR state ($k=2$; $m = M+1$)

$$\Psi_{MR} = \langle \psi(z_1) \cdots \psi(N) \rangle_\psi \langle e^{i\sqrt{m}\varphi(z_1)} \cdots e^{i\sqrt{m}\varphi(z_N)} e^{-i\int\rho} \rangle_\varphi$$

$\langle \psi(z)\psi(z') \rangle = \frac{1}{z-z'}$
ψ = Majorana fermion
– produces Pfaffian

$\langle \varphi(z)\varphi(z') \rangle = -\ln(z-z')$
massless scalar or Coulomb gas
→ produces Laughlin factor

Further, ope's $\psi(z)\psi(0) \sim \frac{1}{z} + \cdots$, $\psi(z)1 \sim \psi(0) + \cdots$

$$e^{i\sqrt{m}\varphi(z)} e^{i\sqrt{m}\varphi(0)} \sim z^m e^{2i\sqrt{m}\varphi(0)}$$

imply $\Psi_{MR} \to 0$ when $z_1, z_2 \to z_3$ (for $m=1$ case)

For RR series, replace Majorana field ψ by parafermions $(SU(2)_k/U(1))$ Zamolodchikov + Fateev (1985)

Fields $\psi_1, \ldots, \psi_{k-1}$, $\psi_\ell^\dagger = \psi_{k-\ell}$

$$\psi_\ell(z)\,\psi_{\ell'}(0) \sim \frac{1}{z^{\Delta_\ell+\Delta_{\ell'}-\Delta_{\ell+\ell'}}}\,\psi_{\ell+\ell'}(0) + \cdots \qquad \ell+\ell' < k$$

$$\psi_\ell(z)\,\psi_{\ell'}^\dagger(0) \sim \frac{1}{z^{\Delta_\ell+\Delta_{\ell'}-\Delta_{\ell-\ell'}}}\,\psi_{\ell-\ell'}(0) + \cdots \qquad \ell' < \ell$$

$$\psi_\ell(z)\,\psi_\ell^\dagger(0) \sim \frac{I}{z^{2\Delta_\ell}} + \cdots$$

$\Delta_\ell = \dfrac{\ell(k-\ell)}{k}$, central charge $c = \dfrac{2(k-1)}{k+2}$

($k=1$ is trivial, $k=2$ is Majorana fermion)

These imply

$$\Psi = \langle \psi_1(z_1) \cdots \psi_1(z_N) \rangle\, \Psi_{LJ}^{M+2/k}$$

has same behavior as RR state as $z_1, \ldots, z_k \to z_{k+1}$

so we knew the function existed at that N_ϕ.

$$\Psi = \Psi_{RR_k}$$

Excitations

Parafermion CFT has primary fields $\sigma_1, \ldots, \sigma_{k-1}, \mathbb{1}$

Insert:

$$\Psi = \langle \psi_1(z_1) \cdots \psi_1(z_N)\,\sigma_1(w_1) \cdots \sigma_1(w_n) \rangle\, \Psi_{LJ}^{M+2/k} \times \prod_{i,\ell} (z_i - w_\ell)^{1/k}$$

still vanishes when any $z_1, \ldots, z_{k+1}$ are equal, and also when any set $z_1, \ldots, z_k$ equal a w_ℓ.
$\to$ zero energy

Nontrivial number of conformal blocks for $k > 1$.

Note

$$\psi_1(z)\,\sigma_1(0) \sim z^{-1/k}\,\sigma_1(0)$$

dictates powers of $z-w$, and implies charge of quasihole is

$$-\nu/k = -\frac{1}{Mk+2}$$

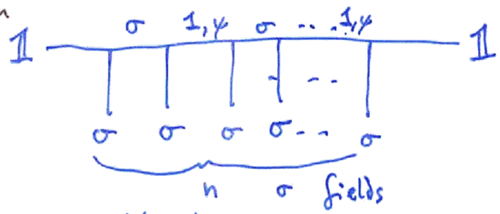
Are these all the zero energy states? (Completeness)

Fusion rules

For $k=2$ case, only (Virasoro) primary fields are $\mathbb{1}, \psi, \sigma=\sigma_1$

$$\boxed{\begin{array}{ll} \mathbb{1}\times\mathbb{1}=\mathbb{1} & \psi\times\sigma=\sigma \\ \mathbb{1}\times\psi=\psi & \psi\times\psi=\mathbb{1} \\ \mathbb{1}\times\sigma=\sigma & \sigma\times\sigma=\mathbb{1}+\psi \end{array}}$$

Then



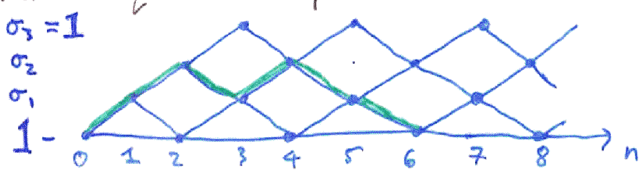
Hence $2^{n/2-1}$ conformal blocks

p_x+ip_y superconductor (BCS paired state)

N.R. + Green 2000

Same number with charge sector included (MR QH states)

$k=3$: For n qholes on sphere



Paths ending at top or bottom

$\longrightarrow$ # blocks $=F_{n-1}$, Fibonacci numbers $F_i = 1, 1, 2, 3, 5, 8, 13, \ldots$

Chiral algebra

Charge sector included, still only finite no of quasiparticle types. Why?

"Invisible" ptcles inc underlying ptcles — electrons or bosons

$$\psi_1(z)\, e^{\pm i\sqrt{M+2/k}\,\varphi(z)} \quad \text{in CFT}$$

These generate chiral algebra under ope's. (Most relevant for edge — same CFT).

For $M=0$ (bosons), ch. alg is $SU(2)$ current alg level $k=1, 2, \ldots$

primary fields of $SU(2)_k$ is $k+1$.

For $M=1$ (fermions), ch. alg. is $N=2$ superconf. level $k=1, 2, \ldots$

primaries is $(k+1)(k+2)/2$

These determine no. of deg. ground states on torus also. In general, $(k+1)(Mk+2)/2$

Statistics of quasiparticles

Conformal blocks have branch point behavior in w's. Braiding under analytic continuation (z's fixed).

We need adiabatic transport à la Berry.

Vector potential (connection)

$$\langle \Psi(w_1 \ldots w_n) \,|\, \frac{\partial \Psi}{\partial w_\ell}(w_1 \ldots w_n) \rangle$$

$$= \int \prod d^2 z_i \;\; \Psi^* \frac{\partial \Psi}{\partial w}$$

> Moore–Read conjecture: exchange of q'ples by adiabatic transport is same as analytic continuation of conformal blocks

Moore + N.R. 1991

Believed but not proven in general (except abelian cases, and a family of $SU(2)$ singlet states Blok + Wen 1992)

Numerical work on 2-body interactions

Do the phases occur for realistic Hams?

$k=2$, $M=1$ (Moore-Read state for electrons)

– well-supported for ints for $n=1$ Landau level

i.e. for $\nu = 2 + \frac{1}{2} = \frac{5}{2}$

and $3 + \frac{1}{2} = \frac{7}{2}$

Morf 1999, Rezayi + Haldane 2000

Also for bosons ($M=0$) at $\nu = 1$

Cooper, Wilkin + Gunn 2001

$k=3$:

$M=0$ (bosons) $\nu = 3/2$

– looks good

Cooper, Wilkin, Gunn 2001

Rezayi, N.R., Cooper 2005

$M=1$ (electrons) $\nu = 2 + \frac{3}{5}$ or $2 + \frac{2}{5}$

$= \frac{12}{5}, \frac{13}{5}$

R.R. 1999 – strong indications

Here – $n=1$ LL with finite thickness (Fang-Stern)

Rezayi + N.R. 2006

and varying V_1, V_3 pseudopotentials

– torus

<u>Ground states on torus</u> ie periodic b.c.

<u>All</u> states at $\nu = \frac{p}{q}$, $(p,q) = 1$
have exact degeneracy of q for
trans. inv Hamiltonian (Haldane 1985)

RR series have total deg $= (k+1)(Mk+2)/2$

– should be exact in thermo limit in general
– exact for trial $k+1$ body Ham in finite size

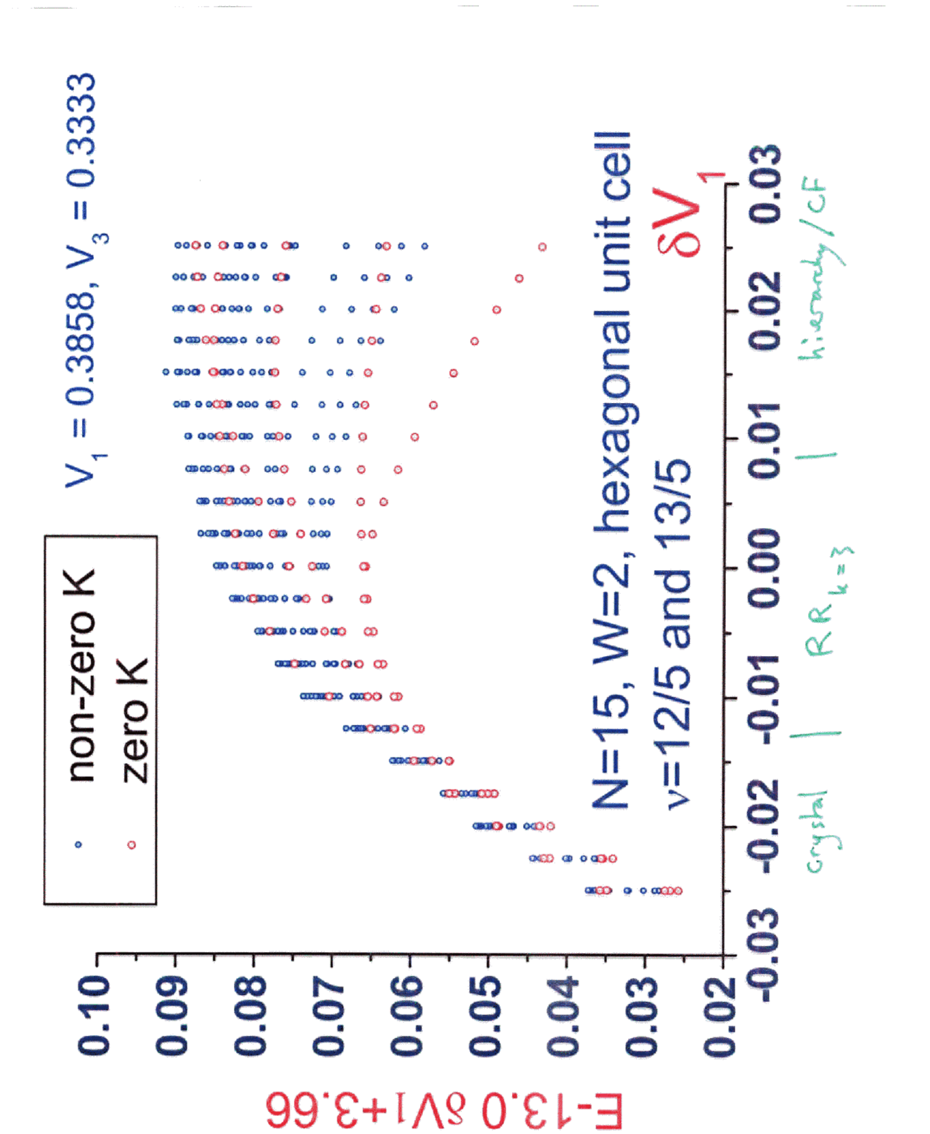
E.g. $k+1$ for $M=0$

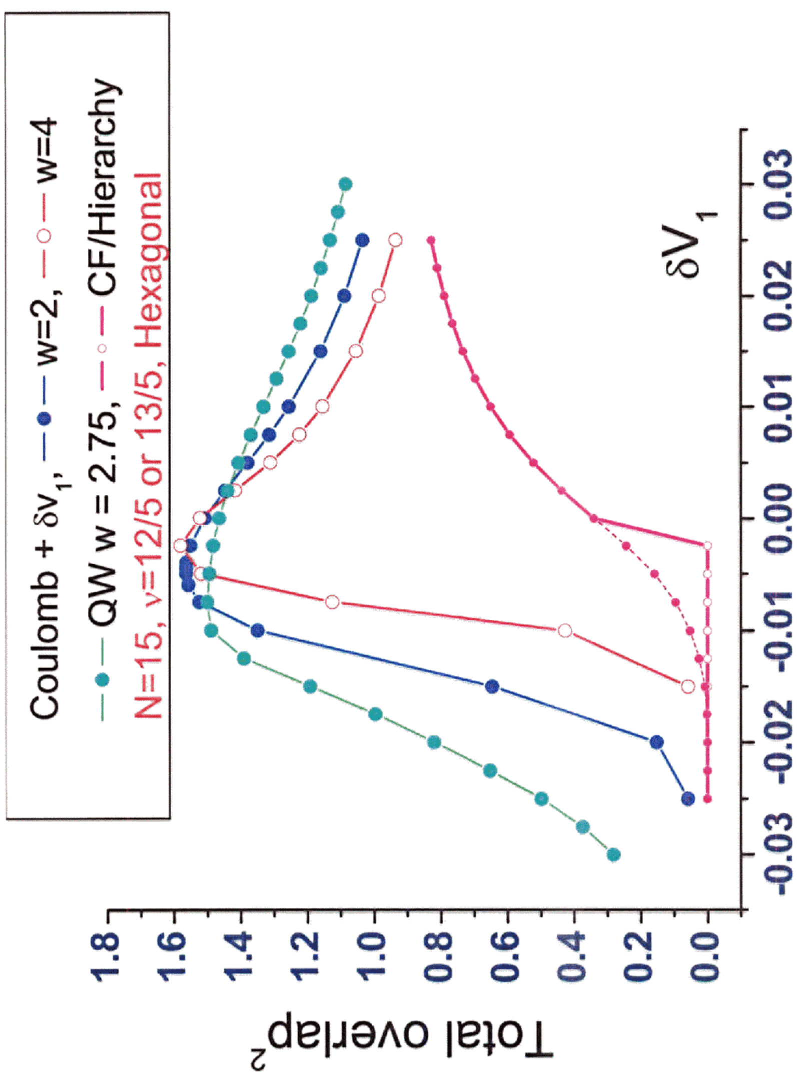
For $k=3$, this is $4 = 2 \times 2$

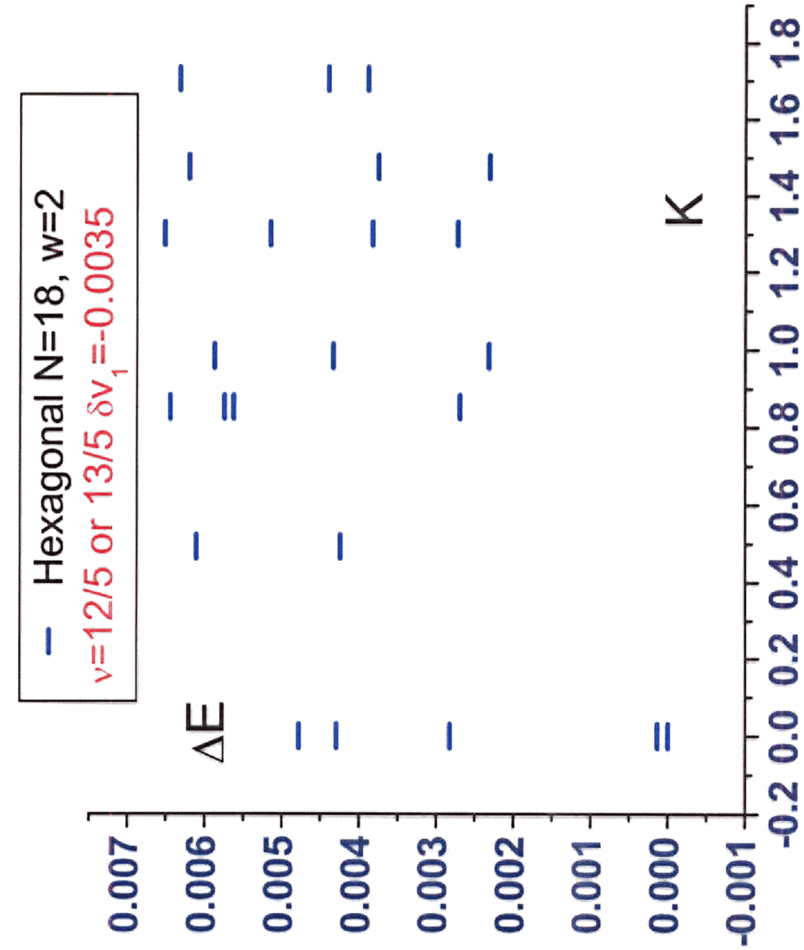
trans symm

"extra" (not exact in finite size in general)

Also states can be labelled by "pseudomomentum" $\underline{K}$ from trans symmetry analysis.




Coulomb + δv1, w=2, w=4
QW w = 2.75, CF/Hierarchy
N=15, ν=12/5 or 13/5, Hexagonal
δV1
Total overlap²
-0.03 -0.02 -0.01 0.00 0.01 0.02 0.03
0.0 0.2 0.4 0.6 0.8 1.0 1.2 1.4 1.6 1.8


Hexagonal N=18, w=2
ν=12/5 or 13/5 δv1=-0.0035
ΔE
K
-0.2 0.0 0.2 0.4 0.6 0.8 1.0 1.2 1.4 1.6 1.8
-0.001 0.000 0.001 0.002 0.003 0.004 0.005 0.006 0.007

Counting quasihole trial states on sphere

ie zero-energy estates of $k+1$-body int ($M=0$, bosons) N ptcles, N_ϕ flux

Let $z_1,..,z_k \to Z_1$. If non-zero, repeat at Z_2. If such a "clustering" limit is zero, don't let these approach. Then these non-vanishing residues must take form

$$\Psi_{N,N_\phi,k} \to \prod_{p<p'} (Z_p - Z_{p'})^{2k} \prod_{p,l} (Z_p - w_l) \prod_{p,m} (Z_p - z_{N-F+m})^2$$

vanishes when any $k+1$ z's equal

$$\times \Psi_{F,\, N_\phi - \frac{2(N-F)}{k},\, k-1}$$

vanishes when any k z's equal

Here w's parametrize as "coherent states"

$l = 1,..,n$ where $N_\phi = \frac{2N}{k} + \frac{n}{k} - 2$

$p,p' = 1,2,\ldots \frac{(N-F)}{k}$

F = no. of unclustered ptcles

Recursion:

$$\dim V_{N,N_\phi,k} = \#_{N,N_\phi,k} = \sum_{F:\, F \equiv N \,(\mathrm{mod}\, k)} \binom{\frac{N-F}{k} + n}{n} \#_{F,N_\phi',k'}$$

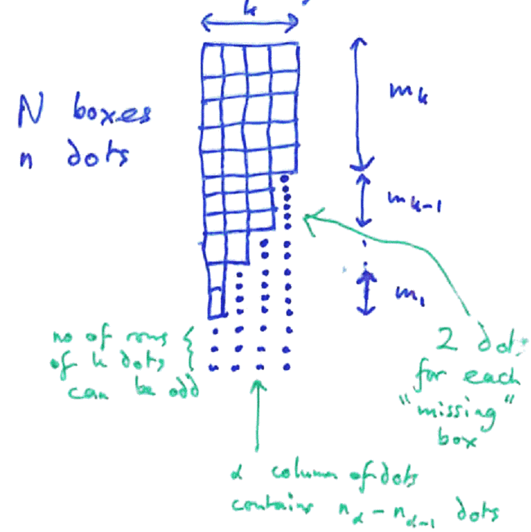
$$N_\phi' = N_\phi - \frac{2}{k}(N-F) = \frac{2F+n}{k} - 2$$

$$k' = k-1$$

Solve:

$$\#_{N,N_\phi,k} = \sum_{\substack{m_1,m_2,..m_k:\\ \sum_\alpha \alpha m_\alpha = N}} \prod_{\alpha=1}^{k} \binom{m_\alpha + n_\alpha}{n_\alpha}$$

Ardonne 2002



where

$$2\sum_{\beta:\beta\geq\alpha} m_\beta + n_\alpha - n_{\alpha-1} = \frac{2N+n}{k}$$

So same degree (N_ϕ) in all z_i's.

Solve:

$$n_\alpha = \alpha\left(\frac{2N+n}{k} - \sum_{\beta:\beta\geq\alpha} 2m_\beta\right) - \sum_{\beta:\beta<\alpha} 2\beta m_\beta$$

To show there is really a state for each of these, construct wfns

(cf Ardonne, Kedem, Stone, 2005)

$$\Psi^{(m_1,\ldots,m_k)}_{N,N_\phi,k} = \zeta\Big\{\prod_{(\alpha,i)>(\beta,i')}\prod_{j=1..\beta}\left(z_{i;j}^{(\alpha)}-z_{i';j}^{(\beta)}\right)\left(z_{i;j+1}^{(\alpha)}-z_{i';j}^{(\beta)}\right) \times\prod_{\alpha}\prod_{i,j}\prod_{\ell}{}^{(\alpha,j)}\left(z_{i;j}^{(\alpha)}-w_\ell^{(\alpha)}\right)\Big\}$$

$$\prod_\ell{}^{(\alpha,j)}:\quad \ell = n_{j-1}+(1-\delta_{j,1})\sum_{\beta:\,j-1\leq\beta<\alpha} m_\beta + 1$$

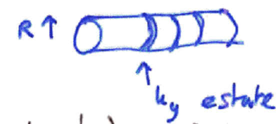
$$\vdots$$

$$n_j + \sum_{\beta:\,j-1<\beta<\alpha} m_\beta$$

(conf blocks contain extra fns of $w_\ell^{(k)}$'s)

One-dim connection

Recent work on cylinder/torus



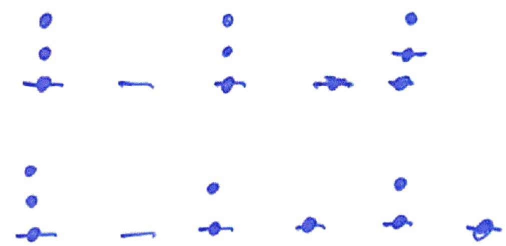
– take limit $R\to 0$ of $k+1$-body int

Haldane 2006

(k=2) { Bergholtz et al 2006, Seidel+Lee 2006

– zero energy only if occupation numbers of adjacent orbitals $\leq k$!

$k=3$



This counting problem solved by Warnaar (1997)

– same formula !

(Fermions: $\leq k$ ptcles in $k+2$ adjacent orbitals)

Conclusion

1) Prospects ($\nu = \frac{5}{2}, \frac{7}{2}, \frac{12}{5}$) look bright
2) Many properties under control
3) Adiabatic statistics still not proved generally