Topological Protection by small Josephson Junction Arrays.

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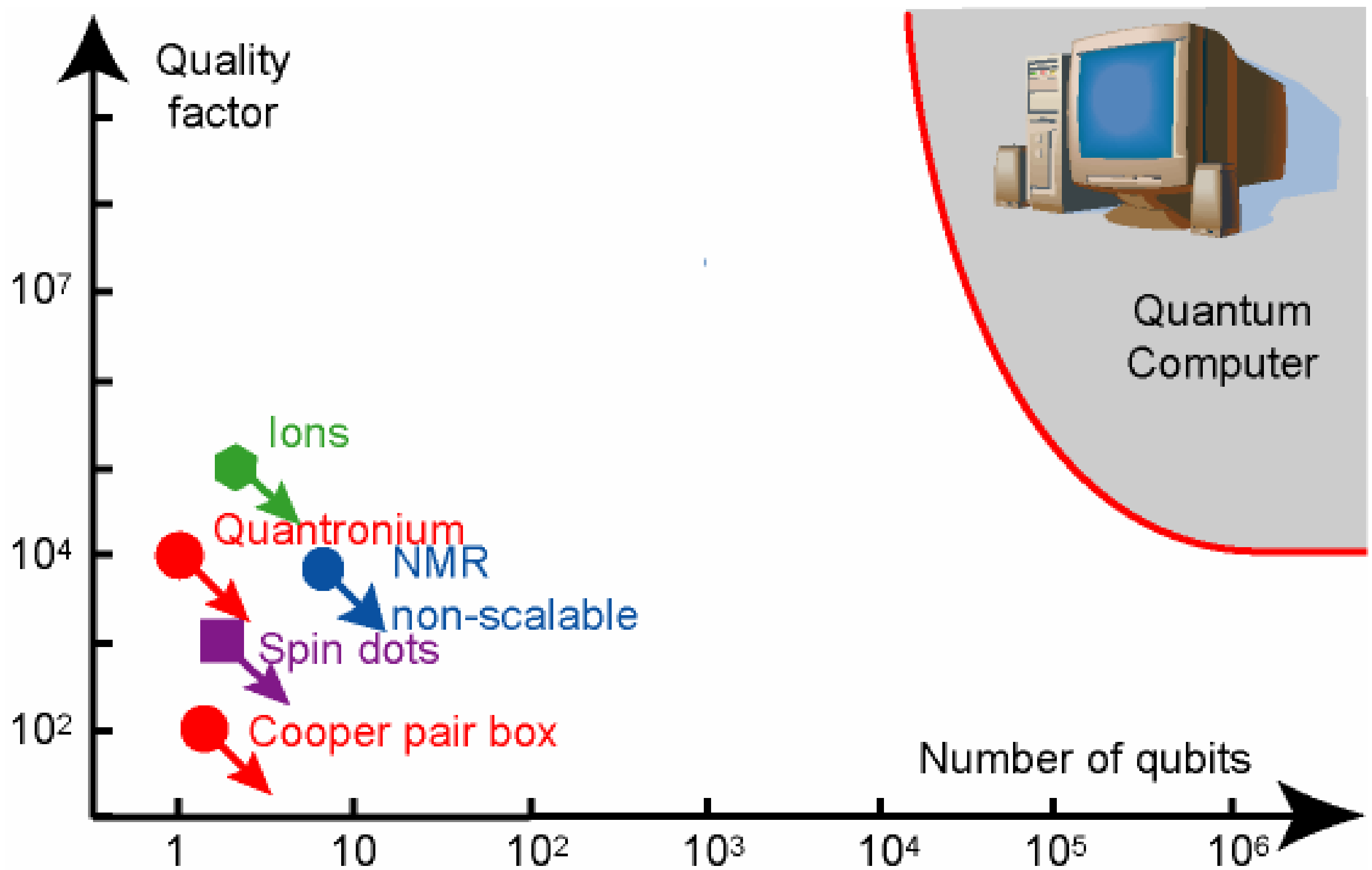
Quality factors: How far are we from quantum computation?



Why Quantum Computation is so difficult?

For quantum computer one needs

1. Many $(10^4 - 10^6)$ individual bits (scalable design!)
2. Short operation time τ_0 (set by inverse energy gap in the spectrum)
3. Long decoherence time (time per single quantum error) $\tau_d / \tau_0 \approx 10^6 - 10^8$

Two types of errors:

a. Single bit flip (classical error $\Psi = \sigma^x \Psi_0$) $\Psi = \exp(-i\sigma^x \int R(t)dt)\Psi_0$

Fluctuates

b. Phase error $\Psi = \exp(-i\sigma^z \int E(t)dt)\Psi_0$

Fluctuates

The measurement projects and discretizes the errors: $\Psi \to \sigma^x \Psi_0$ or $\Psi \to \sigma^z \Psi_0$

Correction of both types of errors is very difficult!

Quantum software error correction requires parallel operations on N>>1 qubits (unrealistic at present): no quantum analogs of Low Density Parity Check codes.

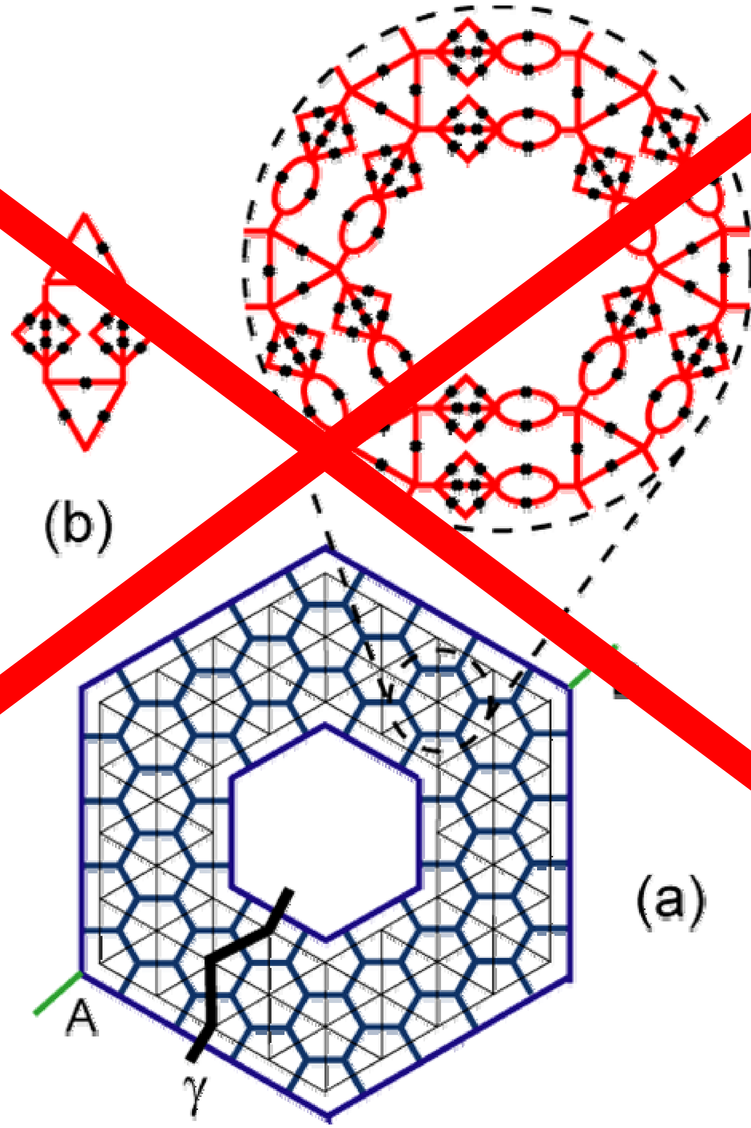
Need hardware error correction ⟺ <u>some</u> error correction by the Hamiltonian

Plan

- Design simplest, not too large Josephson junction arrays that provide a significant (but not perfect) amount of protection, almost perfect in one channel (e.g. phase) and reasonable in another.
- Develop more efficient error correction schemes that are suited for these systems.

Non Abelian Gauge Models

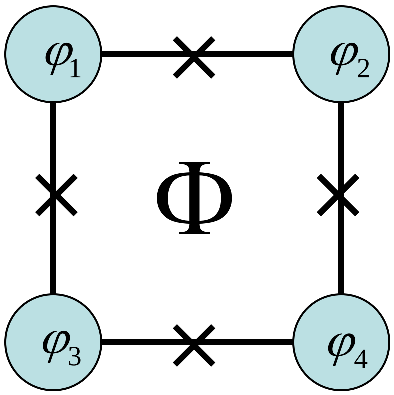
Too far away



S_3 gauge theory realization

Josephson Arrays

Elementary building block



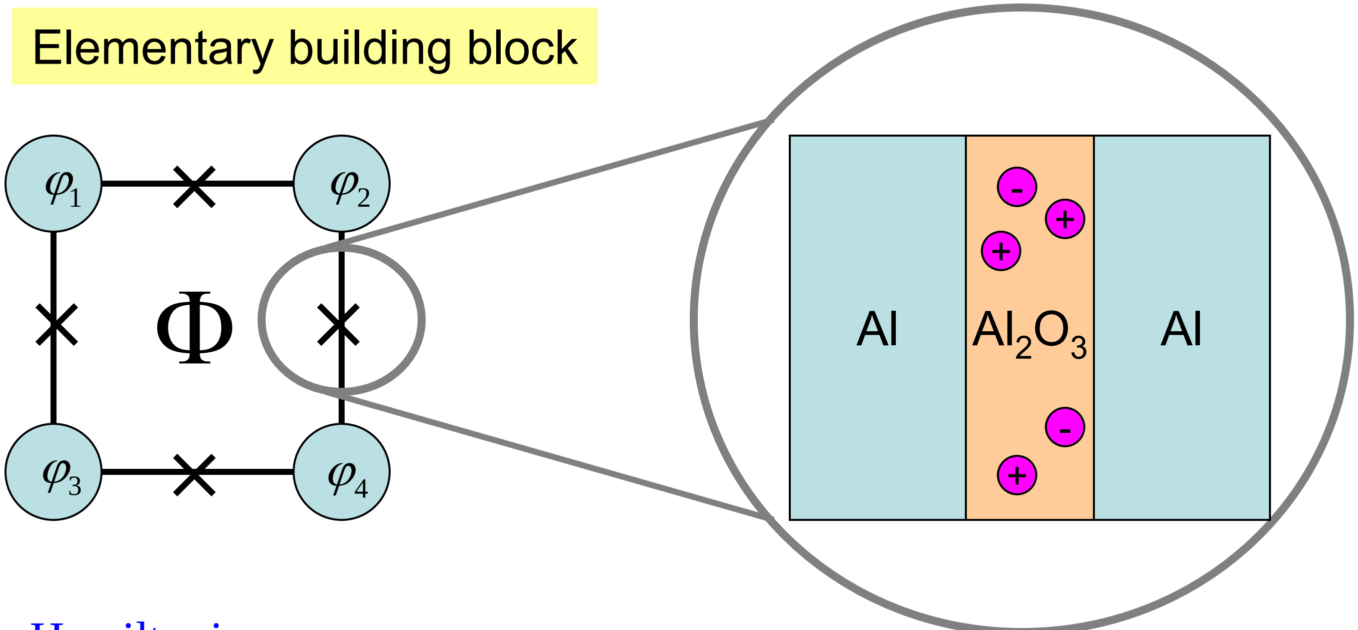
Ideal Hamiltonian:

$$H = \frac{1}{2}\sum_{i,j} C_{ij}^{-1} q_i q_j + E_J \cos(\varphi_i - \varphi_j - 2\pi \frac{\Phi_{ij}}{\Phi_0}) \quad q_i = 2e\, i \frac{d}{d\varphi_i}$$

C_{ij} - capacitance matrix E_J - Josephson energy

Josephson Arrays

Elementary building block



More realistic Hamiltonian:

$$H = \frac{1}{2}\sum_{i,j} C_{ij}^{-1}(q_i + Q_i)(q_j + Q_j) + (E_J + \delta E_J)\cos(\varphi_i - \varphi_j - 2\pi\frac{\Phi_{ij} + \delta\Phi}{\Phi_0}) \quad q_i = 2e\, i\frac{d}{d\varphi_i}$$

C_{ij} - capacitance matrix E_J - Josephson energy

$Q_i = Q_i^0 + Q_i(t)$ - induced charge (static and fluctuating)

$\delta\Phi = \delta\Phi^0 + \delta\Phi(t)$ - static flux due to area scatter and flux noise

$\delta E_J = \delta E_J^0 + \delta E_J(t)$ - static scatter of Josephson energies and their time dependent fluctuations.

Josephson Arrays – realistic parameters

$$H = \frac{1}{2}\sum_{i,j} C_{ij}^{-1}(q_i + Q_i)(q_j + Q_j) + (E_J + \delta E_J)\cos(\varphi_i - \varphi_j - 2\pi\frac{\Phi_{ij} + \delta\Phi}{\Phi_0}) \quad q_i = 2e\, i\frac{d}{d\varphi_i}$$

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"Realistic" assumptions:

Q_i^0 - random (no control)

$$S_Q(t) = \int e^{i\omega t}\left\langle Q_i(t)Q_i(0)\right\rangle dt = \alpha_Q^2 \frac{e^2}{\omega} \qquad \alpha_Q \approx 10^{-2} - 10^{-3}$$

$\delta\Phi^0 / \Phi \approx 0.01 - 0.02$

$$S_\Phi(t) = \int e^{i\omega t}\left\langle \delta\Phi_i(t)\delta\Phi_i(0)\right\rangle dt = \alpha_\Phi^2 \frac{\Phi_0^2}{\omega} \qquad \alpha_\Phi \approx 10^{-5} - 10^{-6}$$

$\delta E_J^0 / E_J \approx 0.2$

$\delta E_J(t)$ - similar to $\delta\Phi_i(t)$ and gives the same effect.

Problem setup.

How to build coherent device from these faulty Josephson elements subject to a considerable noise in all channels?

- Use topological (non-local) protection (similar to spin models).
- The charge noise is the strongest noise but it has less effect in the regime $E_J > E_C$, Use this regime!
- Geometry is the most controlled quantity, use it to frustrate the circuits by magnetic field and increase the quantum fluctuations

What is most general way to get a protected doublet?

Doublet is not a coincidence must follow from the symmetry!

If $\left[P,Q\right]\left|\Psi\right\rangle \neq 0$ and $\left[P,H\right]=0$, $\left[Q,H\right]=0$ for all Ψ all states are doubly degenerate.
Proof. Consider eigenstates of $P\Psi=\lambda\Psi$. Act on them with $Q\Psi=\mu\Psi'$, with $\Psi'\neq\Psi$ but with the same energy.

Noise adds some local $H_{noise}=\sum_i H_i,\ \exists i:\left[P,H_i\right]\neq 0$ splits degeneracy

However, if there is a ***set*** of such $\{P_i\},\{Q_i\}$, for any local noise H_i we can find P_k,Q_k so that

$$\left[P_k,H_i\right]=0,\ \left[Q_k,H_i\right]=0$$

Degeneracy is preserved!

If each set contains L operators, degeneracy is lifted in the ***Lth other*** of the perturbation theory in P_k,Q_k

Parameter of the perturbation theory (H_i/Δ) $\Delta-$ gap to low energy excitations

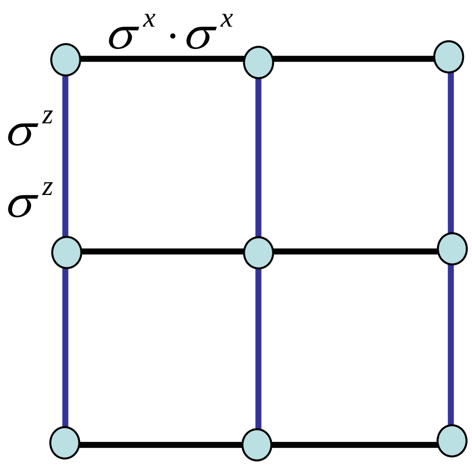
⇨ Gapless excitations are dangerous

Another danger: higher degeneracy. To avoid them, we need

$$\left[P_iP_j,Q_k\right]=0,\ \left[P_i,Q_kQ_l\right]=0$$

Spin model

Can be mapped to Z_2 Chern Simons Theory.



$$H = J_x \sum_{i,j} \sigma_{ij}^x \sigma_{ij+1}^x + J_z \sum_{i,j} \sigma_{ij}^z \sigma_{i+1j}^z$$

Integrals of motion

$P_i = \prod_j \sigma_{ij}^z$ row product

$Q_j = \prod_i \sigma_{ij}^x$ column product

because $[P_i, H] = 0 \quad [Q_j, H] = 0$

$\{P_i, Q_j\} = 0 \quad [P_i, Q_j]^2 = 4 \quad \Longrightarrow \quad [P_i, Q_j]\Psi \neq 0$

Further $[Q_i Q_j, P_k] = 0, \; [P_i P_j, Q_k] = 0$

$\Longrightarrow$ Ground state is exactly doubly degenerate, *regardless* of the boundary conditions, this is *not* what one expects in a generic gauge theory.

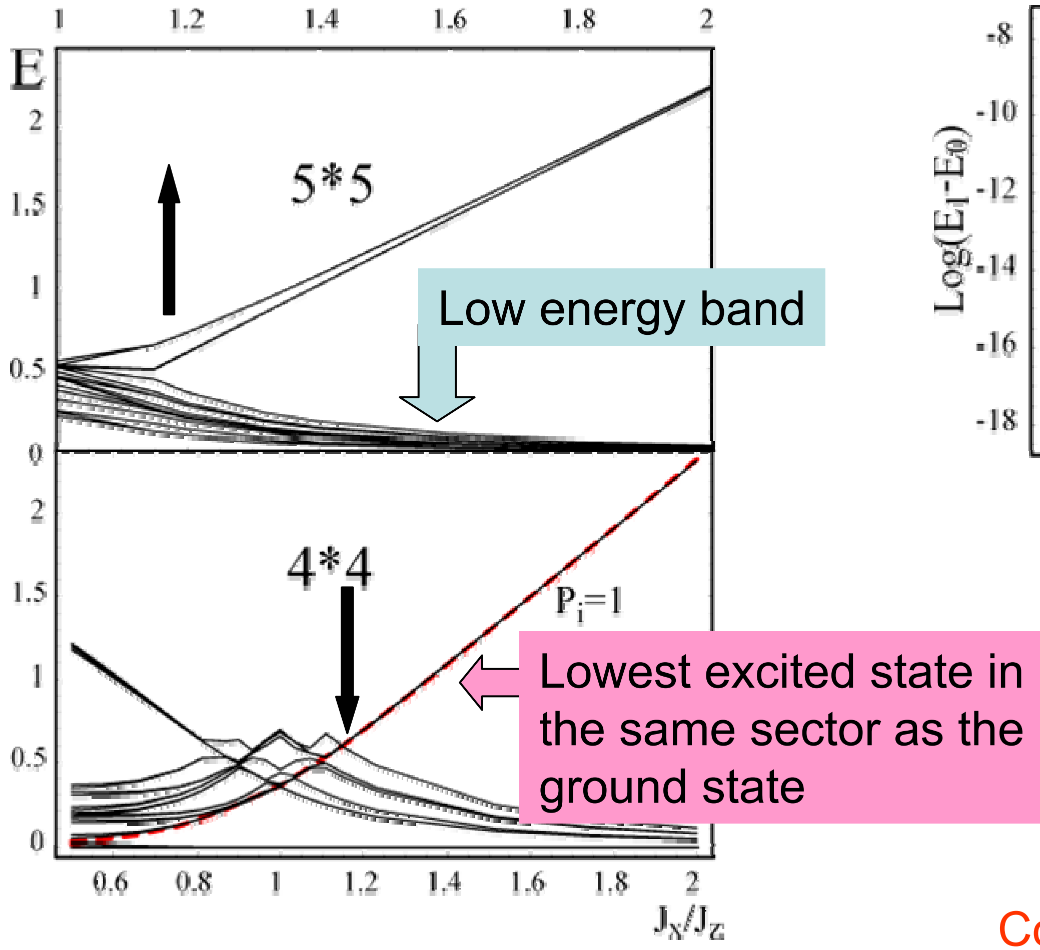
? What is the spectrum of excitations **?**

All eigenstates of P_i have eigenvalues $\lambda_i = \pm 1$. $P_i Q_j P_i = -Q_j$, thus $Q_j |\lambda_i\rangle = |-\lambda_i\rangle$

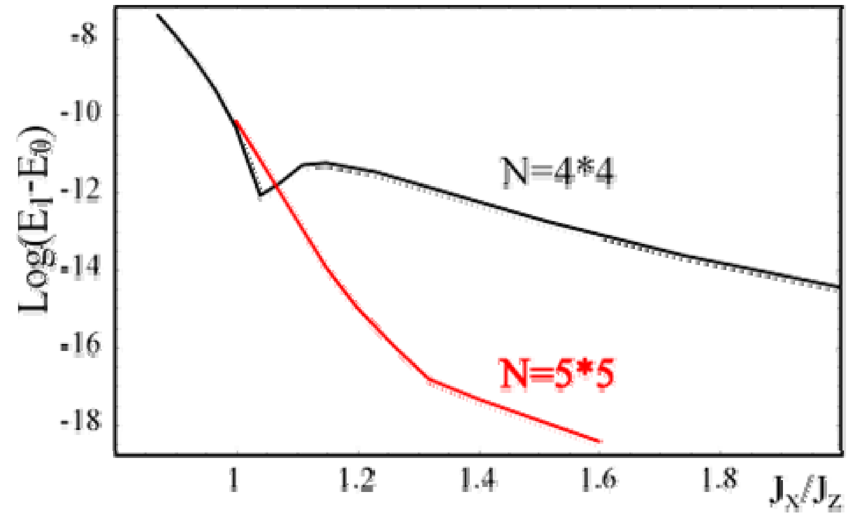
Conjecture: all low energy states are exhausted by different choices of $\lambda_i = \pm 1$ or $\mu_j = \pm 1$ (if we choose to label the states by $Q_j |\mu_j\rangle = \mu_j |\mu_j\rangle$)

Very unusual: number of low energy modes is 2^L not 2^{L^2} and these modes are not associated with the boundary

Numerics



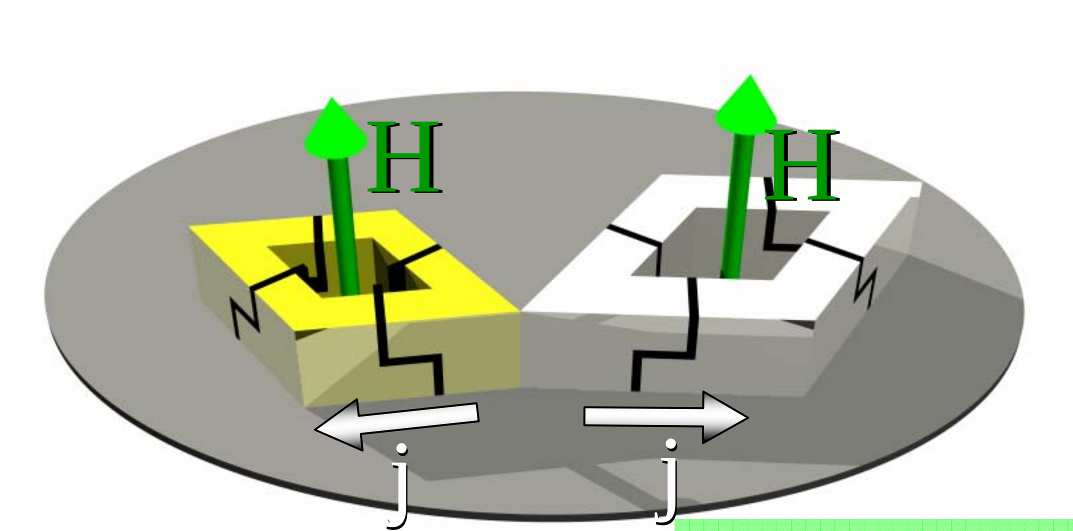
Low energy states for 4*4 and 5*5 spin array. Low energy band contains 2^4 and 2^5 states

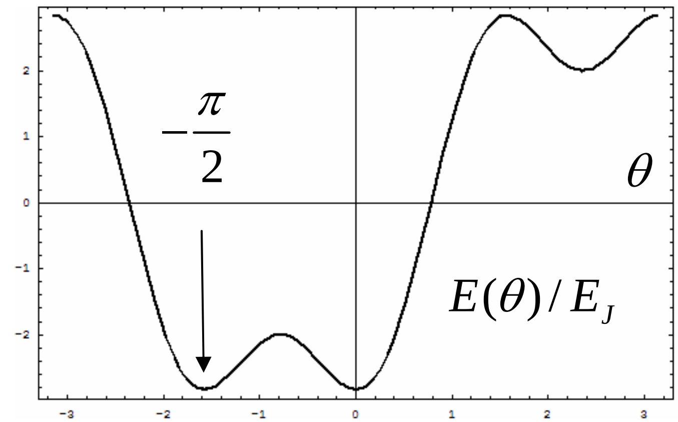


Splitting of two lowest (degenerate) levels by random field applied to each spin and distributed in interval (-0.05, 0.05).

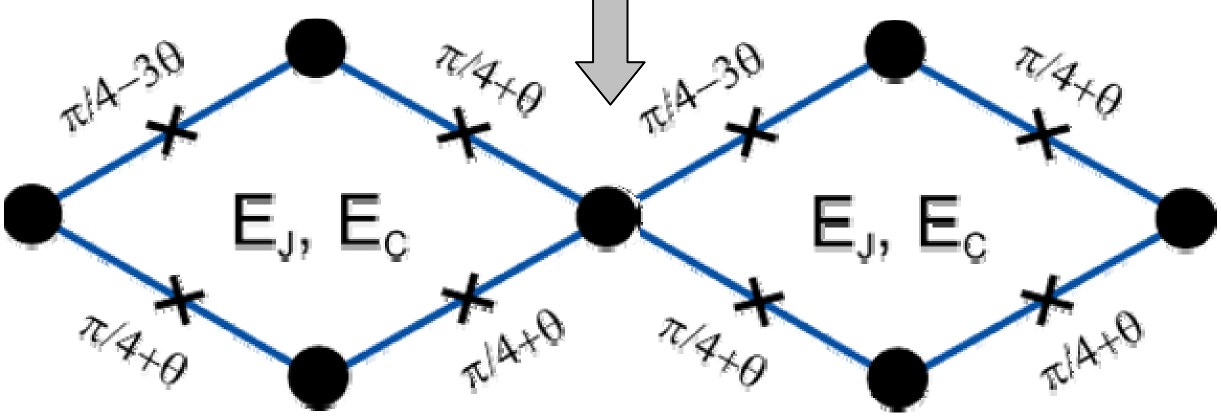
Conclusion: relatively small arrays provide very good protection, especially in one channel!

Emulation of one spin by a Josephson junction array





$$HArea = \Phi = \Phi_0 / 2 = \frac{\pi \hbar c}{e}$$



$\theta = 0$ $\theta = -\frac{\pi}{2}$

Charging energy results in transitions between classically equivalent states:

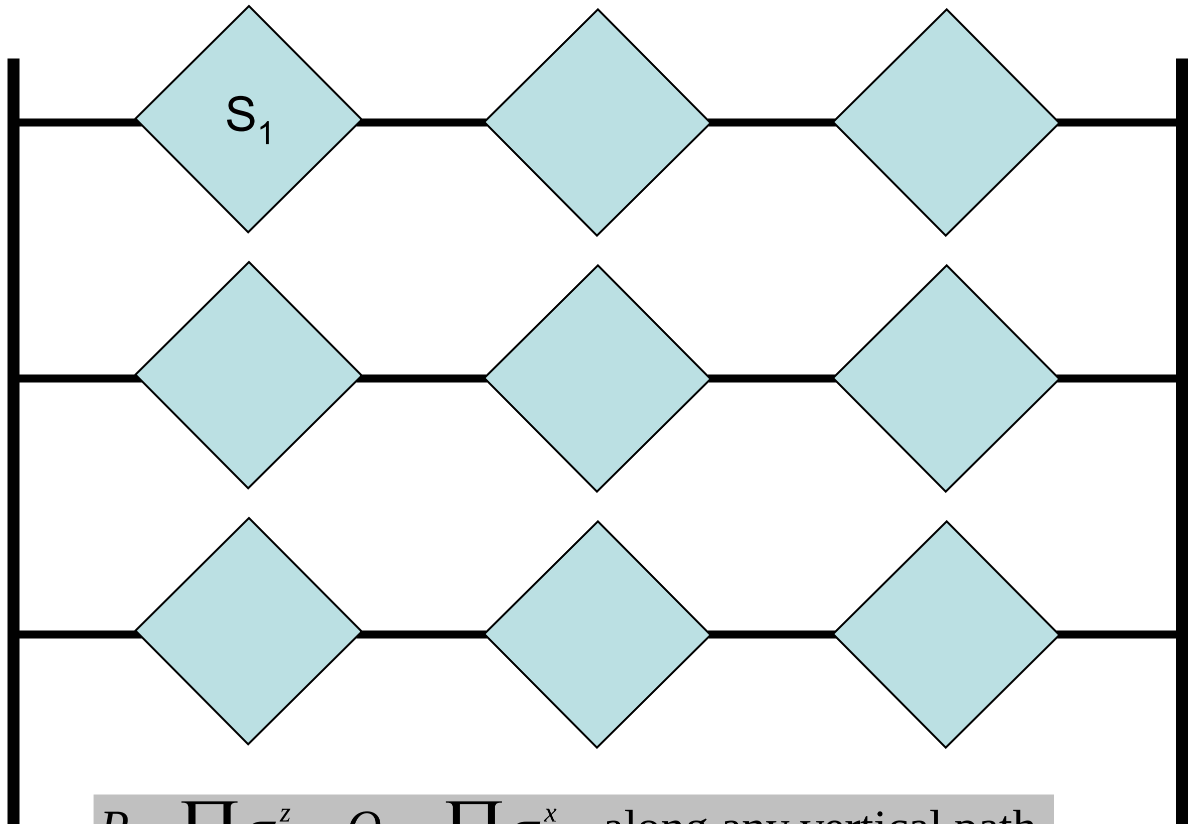
$$H = r(\sigma_1^x + \sigma_2^x)$$

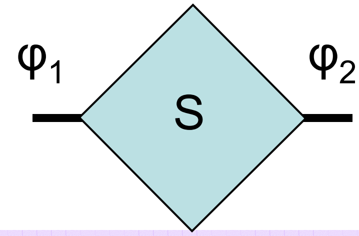
$$r \approx E_J^{3/4} E_C^{1/4} e^{-1.61\sqrt{E_J / E_C}}$$

Imperfections (area, etc) give additional terms

$$\delta H = \delta h_1 \sigma_1^z + \delta h_2 \sigma_2^z$$

Simplest protected physical device





Josephson element that has two (approximately) degenerate classical states with phase differences

$\phi_2 - \phi_1 = 0$ S^z=1/2

or $\phi_2 - \phi_1 = \pi$ S^z=-1/2

and fluctuates between them → effective 'spin' degree of freedom.

$\phi = 0$

$P_i = \prod_j \sigma_{ij}^z \quad Q_j = \prod_i \sigma_{ij}^x$ - along any vertical path, commute with the effective Hamiltonian

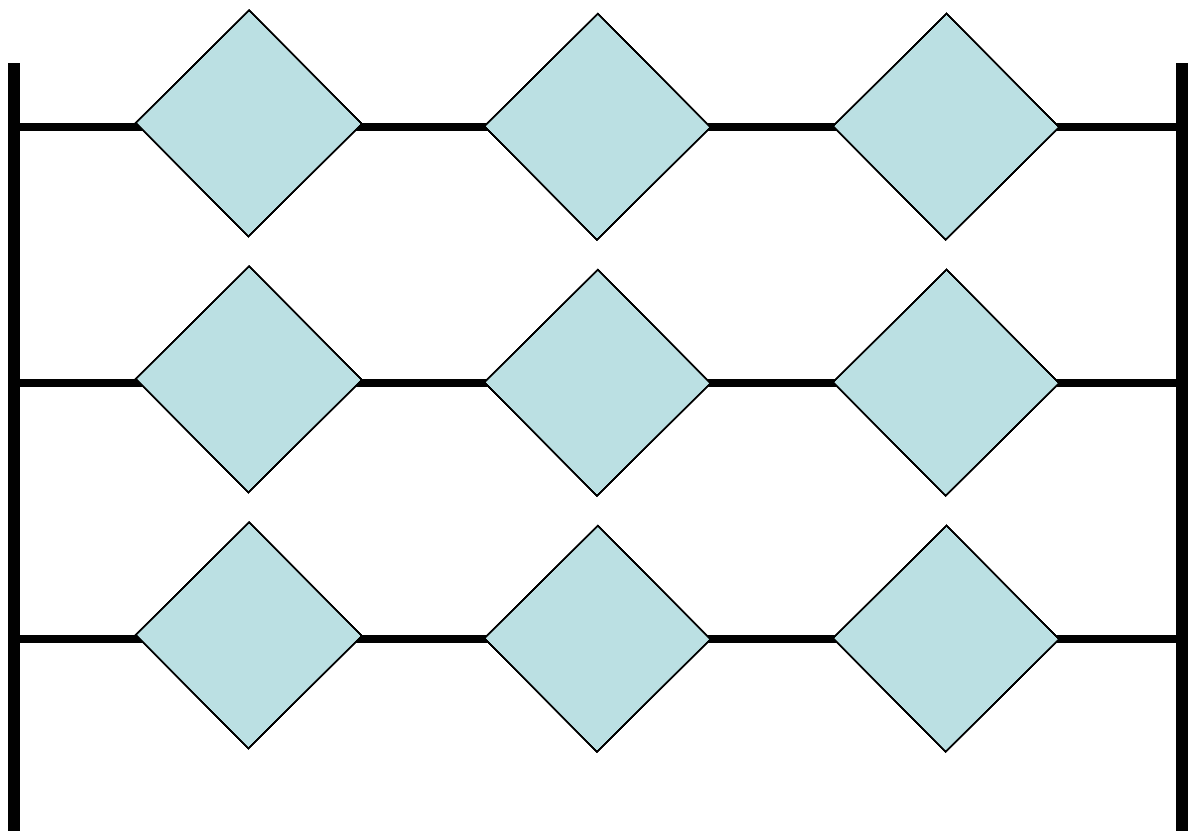
$\phi = 0$ or $\phi = \pi$

No physical operator can be coupled to the absolute value of the phase

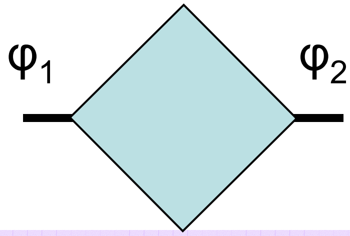
⇒ states with φ=0 and φ=π have the same energy.

⇒ Good protection against "phase" fluctuations.

Simplest protected physical device



$\phi = 0$ $\qquad$ $\phi = 0$ or $\phi = \pi$

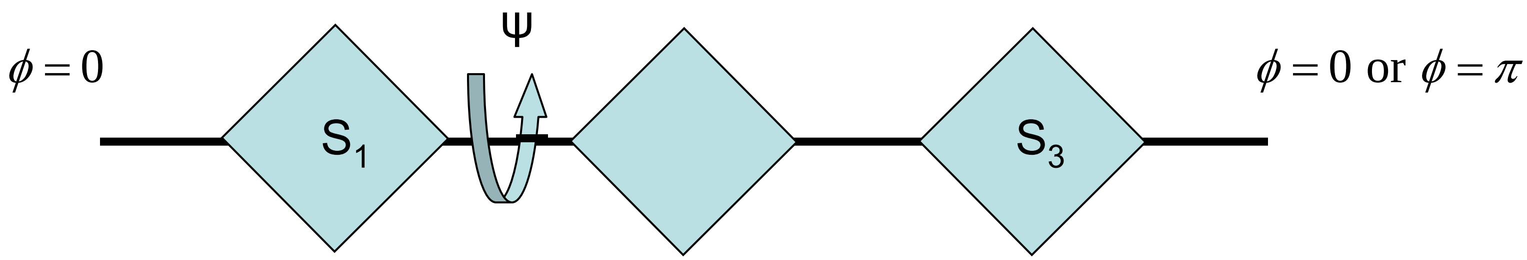


Josephson element that has two (approximately) degenerate classical states with phase differences $\phi_2 - \phi_1 = 0$ or $\phi_2 - \phi_1 = \pi$ and fluctuates between them → effective ‘spin’ degree of freedom.

In order to protect against flip errors one needs large potential barrier in the potential $V(\varphi)$ that separates states with $\varphi=0$ and $\varphi=\pi$.

⇒ Need M~N chains to add these potentials and decrease charging energy of φ

Ideal Hamiltonian of individual chains



Relevant variables: 'spins' of each rhombus and 'phonon' mode of ψ

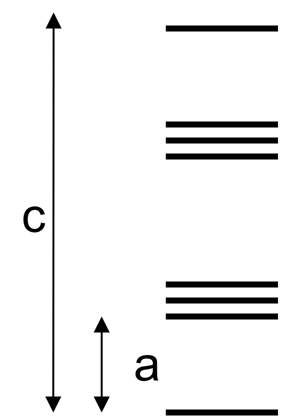
$$H=\alpha\left[\sum_{\mathrm{i}} S_i^x\right]^2+\beta\sum_{\mathrm{i}} S_i^x(a^\dagger+a)+\gamma(a^\dagger a) \qquad \prod_i S_i^z\tau^z\left|0\right\rangle=\left|0\right\rangle$$

Elastic energy of ψ

Operator for the whole array

Integrate out phase modes for large elastic energy, sufficient for not too long chains:

$$H=-a\left[\sum_{\mathrm{i}} S_i^x\right]^2-c_m\prod_i S_i^z\tau^z \quad c_m\gg a$$



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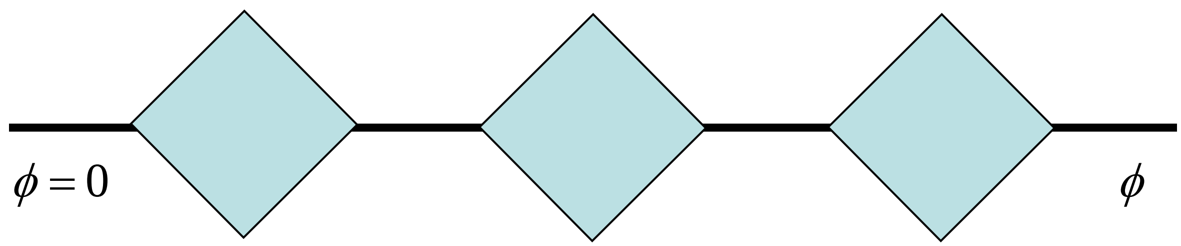
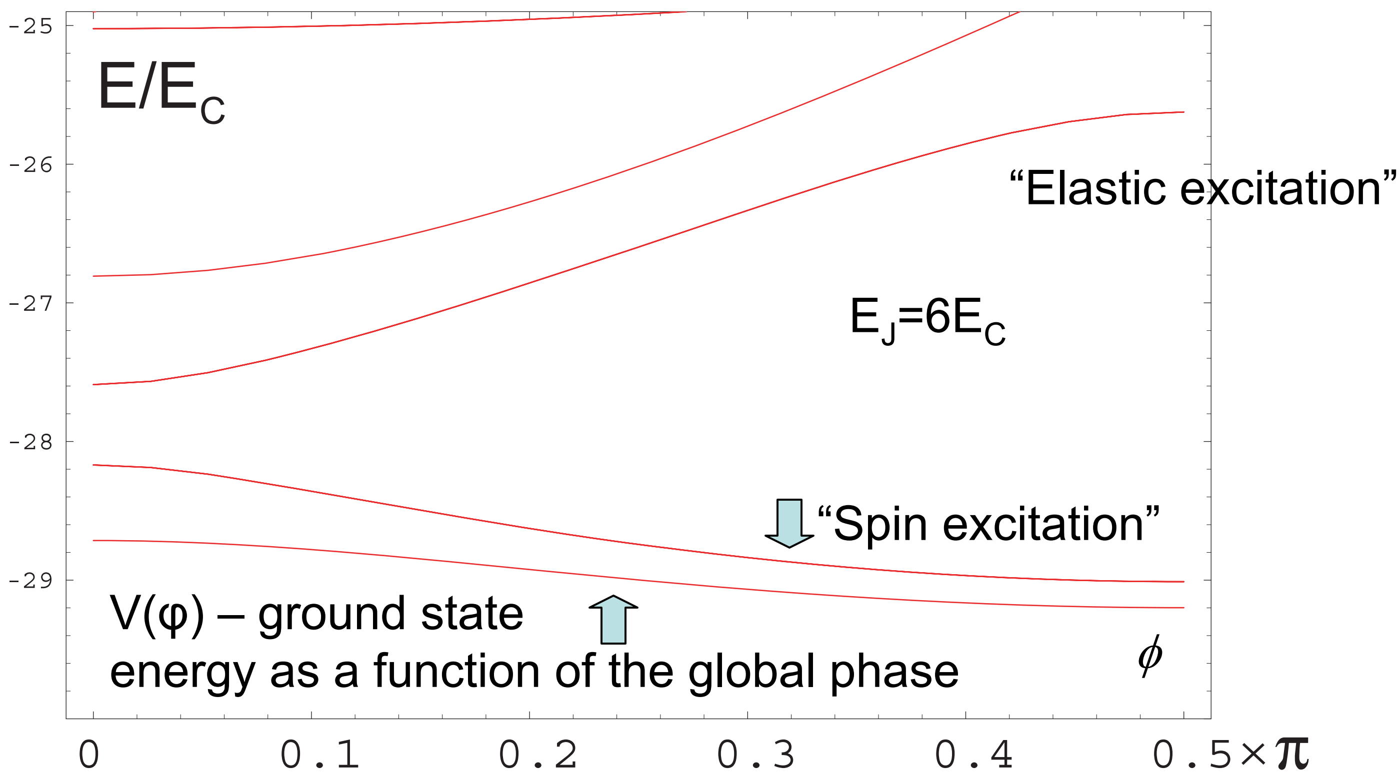
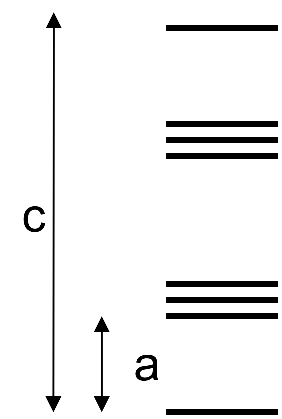
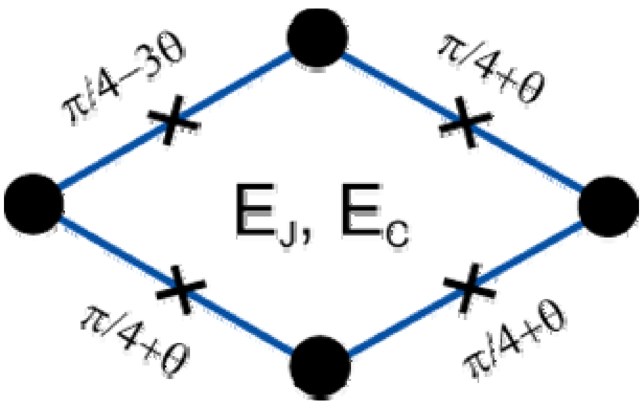






Realistic Hamiltonian of individual chain



Effect of offset charges

$$L_c = \frac{1}{2}\sum_{i,j}\frac{1}{4e^2}C_{ij}(\dot{\varphi}_i - \dot{\varphi}_j)^2 + \frac{1}{2e}(\dot{\varphi}_i - \dot{\varphi}_j)q_{ij}$$

change the amplitude of the transition of individual rhombi by $e^{i(q/2e)\Delta\varphi}$

Amplitude remains real because processes with $\Delta\varphi$, $-\Delta\varphi$ add

Effective Hamiltonian:

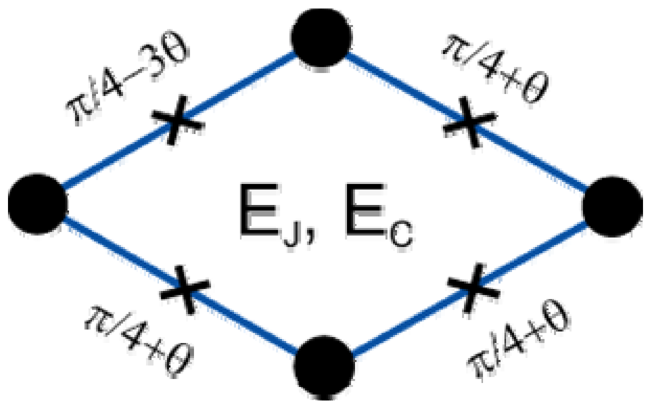
$$\mathrm{H}=\alpha\left[\sum_{\mathrm{i}} k_i S_i^x\right]^2 + \beta\sum_{\mathrm{i}} k_i S_i^x (a^\dagger + a) + \gamma(a^\dagger a) \qquad \prod_i S_i^z \tau^z \left|0\right\rangle = \left|0\right\rangle$$

$k_i \subset$ (-1,1) random variables

The sign of k_i can be eliminated by the transformation $S^X \to -S^x$

The degeneracy of excited states is lifted but the gap is not affected significantly.

Realistic Hamiltonian of individual chain



Effect of Junction parameters scatter:

$$\frac{\pi}{4} \to \frac{\pi}{4}(1-\delta\theta) \quad \delta\theta \approx \frac{\delta E_J}{E_J}$$

Two rhombi with the same sign of $\delta\theta$ prefer to have opposite 'spins'

Effective Hamiltonian:

$$\mathrm{H}=\alpha\left[\sum_{\mathrm{i}} k_i S_i^x\right]^2 + \beta\sum_{\mathrm{i}} k_i S_i^x (a^\dagger + a) + \gamma(a^\dagger a) \qquad \prod_i S_i^z \tau^z \left|0\right\rangle = \left|0\right\rangle$$

$$\delta H = \sum_{ij} \delta E_{ij} S_i^z S_j^z \quad \delta E_{ij} \approx V(\theta_i + \theta_j) \ll \Delta \text{ (gap to the first excited state)}$$

$k_i \subset$ (-1,1) random variables

The sign of k_i can be eliminated by the transformation $S^X \to -S^x$
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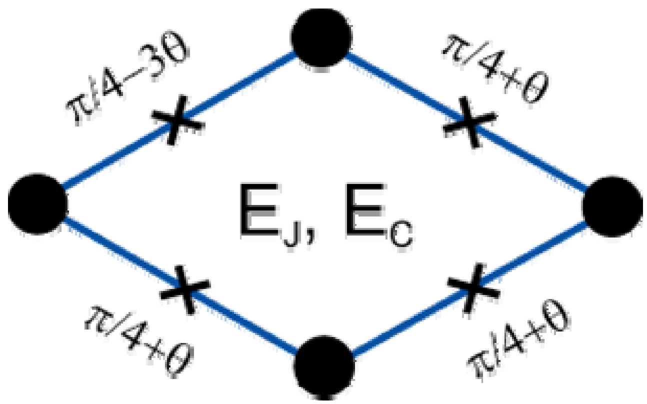
Realistic Hamiltonian of individual chain



Effect of geometry scatter:

two rhombus states are no longer degenerate

because $\Phi \neq \Phi_0 / 2$

$$\delta E_J \approx \frac{\delta\Phi}{\Phi_0} E_J$$

Effective Hamiltonian:

$$\mathrm{H}{=}\alpha\left[\sum_{\mathrm{i}} k_i S_i^x\right]^2 + \beta \sum_{\mathrm{i}} k_i S_i^x (a^\dagger + a) + \gamma (a^\dagger a) \qquad \prod_i S_i^z \tau^z \left|0\right\rangle = \left|0\right\rangle$$

$$\delta H = \sum_{ij} \delta E_{ij}^J S_i^z S_j^z \quad \delta E_{ij} \approx V(\theta_i + \theta_j) \ll \Delta \text{ (gap to the first excited state)}$$

$$\delta H = \sum_i \delta E_i^\Phi S_i^z \qquad \delta E_i^\Phi \approx \frac{\delta\Phi}{\Phi_0} E_J \ll \Delta$$

$k_i \subset$ (-1,1) random variables

Realistic Hamiltonian of individual chain

Effective Hamiltonian:

$$\mathrm{H}=\alpha\left[\sum_{\mathrm{i}} k_i S_i^x\right]^2+\beta\sum_{\mathrm{i}} k_i S_i^x(a^\dagger+a)+\gamma(a^\dagger a) \qquad \prod_i S_i^z \tau^z\left|0\right\rangle=\left|0\right\rangle$$

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$$\delta H=\sum_i \delta E_i^\Phi S_i^z \qquad \delta E_i^\Phi\approx\frac{\delta\Phi}{\Phi_0}E_J\ll\Delta$$

$k_i \subset$ (-1,1) random variables

Lifting the global degeneracy by the disorder terms appears only the N^{th} order of the perturbation theory:

$$\delta\varepsilon=\varsigma\prod_i \delta E_i^\Phi/\Delta^{N-1} \qquad \varsigma\sim O(1) \text{ combines combinatorical factors and matrix elements}$$

Dephasing of the global state

Charge noise results in the $k_i(t)$ dependence of the main terms:

$$\mathrm{H}=\alpha\left[\sum_{\mathrm{i}} k_i(t) S_i^x\right]^2+\beta\sum_{\mathrm{i}} k_i(t) S_i^x(a^\dagger+a)+\gamma(a^\dagger a) \qquad \prod_i S_i^z\tau^z\left|0\right\rangle=\left|0\right\rangle$$

$k_i(t)=k_i(1+\xi_i^q)$ with $\xi_i^q\sim 10^{-2}-10^{-3}$

Flux noise give time dependence to the correction terms:

$$\delta H=\sum_i \delta E_i^\Phi(t) S_i^z \qquad \delta E_i^\Phi\approx\frac{\delta\Phi}{\Phi_0}(1+\xi_i^\Phi) \text{ with } \xi_i^q\sim 10^{-3}-10^{-4}$$

Lifting the global degeneracy lead to the dephasing because the gap acquires time dependence and because δE^Φ fluctuates:

$$\delta\varepsilon=\varsigma\prod_i \delta E_i^\Phi(t)/\Delta(t)^{N-1} \quad \Delta(t)\approx\Delta_0(1+\sum\xi_{\mathrm{i}}^{\mathrm{Q}}) \quad \delta E_i^\Phi(t)=\delta E_i^\Phi(1+\xi_{\mathrm{i}}^\Phi)$$

Both effects are suppressed by the N^{th} power of $\delta E/\Delta$

Δ grows as N for longer chains leading to even larger suppression.

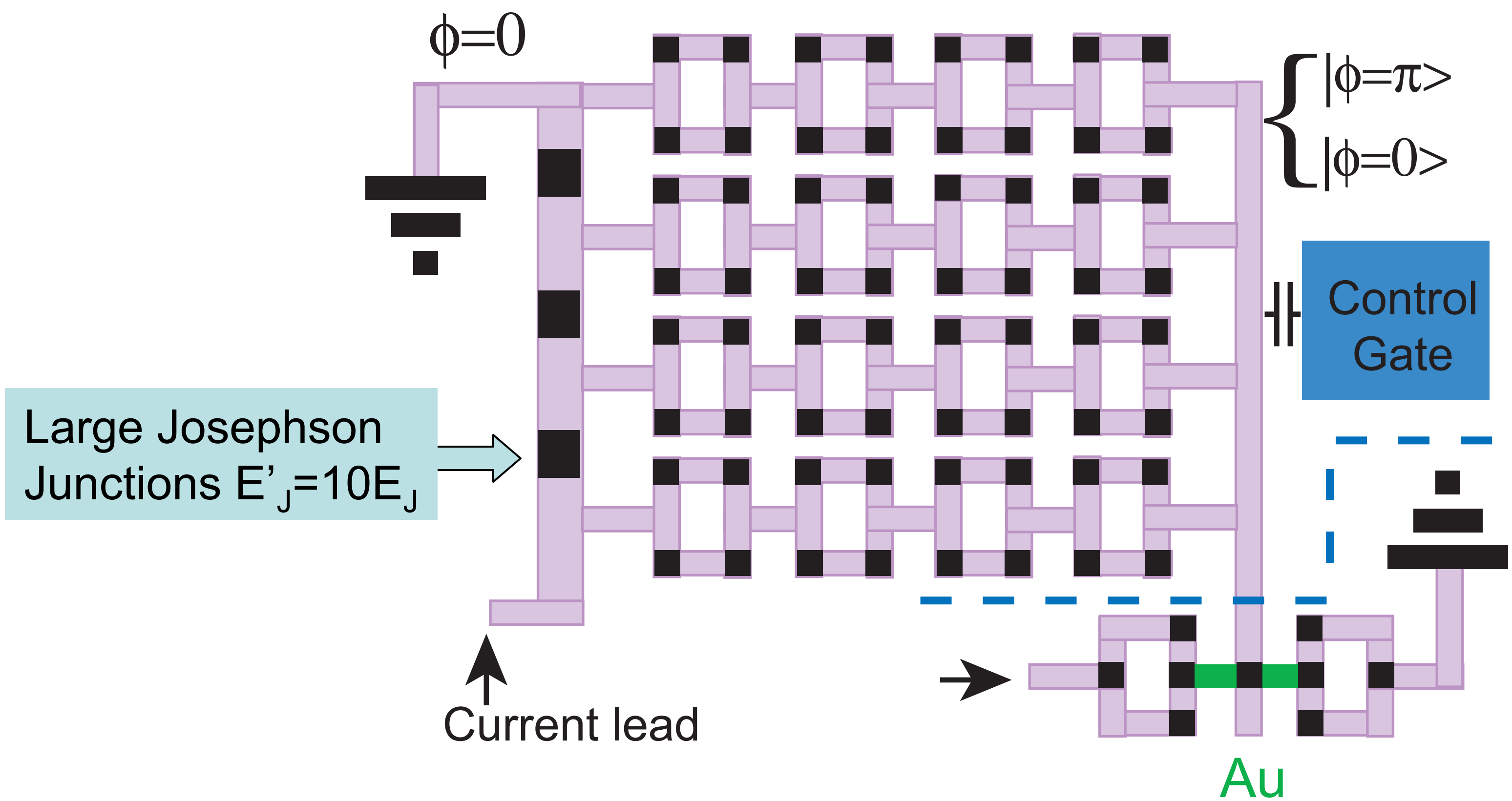
Measurements and manipulations.

- Abelian (Z_2) symmetry → only limited set of precise manipulations
- Most operations require taking the system out of the protected space
- Measurement also requires moving the system out of protected space first and then one can do

    destructive (by critical current)

    nondestructive ($Z(\omega \sim \Delta)$)

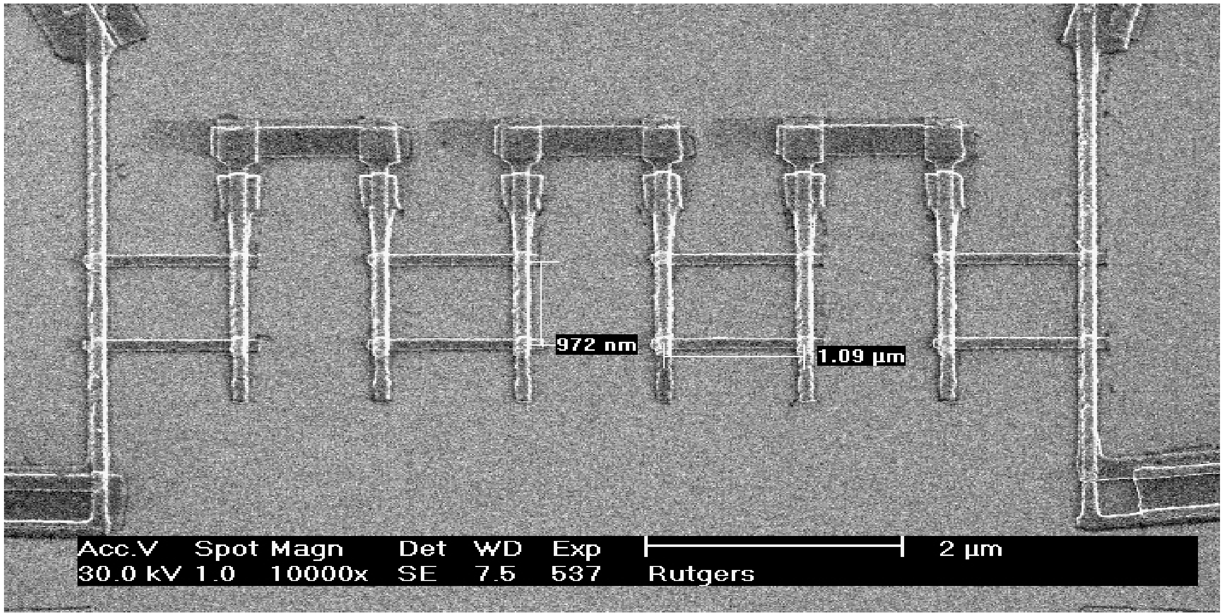
Measurements and manipulations.



Conclusion

- It is possible to built protected qubit with the decoherence time orders of magnitude longer than achieved currently in Josephson junction devices with current technology.

Will it work as we hope?



Algorithms

- Construct efficient error correction algorithms for the systems in which the noise in one channel is much larger than in another.

BCH error correction

Need to add more parity checks so that to correct more errors:

$$P_1 = \left(\begin{pmatrix}0\\0\\1\end{pmatrix} \begin{pmatrix}0\\1\\0\end{pmatrix} \begin{pmatrix}0\\1\\1\end{pmatrix} \begin{pmatrix}1\\0\\0\end{pmatrix} \begin{pmatrix}1\\0\\1\end{pmatrix} \begin{pmatrix}1\\1\\0\end{pmatrix} \begin{pmatrix}1\\1\\1\end{pmatrix} \right) \text{ - original parity check}$$

$$P_2 = \left(f\begin{pmatrix}0\\0\\1\end{pmatrix} f\begin{pmatrix}0\\1\\0\end{pmatrix} f\begin{pmatrix}0\\1\\1\end{pmatrix} f\begin{pmatrix}1\\0\\0\end{pmatrix} f\begin{pmatrix}1\\0\\1\end{pmatrix} f\begin{pmatrix}1\\1\\0\end{pmatrix} f\begin{pmatrix}1\\1\\1\end{pmatrix} \right) \text{ - additional parity check}$$

Additional parity checks should be independent from the original ones

⇨ function f() should be pseudorandom but allows decrypting

Main idea: represent bits as elements of Galois field GF(2^k) (k=3 above) and use

$f(x)=x$ – 1^{st} set of checks

$f(x)=x^3$ – 2^{nd} set of checks

$f(x)=x^5$ – 3^{rd} set of checks…

$$S_1 = \sum_k f_1(\beta_k)$$

Error detection is possible because the set of equations can be solved iff the number of unknowns (errors) is less the number of equations (checks).

$$S_2 = \sum_k f_2(\beta_k)$$

Classical BCH+LDPC = quantum error correction

1. Start with BCH error correction of t errors in a good (Z) channel.

2. Smallest codewords (undetectable errors) in Z channel have length 2t and satisfy

$$0 = \sum_k f_j(\beta_k) \text{ for all j}$$

3. Choose randomly t+1 errors and generate the remaining t errors by solving the equations

$$\sum_{k=1}^{t+1} f_j(\beta_k) = \sum_{k=t+2}^{2t+1} f_j(\beta_k) \text{ for all j}$$

4. Find the subset of the pseudorandom sets of (2t+1) errors with the degree distribution required by good LDPC codes.

5. Use LDPC code in X-channel.

Results

Simulations of errors in the block of 1023 bits GF(2^{10})

1. T=3 code (Total quantum rate R=0.52)

$P_Z = 2.2\ 10^{-4}$ $P_X = 2.2\ 10^{-2}$ $P_B = 10^{-4}$

2. T=4 code (Total quantum rate R=0.44)

$P_Z = 4.3\ 10^{-4}$ $P_X = 4.3\ 10^{-2}$ $P_B = 10^{-4}$

P_B - Block error rate (probability of a single error)

Conclusion: high rate (low redundancy) code can tolerate large noise in one channel provided that the noise in another is reasonably low.