Topological Quantum Compiling

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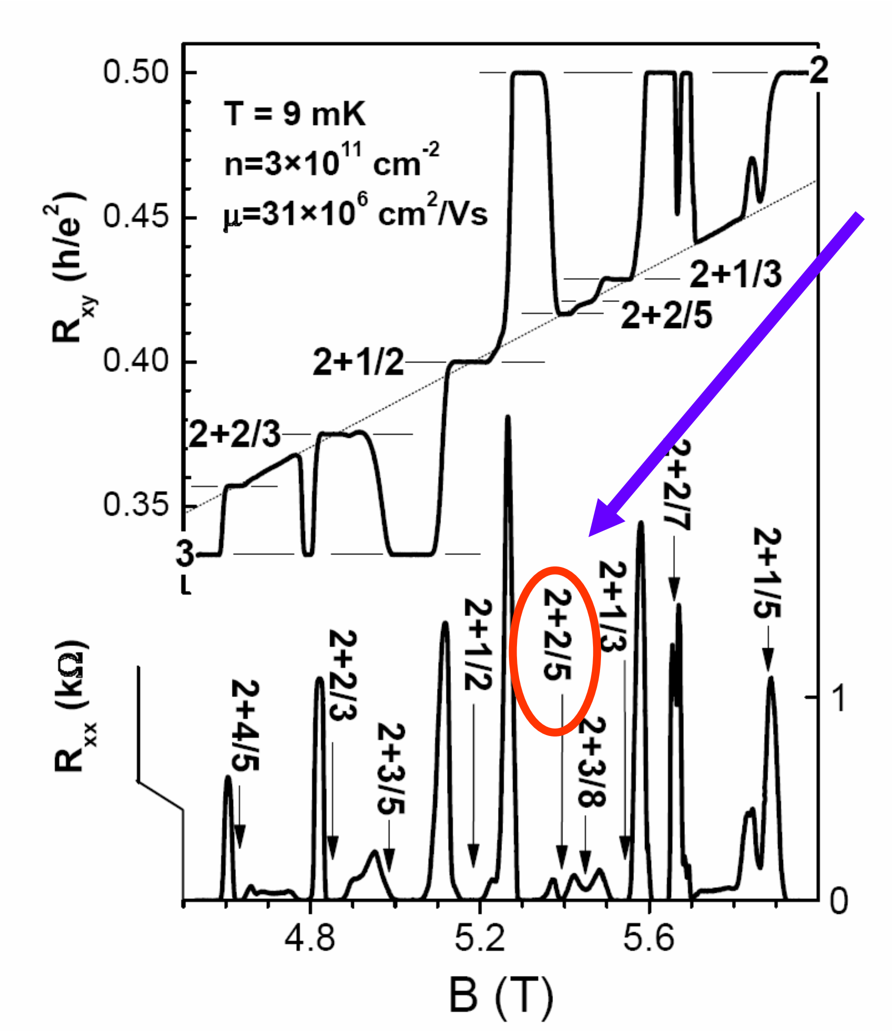
NEB, L. Hormozi, G. Zikos, S.H. Simon, Phys. Rev. Lett. 95 140503 (2005)
S.H. Simon, NEB, M.Freedman, N, Petrovic, L. Hormozi, Phys. Rev. Lett. 96, 070503 (2006).
L. Hormozi, G. Zikos, NEB, and S.H. Simon, In preparation

Support: US DOE

Fibonacci Anyons may actually exist!



J.S. Xia et al., PRL (2004).

ν = 12/5

Possibly a Read-Rezayi k = 3 "Parafermion" state.

Read and Rezyai, '99

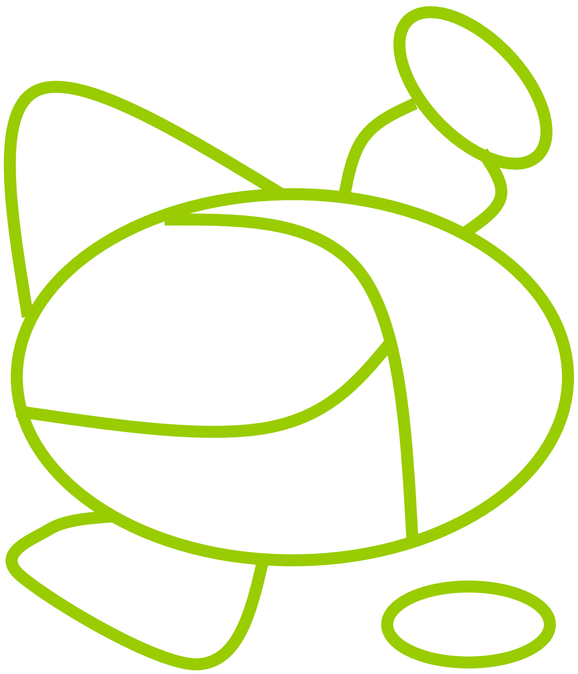
Charge *e/5* quasiparticles with braiding properties described by *SU(2)*$_3$ Chern-Simons Theory.

Slingerland and Bais, '01

Non-Abelian content is that of Fibonacci anyons.

Maybe, one day, **Fibonacci Anyons** will be everywhere!

Bosonic Read-Rezayi states (including **k=3** at **ν = 3/2**) may be realizable in rotating Bose condensates. Rezayi, Read, Cooper '05



Doubled Fibonacci "string-nets" may be found / realized.

Levin and Wen '04

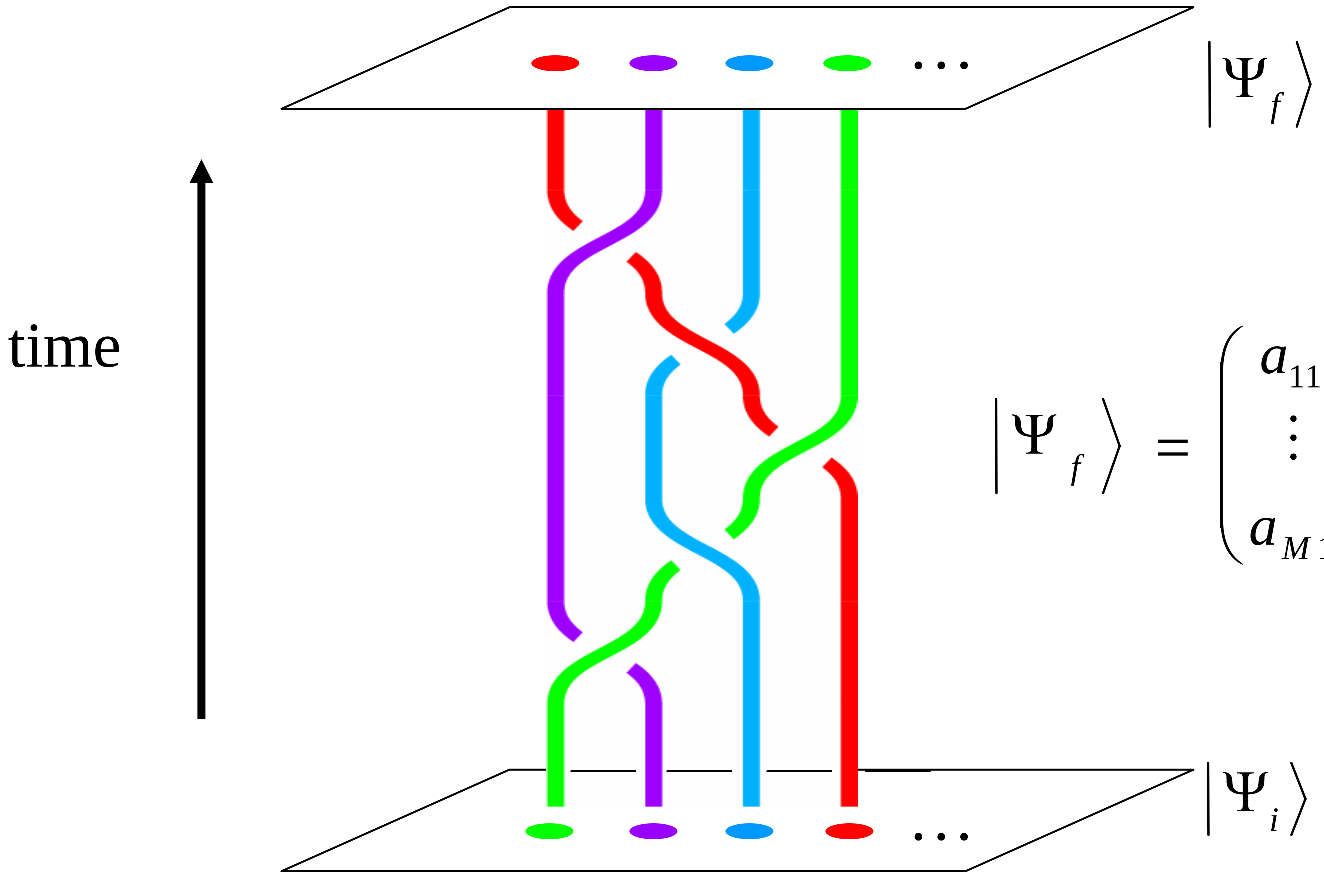
Fendley and Fradkin '05

Freedman, Nayak, Shtengel, Walker and Wang '03

Think Golden!


τ × τ = 1 + τ

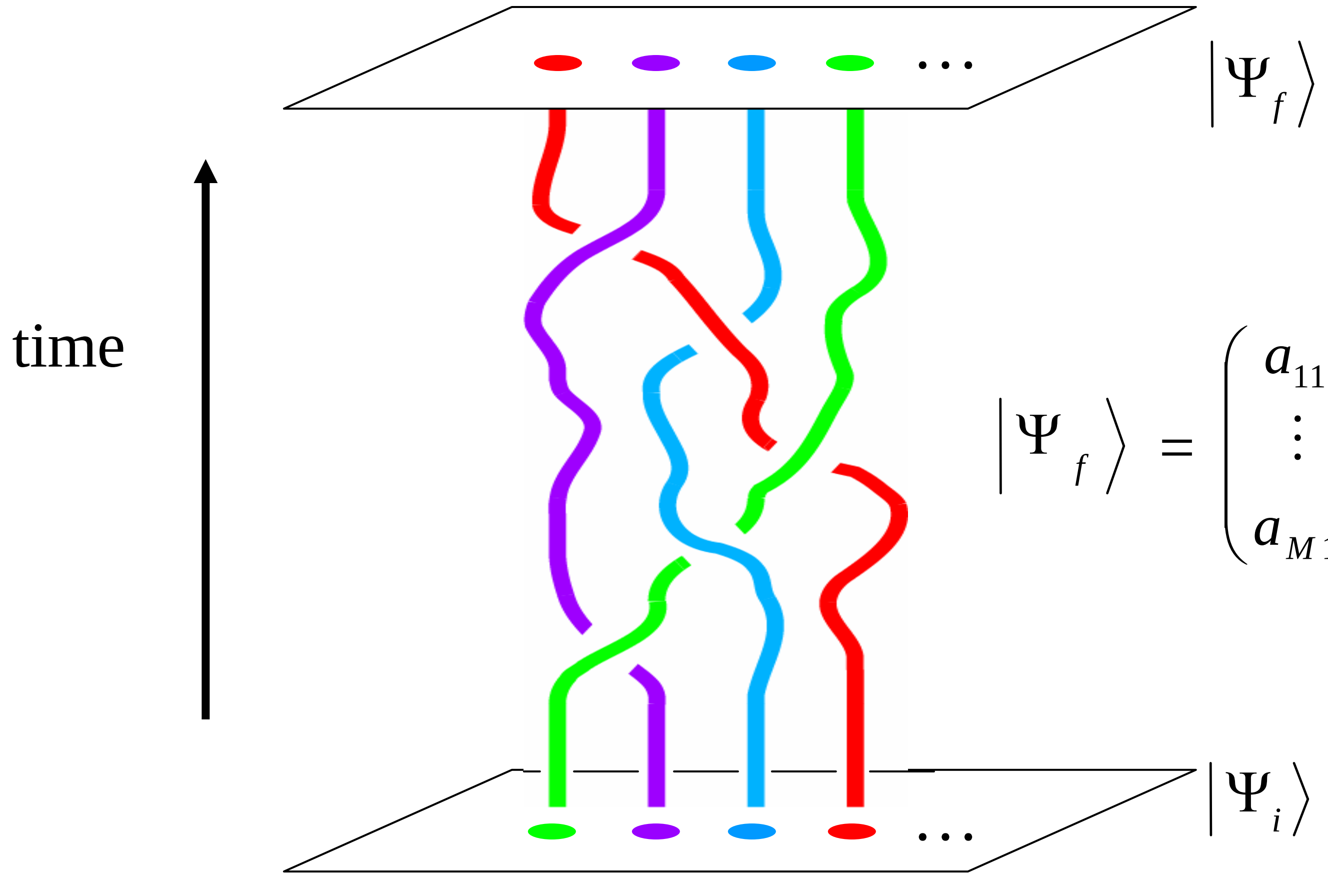
Topological Quantum Computation



$$|\Psi_f\rangle = \begin{pmatrix} a_{11} & \cdots & a_{1M} \\ \vdots & \ddots & \vdots \\ a_{M1} & \cdots & a_{MM} \end{pmatrix} |\Psi_i\rangle$$

When quasiparticles are present there is an exponentially large Hilbert space whose states cannot be distinguished by local measurements. Quasiparticle world-lines forming braids carry out unitary transformations on this Hilbert space.

Topological Quantum Computation



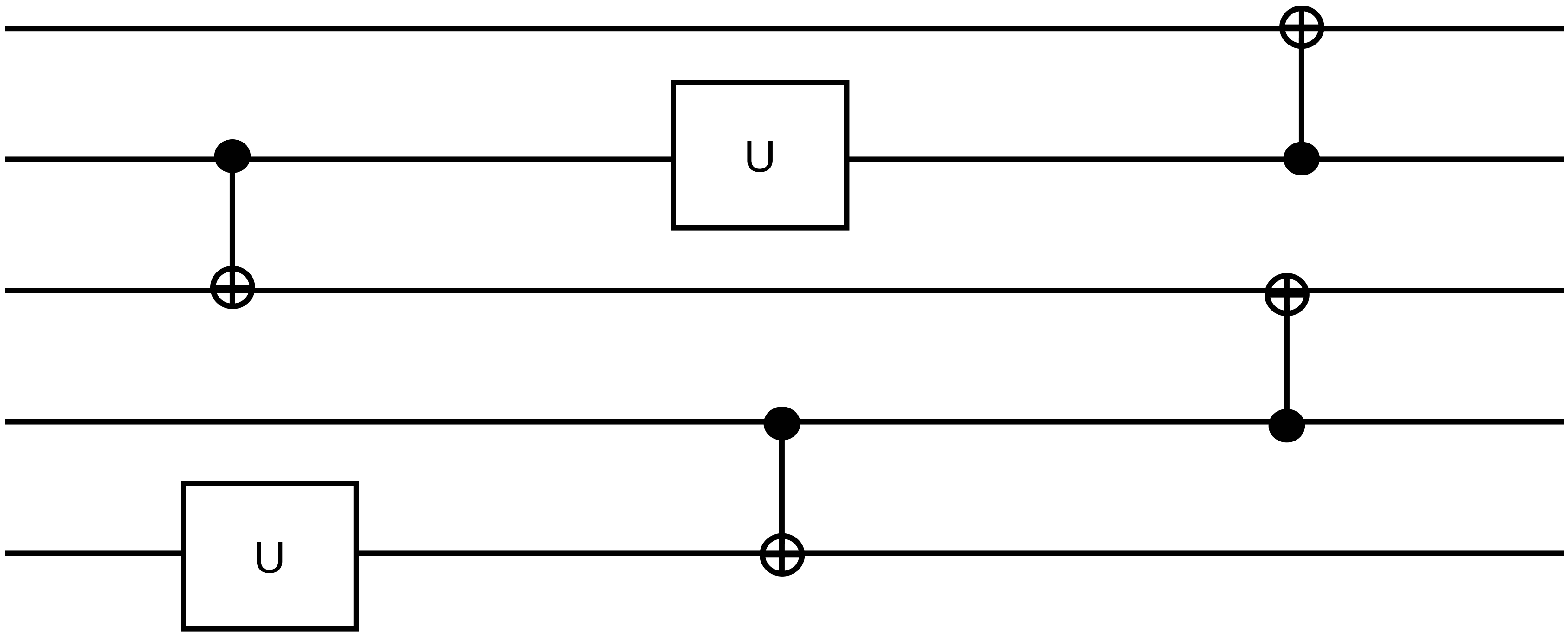
$$|\Psi_f\rangle = \begin{pmatrix} a_{11} & \cdots & a_{1M} \\ \vdots & \ddots & \vdots \\ a_{M1} & \cdots & a_{MM} \end{pmatrix} |\Psi_i\rangle$$

Unitary transformation depends only on the topology of the braid swept out by anyon world lines!

Robust quantum computation?

(Kitaev ’97; Freedman, Larsen and Wang ‘01)

Quantum Circuit



What braid corresponds to this circuit?

Fibonacci Anyon Basics

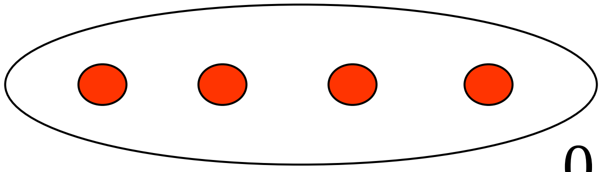
A Fibonacci Anyon → ● Fibonacci →

The laws of Fibonacci anyons:

1. Fibonacci anyons have a quantum attribute I'll call q-spin:

● ← q-spin = 1

2. A collection of Fibonacci anyons can have a total q-spin of either 0 or 1:



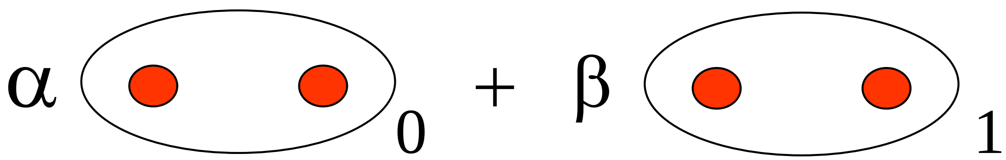
0,1

Notation: Ovals are labeled by total q-spin of enclosed particles.

Fibonacci Anyon Basics

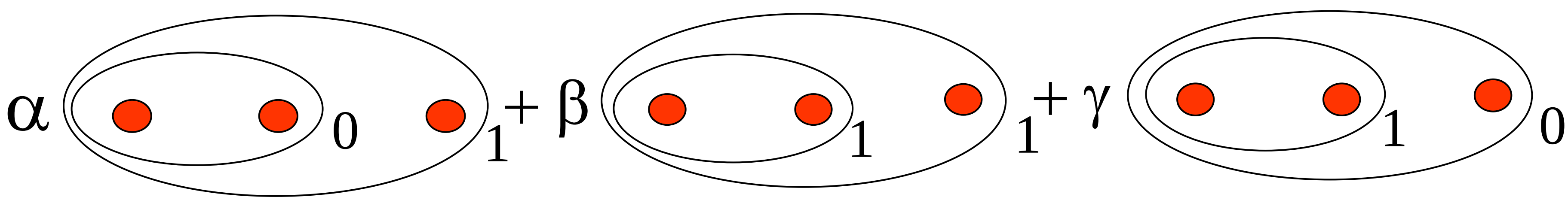
3. The “fusion” rule for combining q-spin is: $\boxed{1 \times 1 \;=\; 0 + 1}$

This means that two Fibonacci anyons can have total q-spin 0 or 1, or be in any quantum superposition of these two states.



Two dimensional Hilbert space

Three Fibonacci anyons → Three dimensional Hilbert space

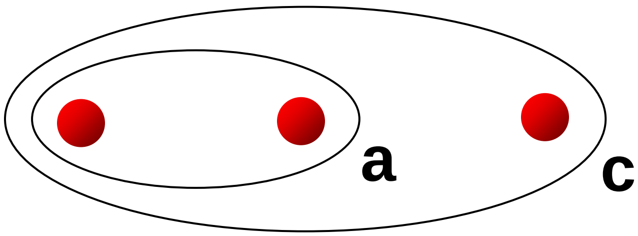


For **N** Fibonacci anyons Hilbert space dimension is **Fib(N-1)**

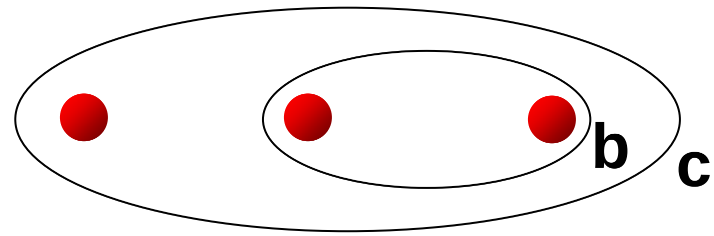
The F Matrix

Changing fusion bases:

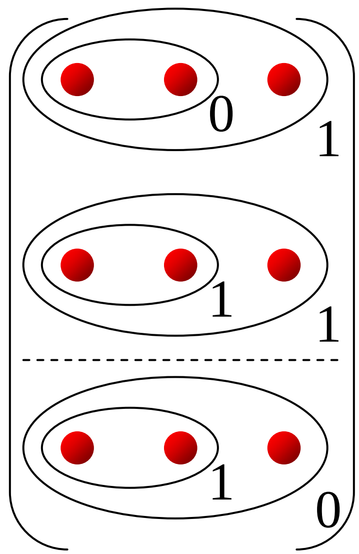
$$\sum_a F^c_{ab}$$



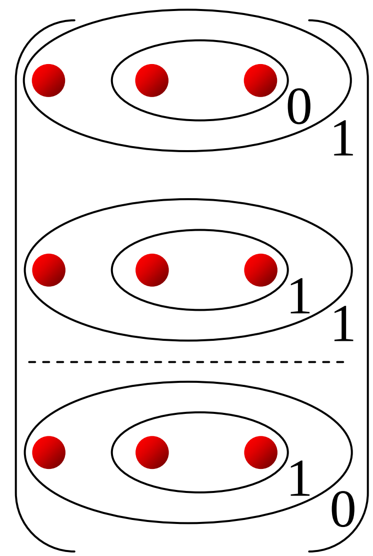
$$=$$



$$\underbrace{\left(\begin{array}{cc|c} -\tau & \sqrt{\tau} & 0 \\ \sqrt{\tau} & \tau & 0 \\ \hline 0 & 0 & 1 \end{array}\right)}_{F^c_{ab}}$$



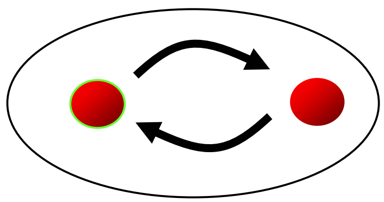
$$=$$



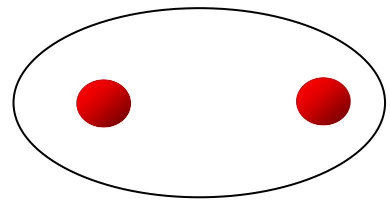
$$\tau = \frac{\sqrt{5}-1}{2}$$

The R Matrix

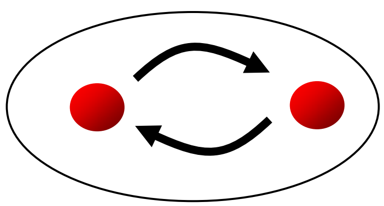
Exchanging Particles:



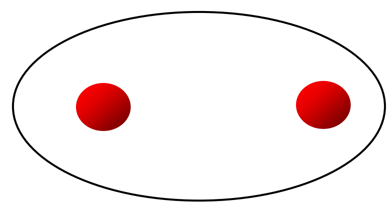
0 $= e^{-i4\pi/5}$



0



1 $= e^{i3\pi/5}$



1

$$R = \begin{pmatrix} e^{-i4\pi/5} & 0 \\ 0 & e^{i3\pi/5} \end{pmatrix}$$

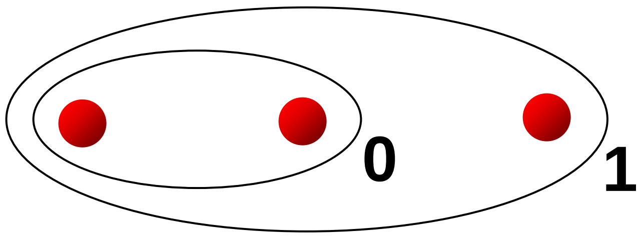
F and *R* must satisfy certain consistency conditions (the “pentagon” and “hexagon” equations). For Fibonacci anyons these equations *uniquely determine* *F* and *R*.

Encoding a Qubit

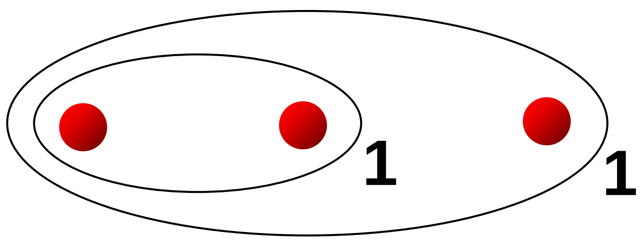
(Freedman, Larsen, and Wang, 2001)

Qubit States

$|0\rangle =$

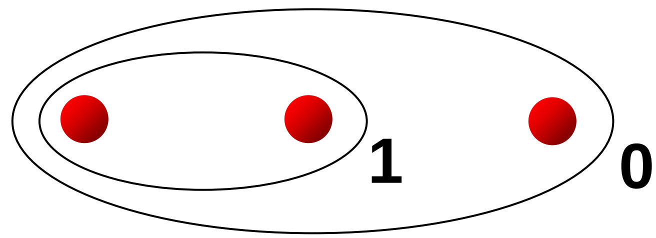


$|1\rangle =$



State of qubit is determined by q-spin of two leftmost particles

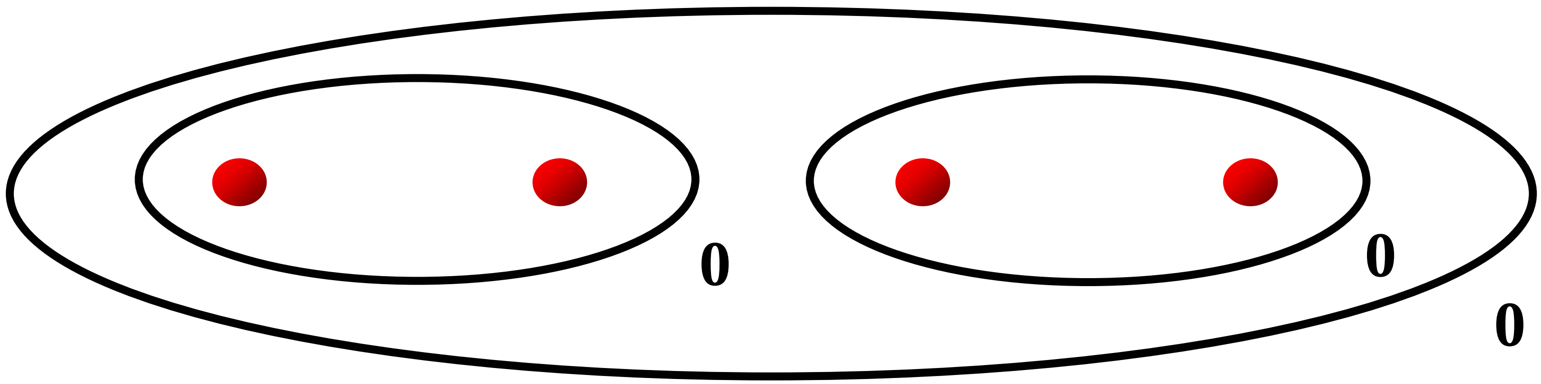
Non-Computational State



Transitions to this state are **leakage errors**

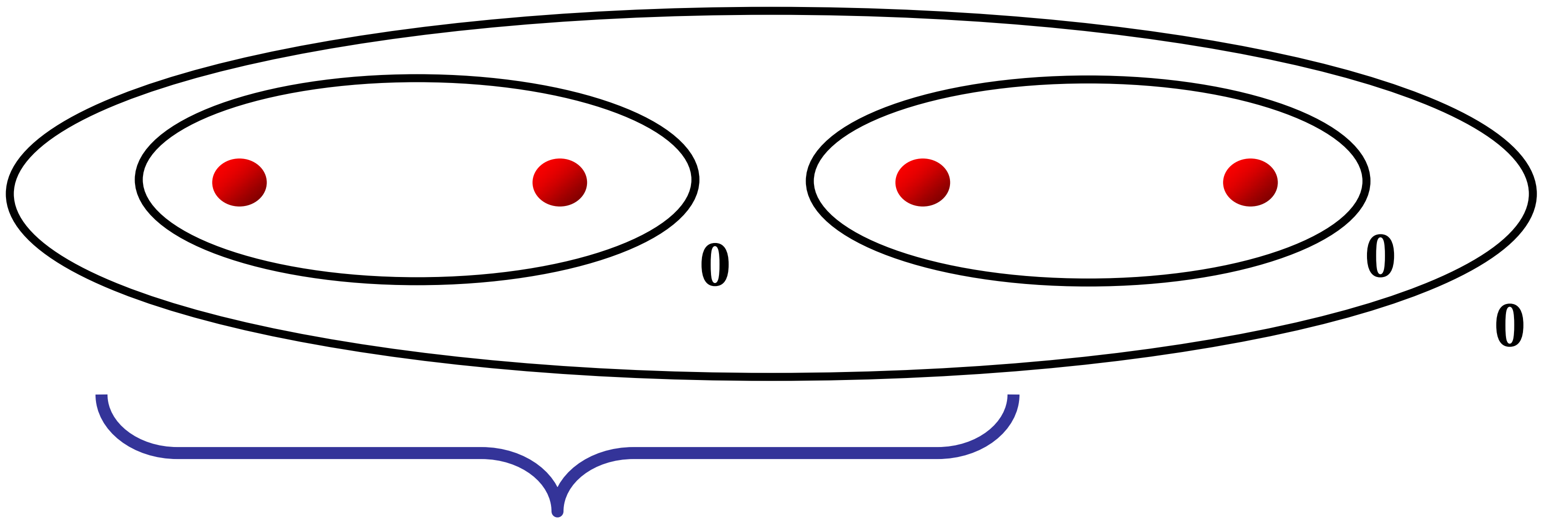
Initializing a Qubit

Pull two quasiparticle-quasihole pairs out of the "vacuum".



Initializing a Qubit

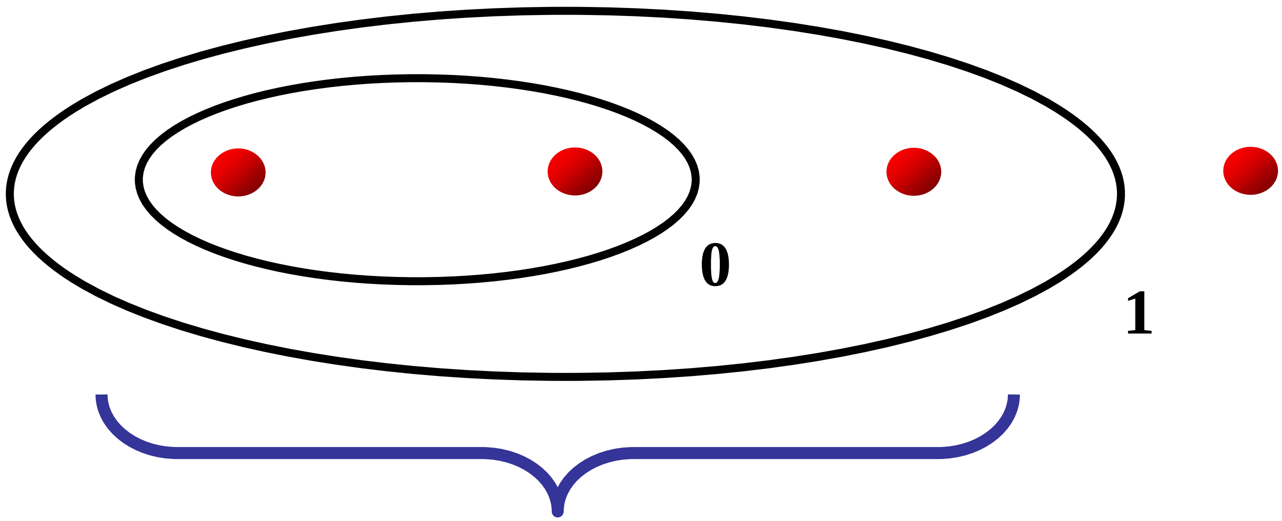
Pull two quasiparticle-quasihole pairs out of the "vacuum".



These three particles have total q-spin 1

Initializing a Qubit

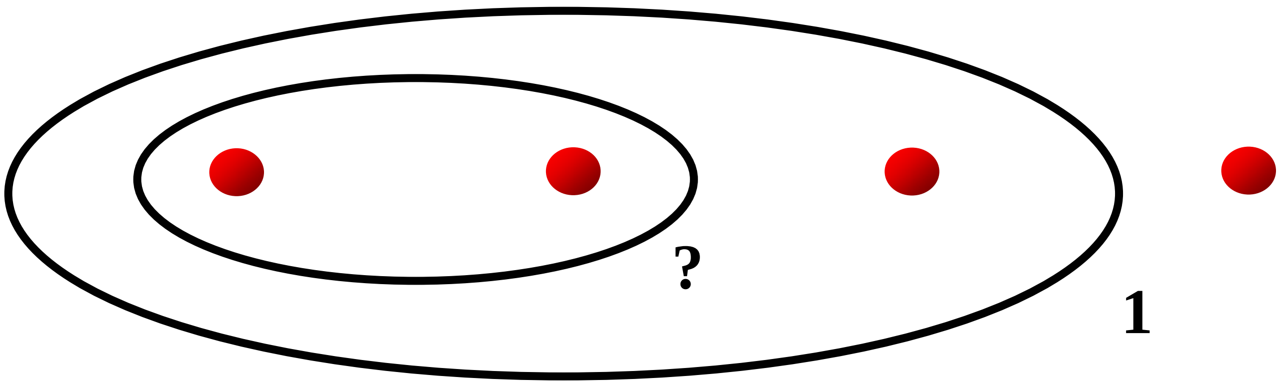
Pull two quasiparticle-quasihole pairs out of the "vacuum".



$$|0\rangle$$

Measuring a Qubit

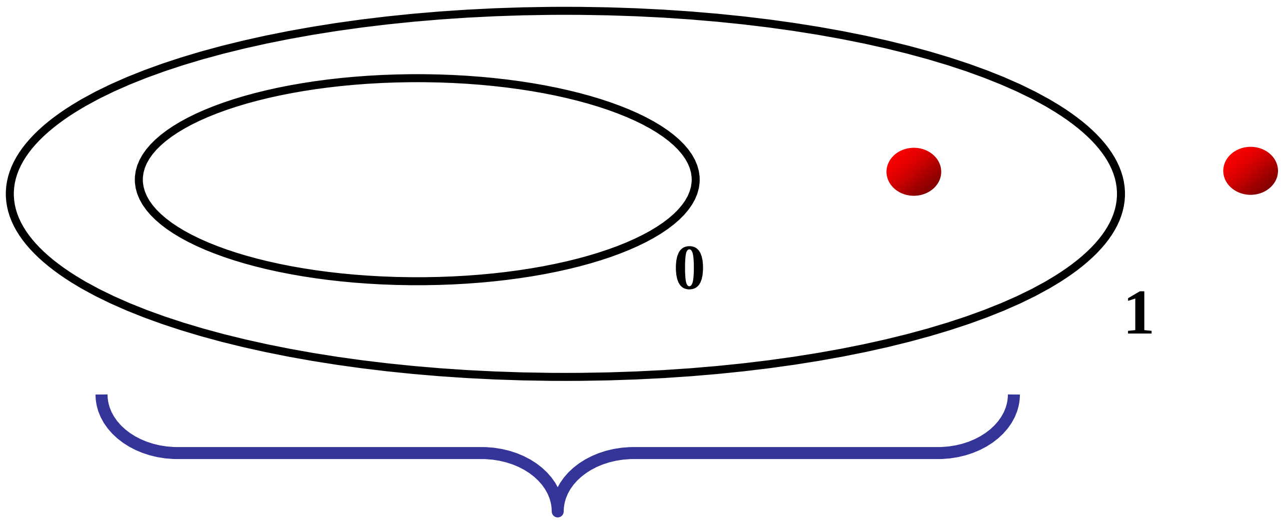
Try to fuse the leftmost quasiparticle-quasihole pair.



$$\underbrace{\hspace{6cm}}_{\alpha\left|0\right\rangle + \beta\left|1\right\rangle}$$

Measuring a Qubit

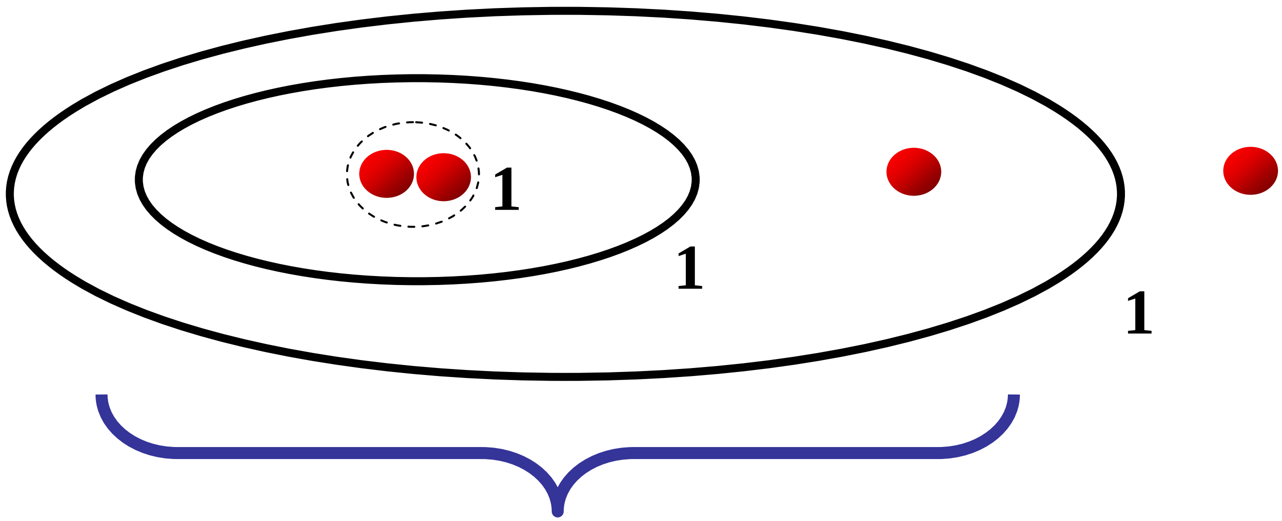
If they fuse back into the “vacuum” the result of the measurement is 0.



$$|0\rangle$$

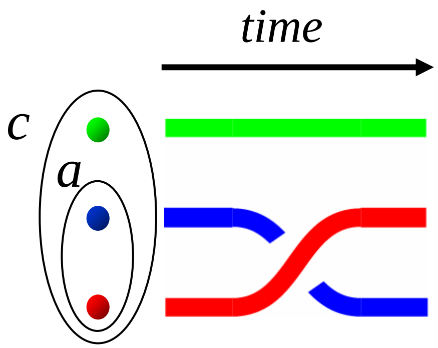
Measuring a Qubit

If they cannot fuse back into the "vacuum" the result of the measurement is 1.



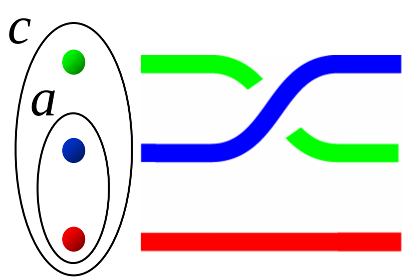
$$|1\rangle$$

Braiding Matrices for 3 Fibonacci Anyons



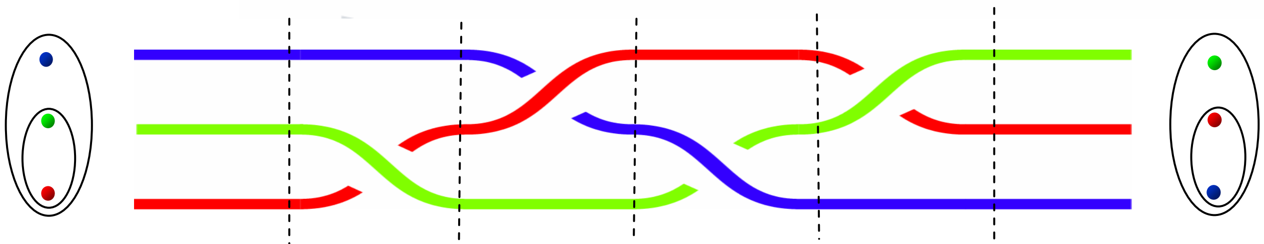
$c = 1 \qquad c = 0$

$$\sigma_1 = \left[\begin{array}{cc|c} e^{-i4\pi/5} & 0 & 0 \\ 0 & -e^{-i2\pi/5} & 0 \\ \hline 0 & 0 & -e^{-i2\pi/5} \end{array}\right]$$



$$\sigma_2 = \left[\begin{array}{cc|c} -\tau\, e^{-i\pi/5} & -i\sqrt{\tau}\, e^{-i\pi/10} & 0 \\ -i\sqrt{\tau}\, e^{-i\pi/10} & -\tau & 0 \\ \hline 0 & 0 & -e^{-i2\pi/5} \end{array}\right] \qquad \tau = \frac{\sqrt{5}-1}{2}$$

$|\Psi_i\rangle$



$|\Psi_f\rangle$

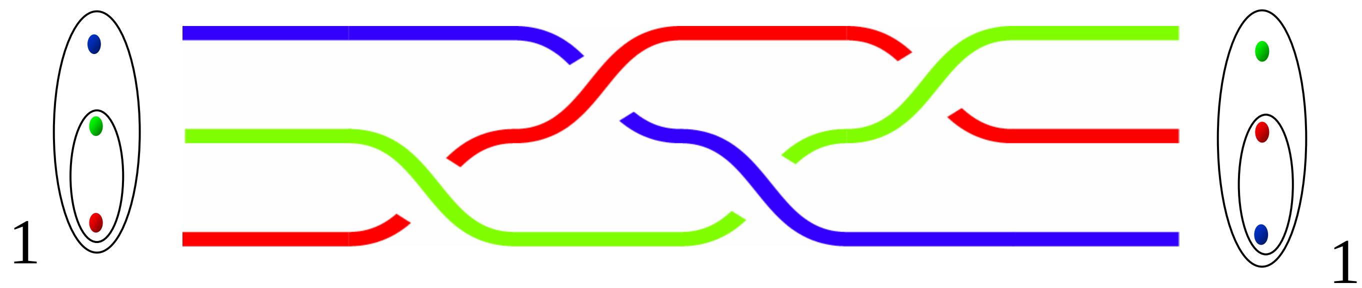
$$\sigma_1^{-1} \quad \sigma_2 \quad \sigma_1^{-1} \quad \sigma_2 = \boldsymbol{M}$$

$$\boxed{|\Psi_f\rangle = \boldsymbol{M}^{-1}\, |\Psi_i\rangle}$$

Single Qubit Operations

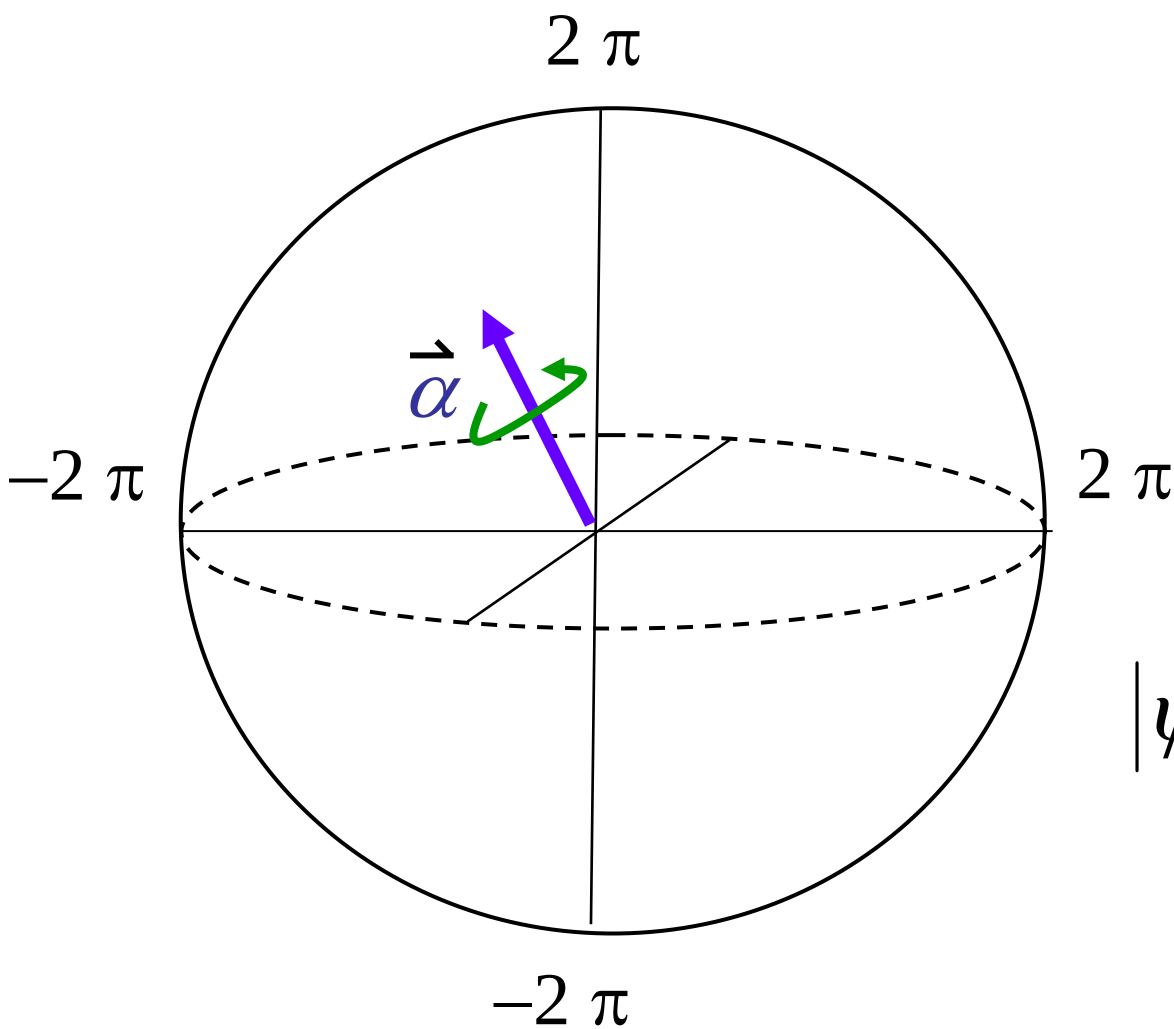
General rule: Braiding inside an oval does not change the total q-spin of the enclosed particles.

Important consequence: As long as we braid *within* a qubit, there is **no leakage error.**



Can we do arbitrary single qubit rotations this way?

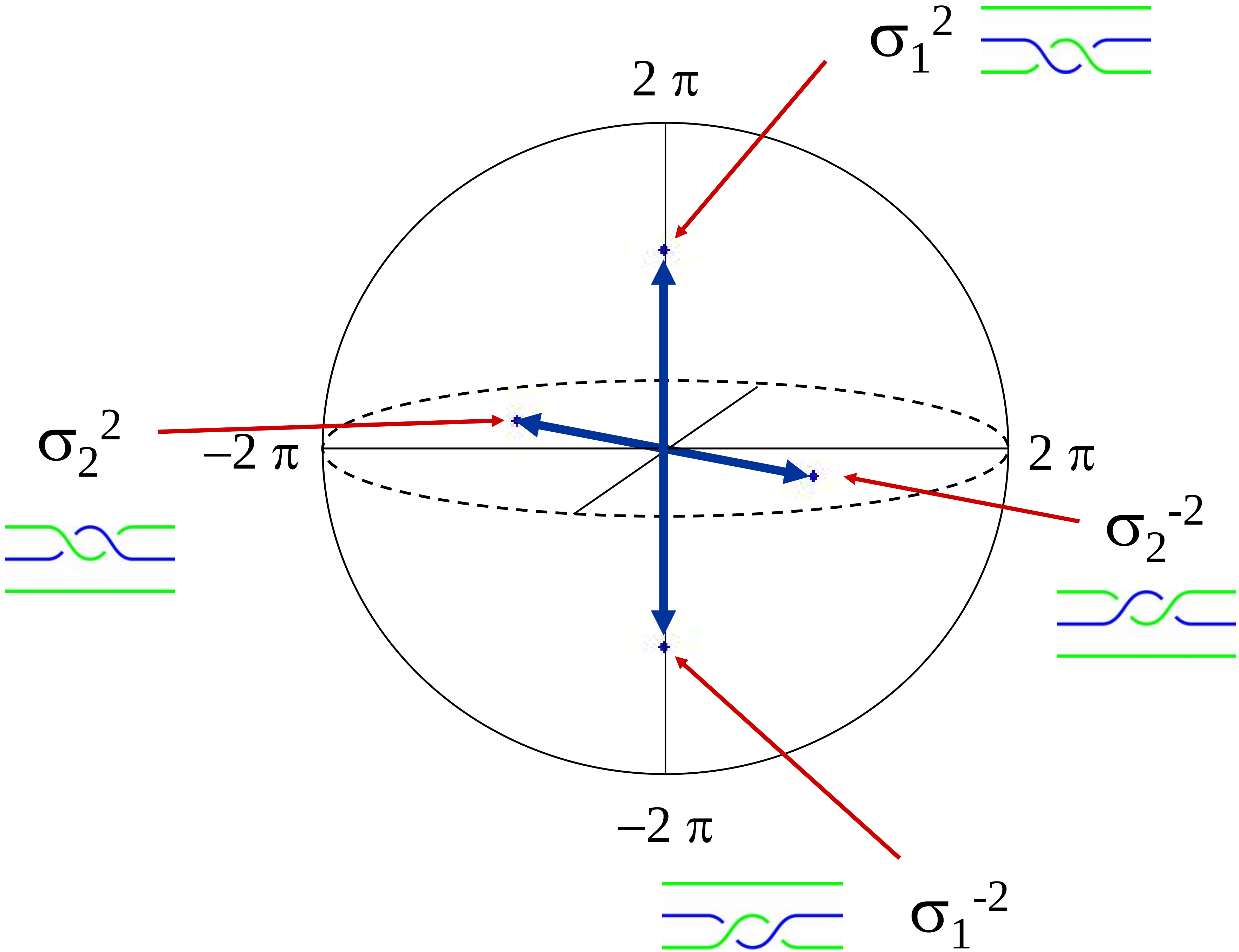
Single Qubit Operations are Rotations



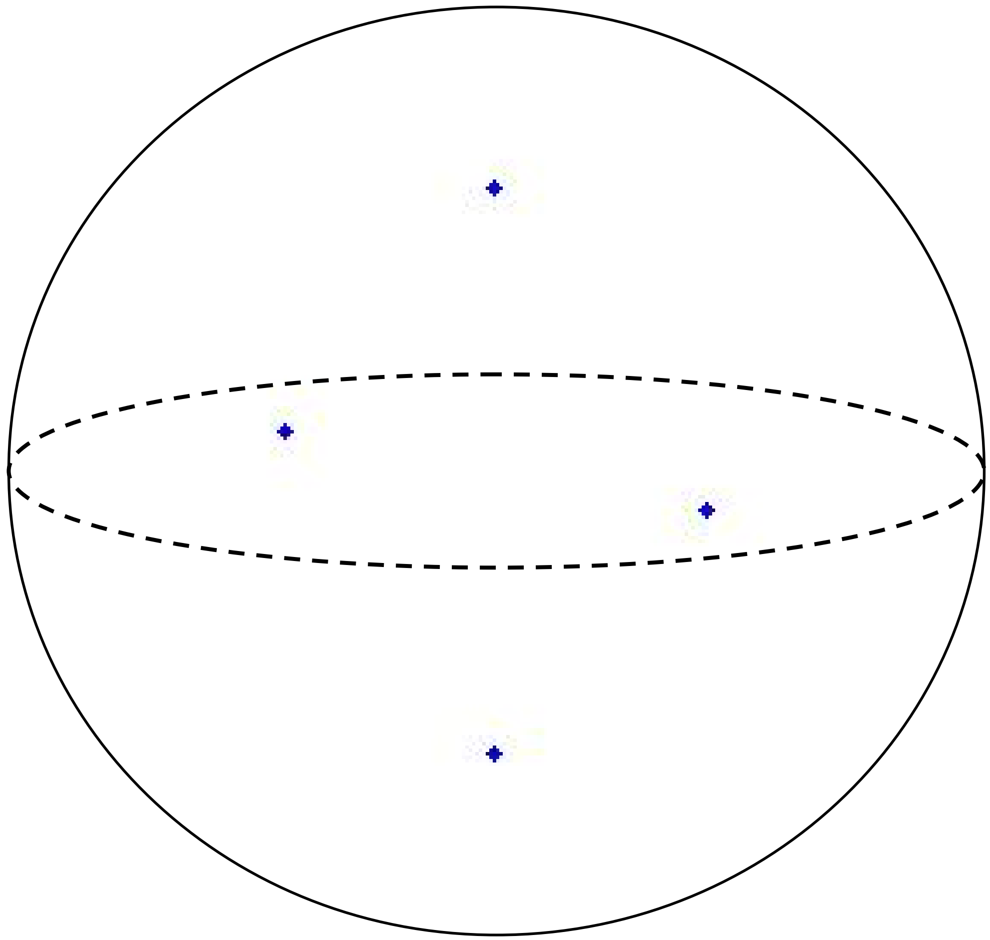
The set of all single qubit rotations lives in a solid sphere of radius 2π.

$$|\psi\rangle \;\text{—}\boxed{U_{\vec{\alpha}}}\text{—}\; U_{\vec{\alpha}}\,|\psi\rangle$$

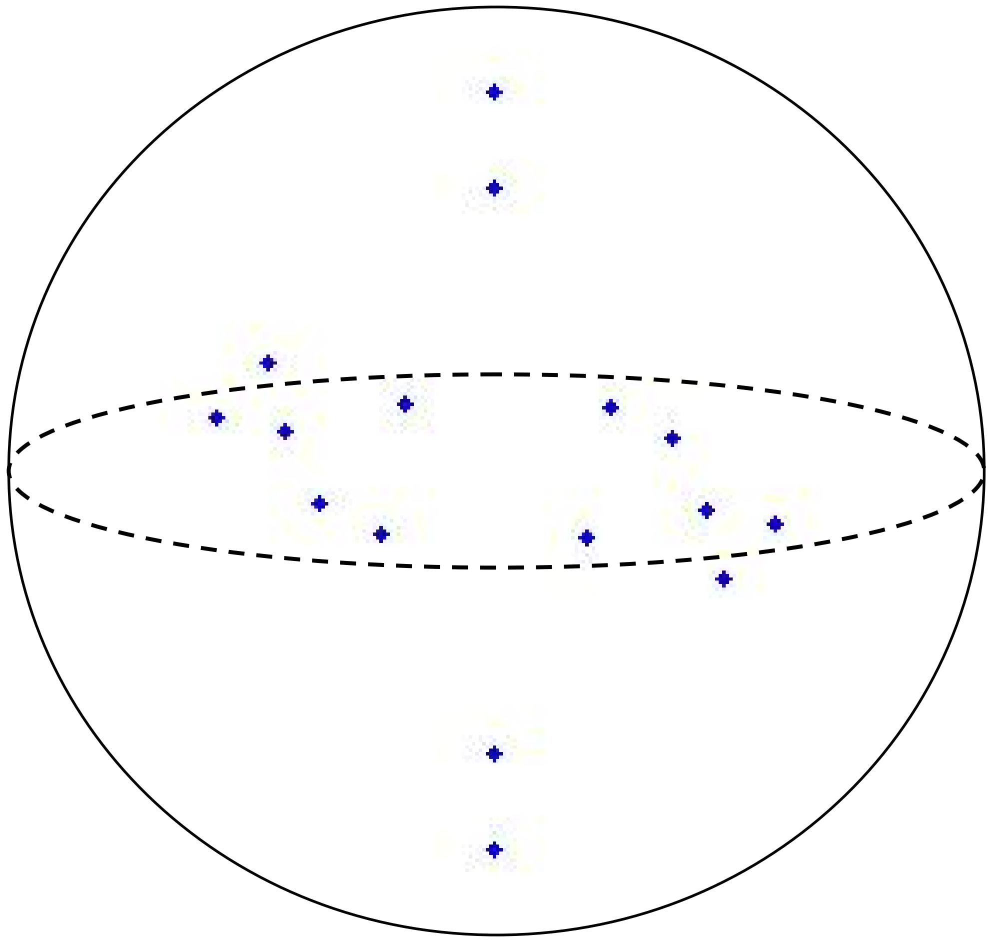
$$U_{\vec{\alpha}} = \exp \frac{i\,\vec{\alpha}\cdot\vec{\sigma}}{2}$$


σ_1^{2}
2π
σ_2^{2}
-2π
2π
σ_2^{-2}
-2π
σ_1^{-2}

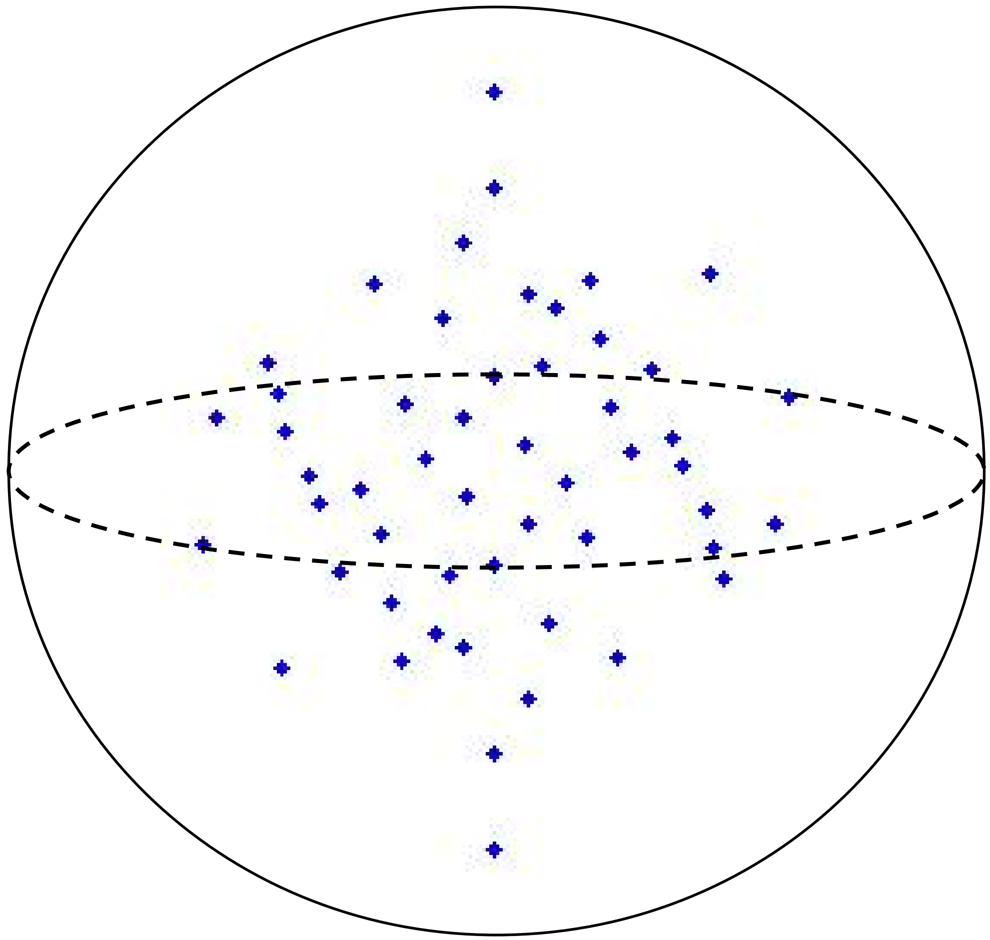
N = 1



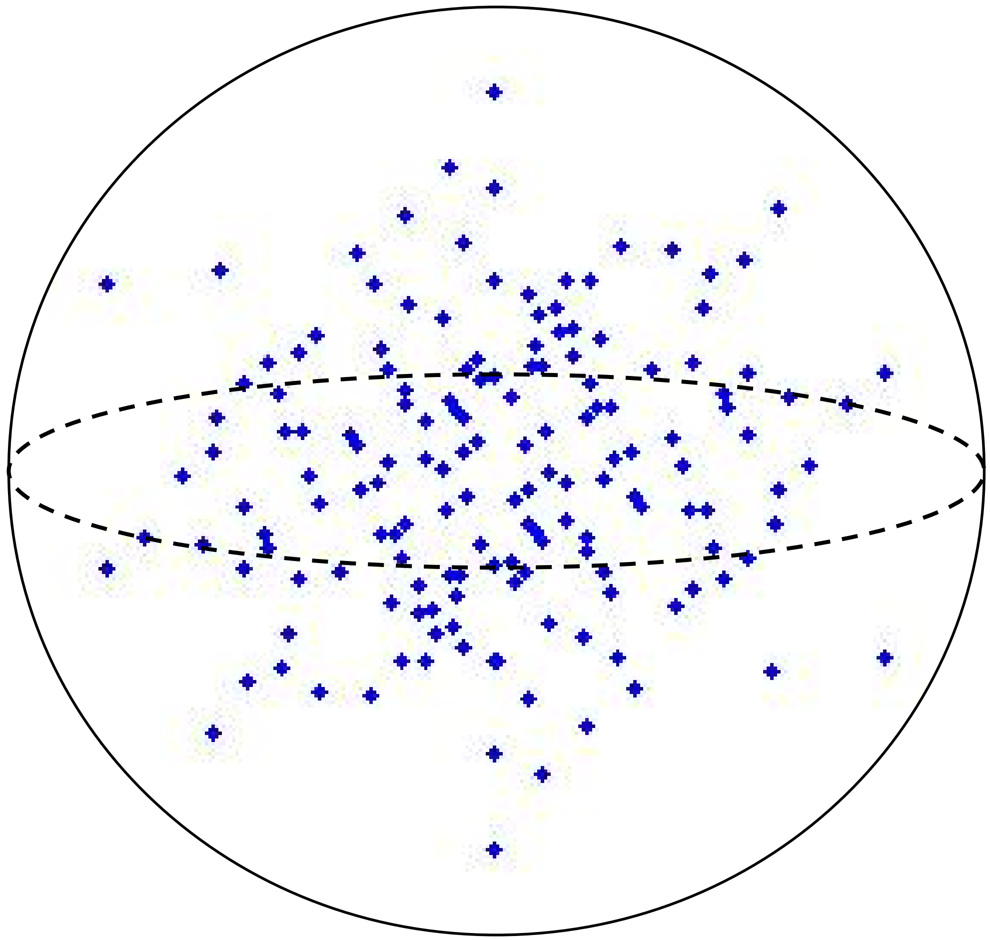
N = 2



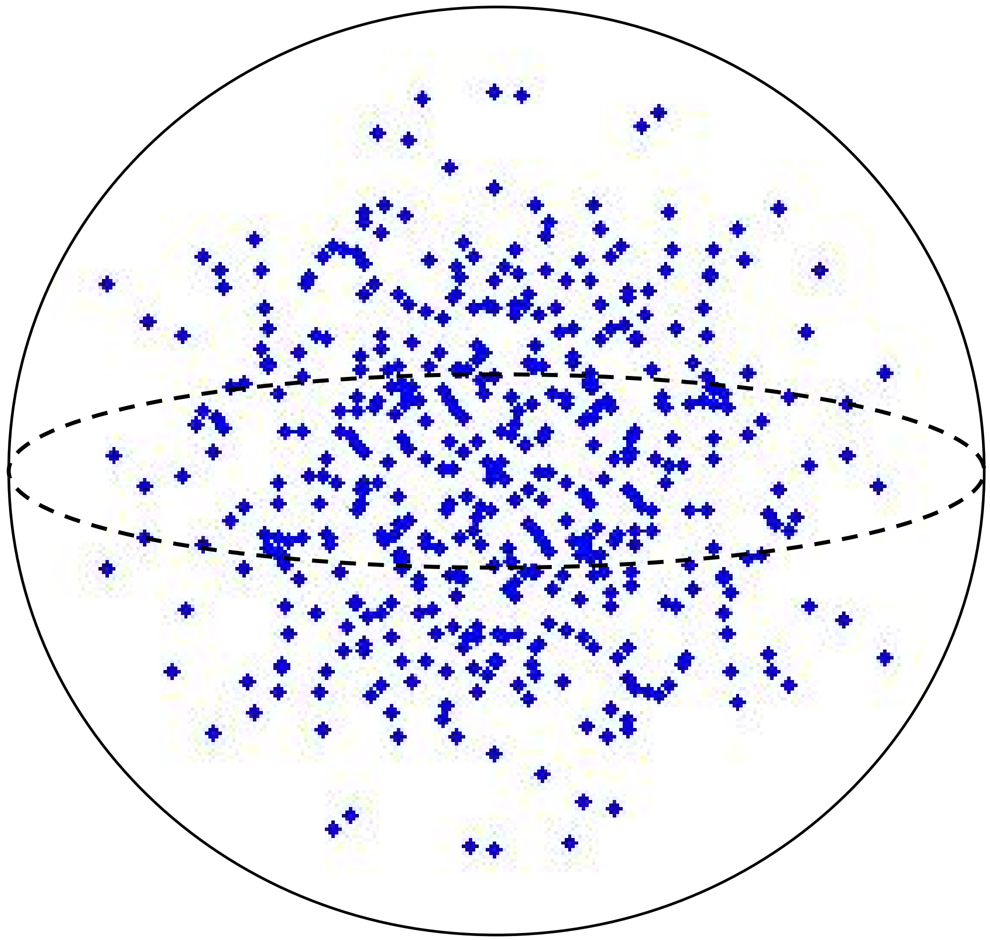
N = 3



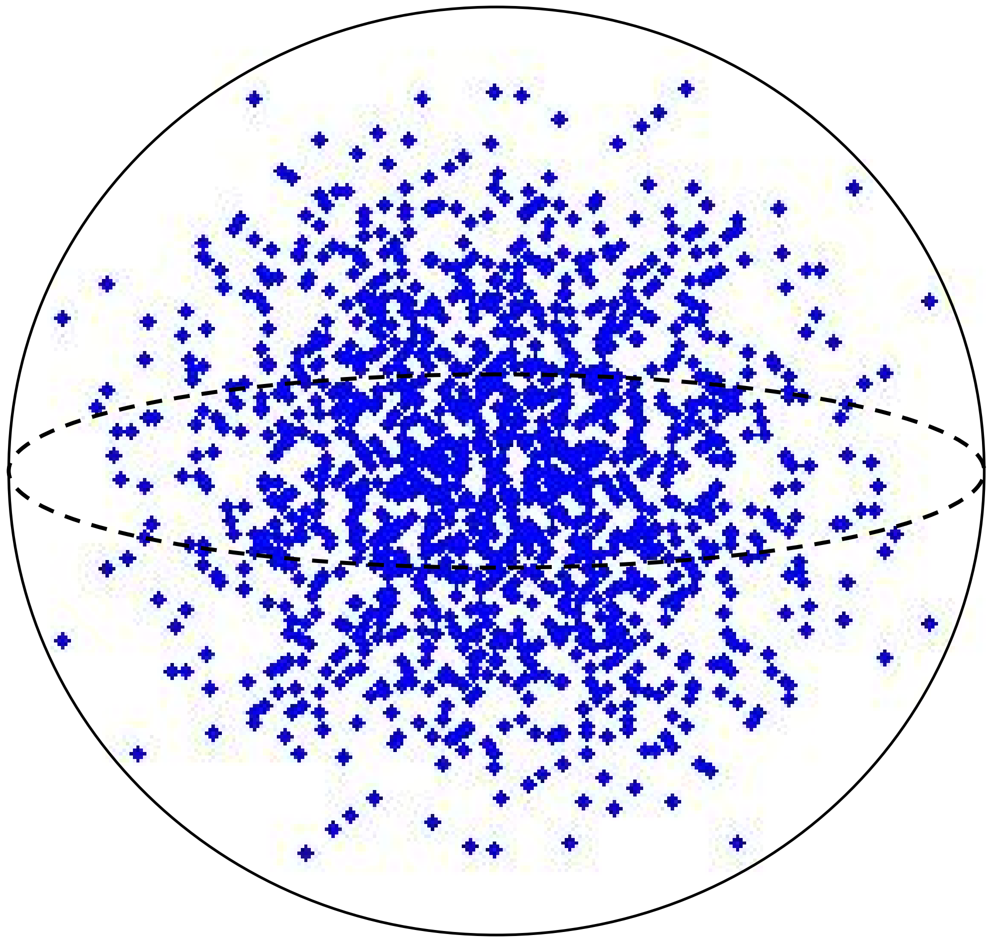
N = 4



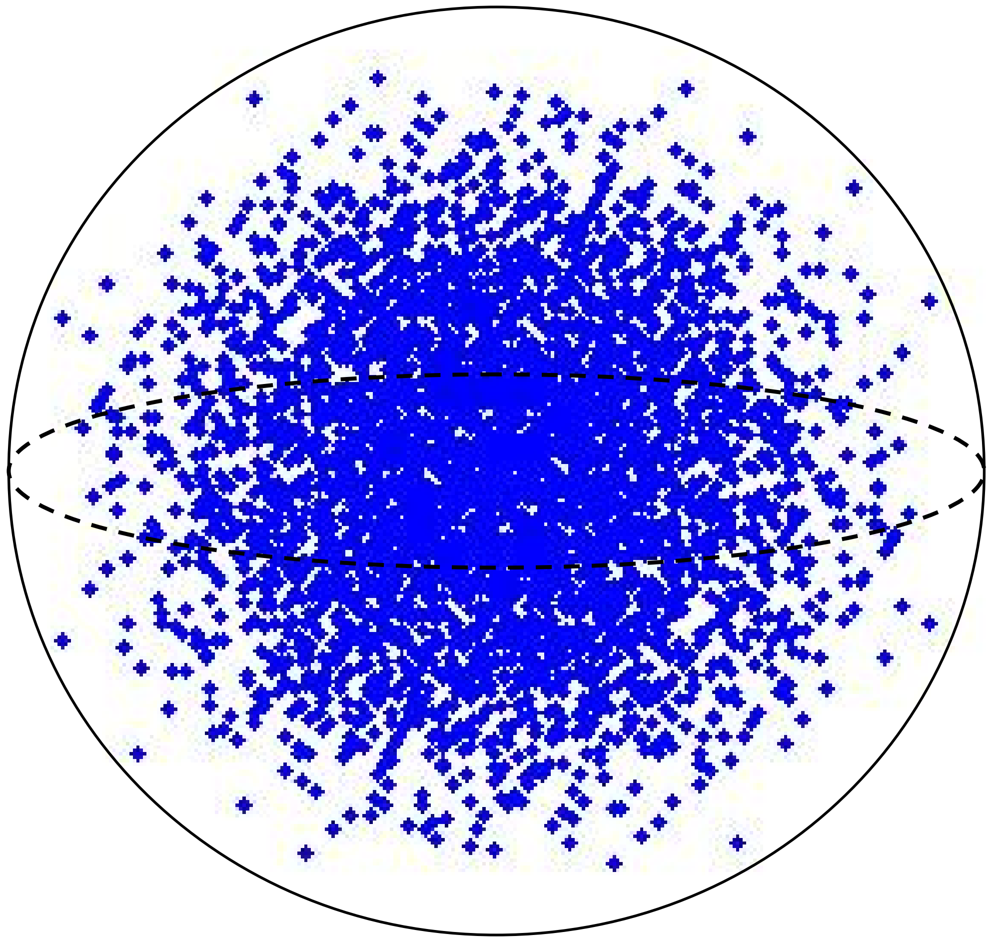
N = 5



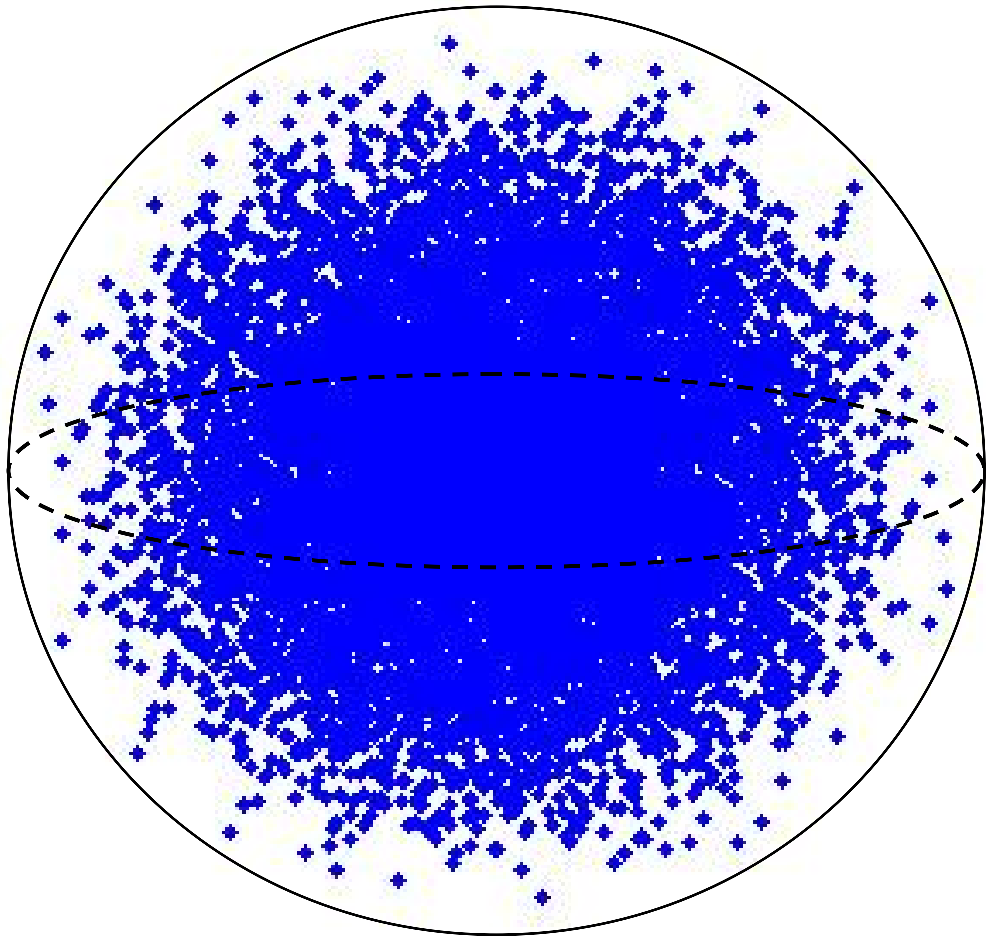
N = 6



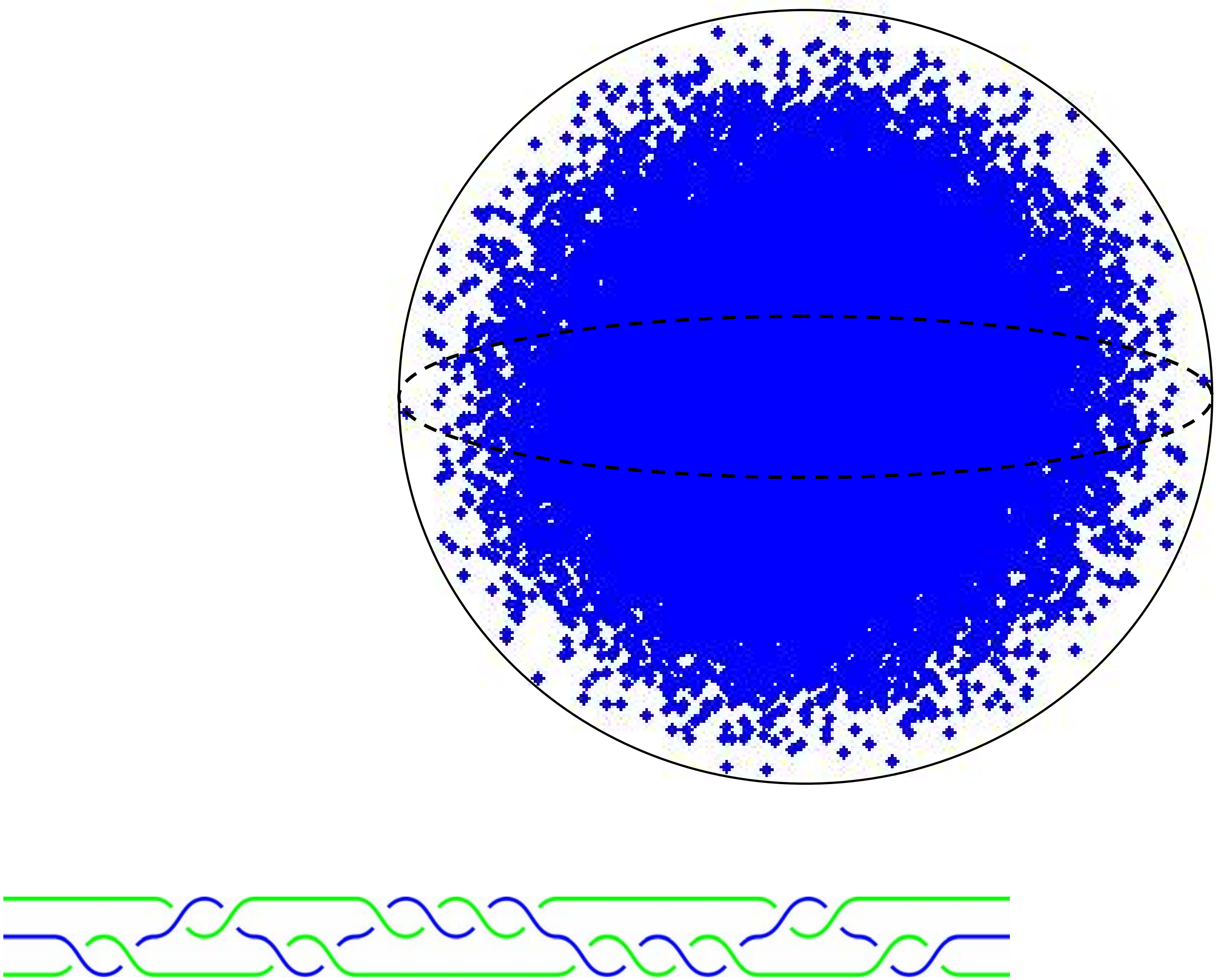
N = 7



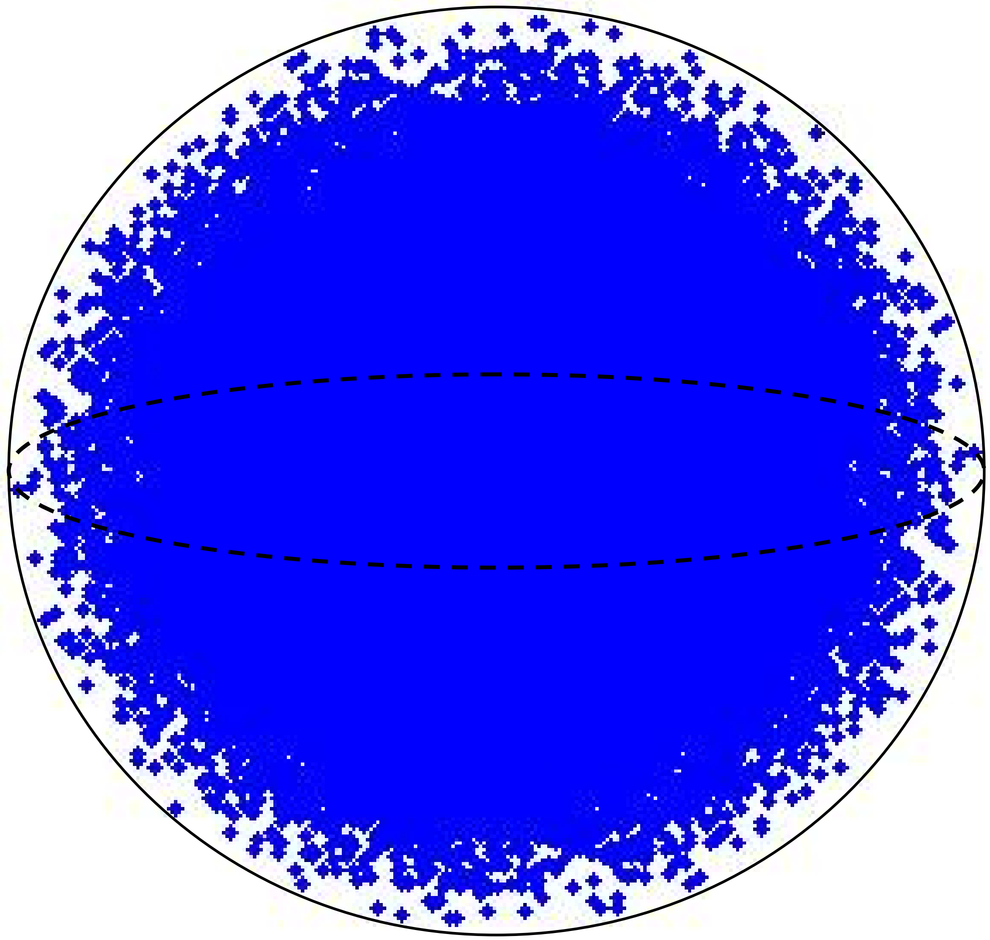
N = 8

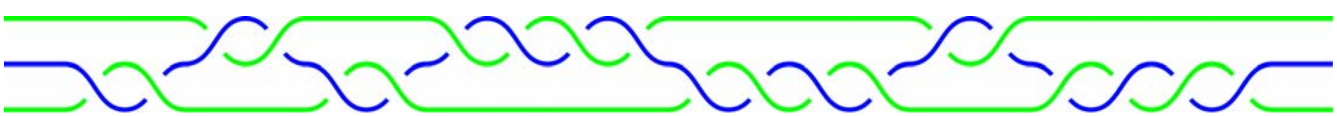


N = 9

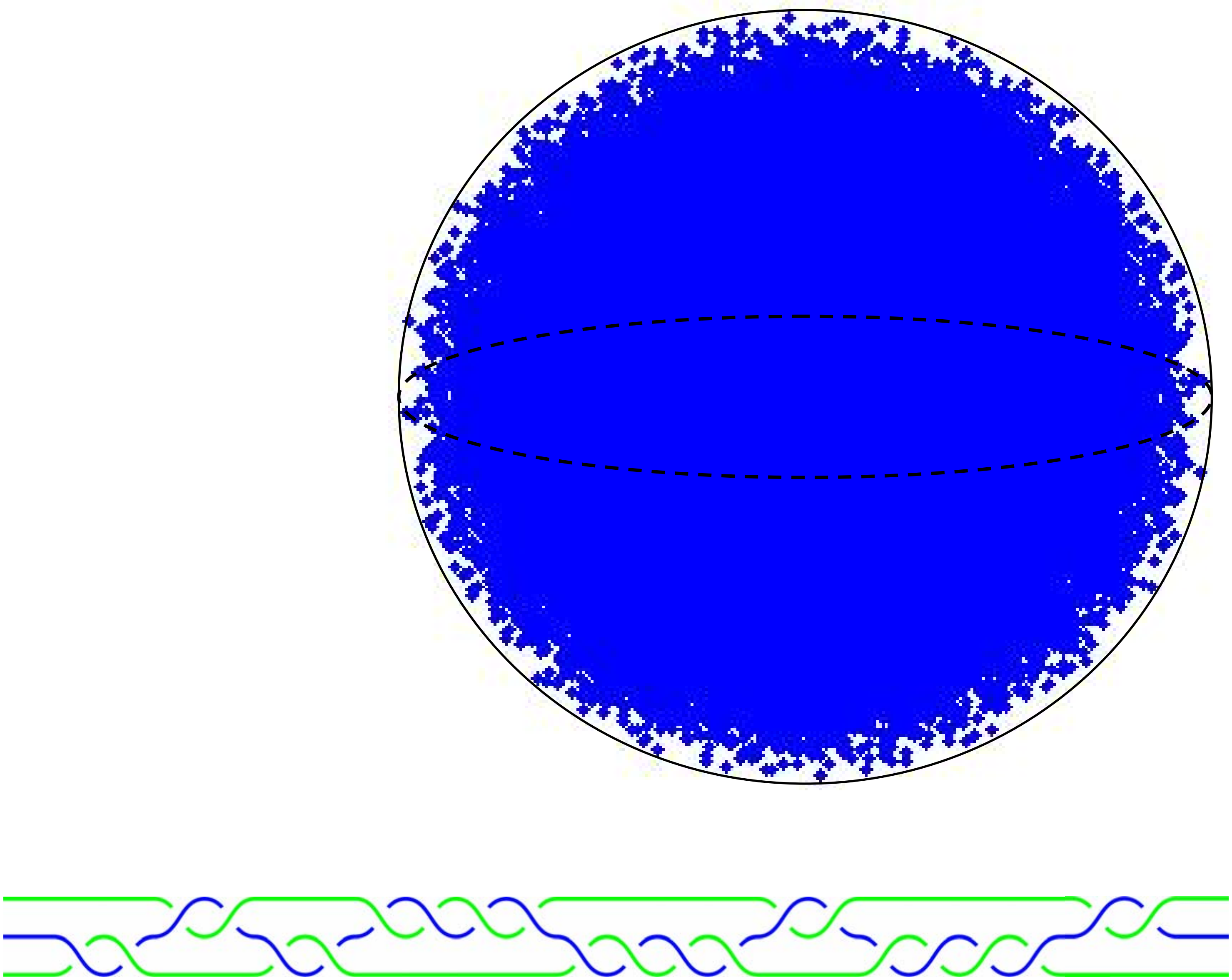


N = 10





N = 11



Brute Force Search

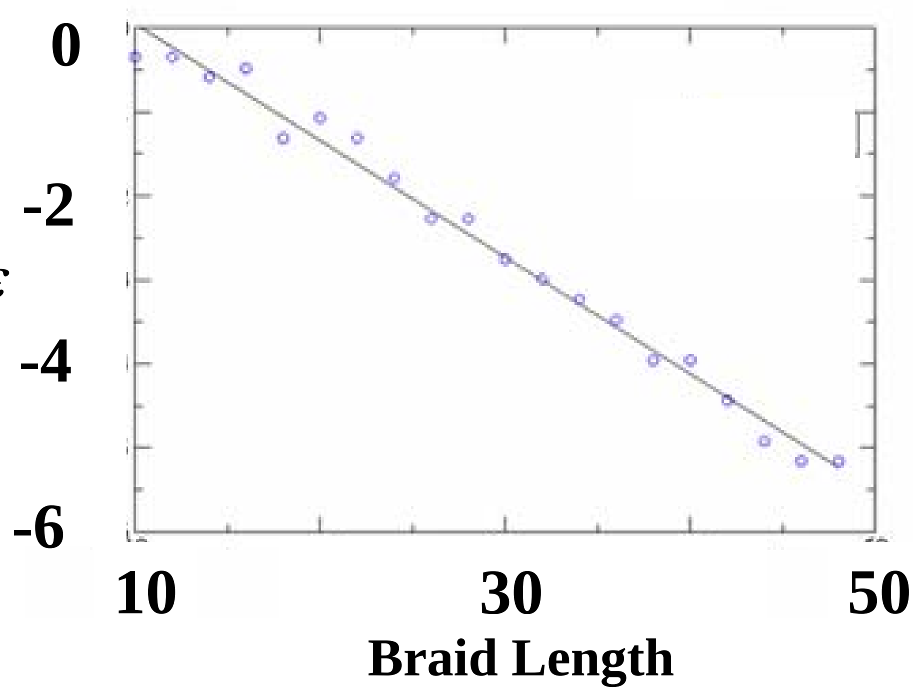
"error"

ε

$$\sigma_1^{-2}\sigma_2^{-4}\sigma_1^{4}\sigma_2^{-2}\sigma_1^{2}\sigma_2^{2}\sigma_1^{-2}\sigma_2^{4}\sigma_1^{-2}\sigma_2^{4}\sigma_1^{2}\sigma_2^{-4}\sigma_1^{2}\sigma_2^{-2}\sigma_1^{2}\sigma_2^{-2}\sigma_1^{-2} = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + O(10^{-3})$$

For brute force search:

$$\text{Braid Length} \sim |\ln \varepsilon|$$



Brute Force Search

$$\sigma_1^{-2}\sigma_2^{-4}\sigma_1^{4}\sigma_2^{-2}\sigma_1^{2}\sigma_2^{2}\sigma_1^{-2}\sigma_2^{4}\sigma_1^{-2}\sigma_2^{4}\sigma_1^{2}\sigma_2^{-4}\sigma_1^{2}\sigma_2^{-2}\sigma_1^{2}\sigma_2^{-2}\sigma_1^{-2} = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + O(10^{-3})$$

Brute force searching rapidly becomes infeasible as braids get longer.

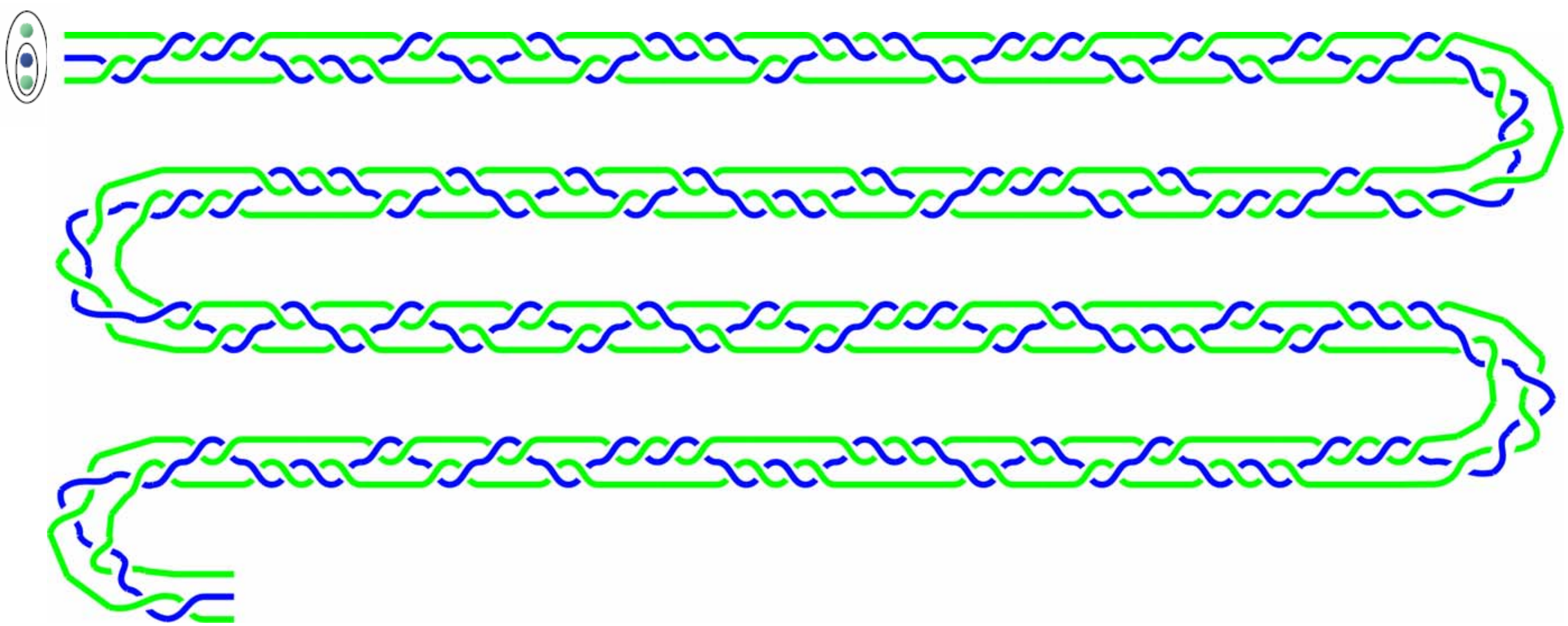
Fortunately, a clever algorithm due to Solovay and Kitaev allows for systematic improvement of the braid given a sufficiently dense covering of *SU(2)*.

Solovay-Kitaev Construction

ε

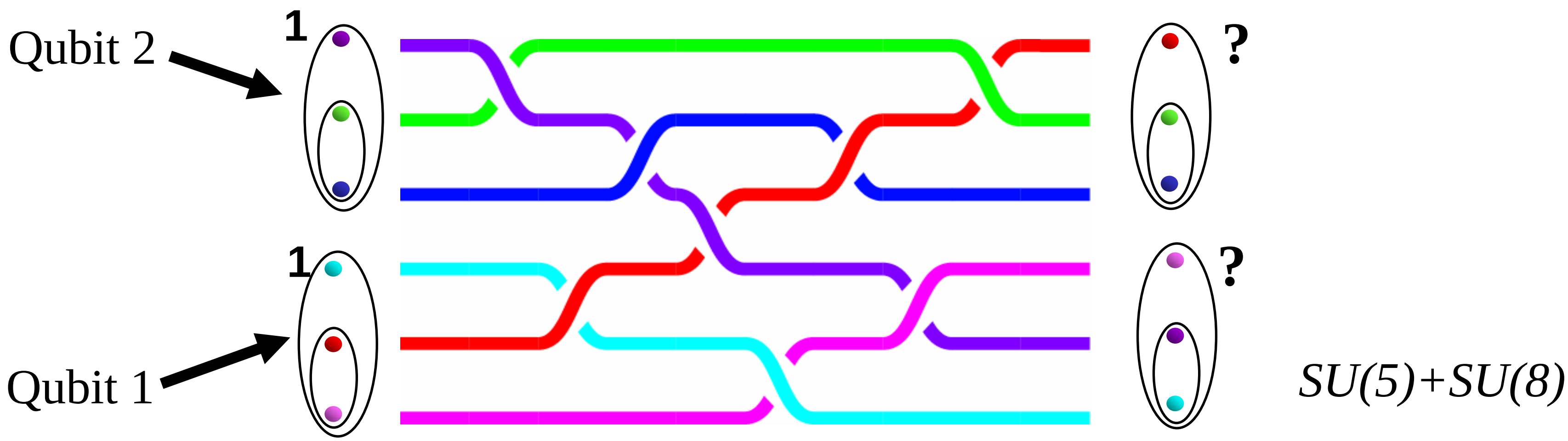
(Actual calculation)

$$\left(\begin{array}{cc|c} 0 & i & 0 \\ i & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) + O(10^{-4})$$



$$\text{Braid Length} \sim |\ln \varepsilon|^{c}, \qquad c \approx 4$$

What About Two Qubit Gates?

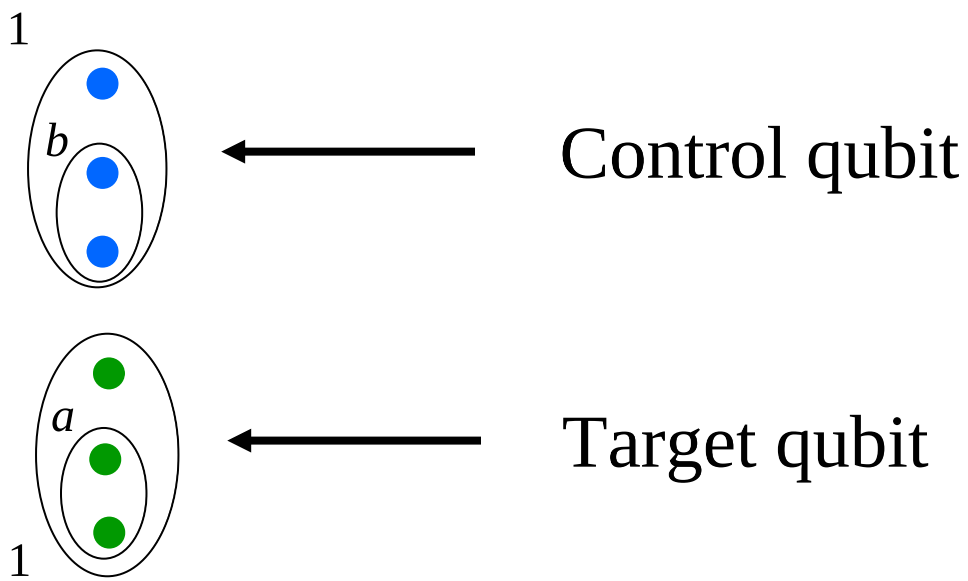


Problems:

1. We are pulling quasiparticles out of qubits: **Leakage error!**

2. **87 dimensional** search space (as opposed to **3** for three-particle braids). Straightforward "brute force" search is problematic.

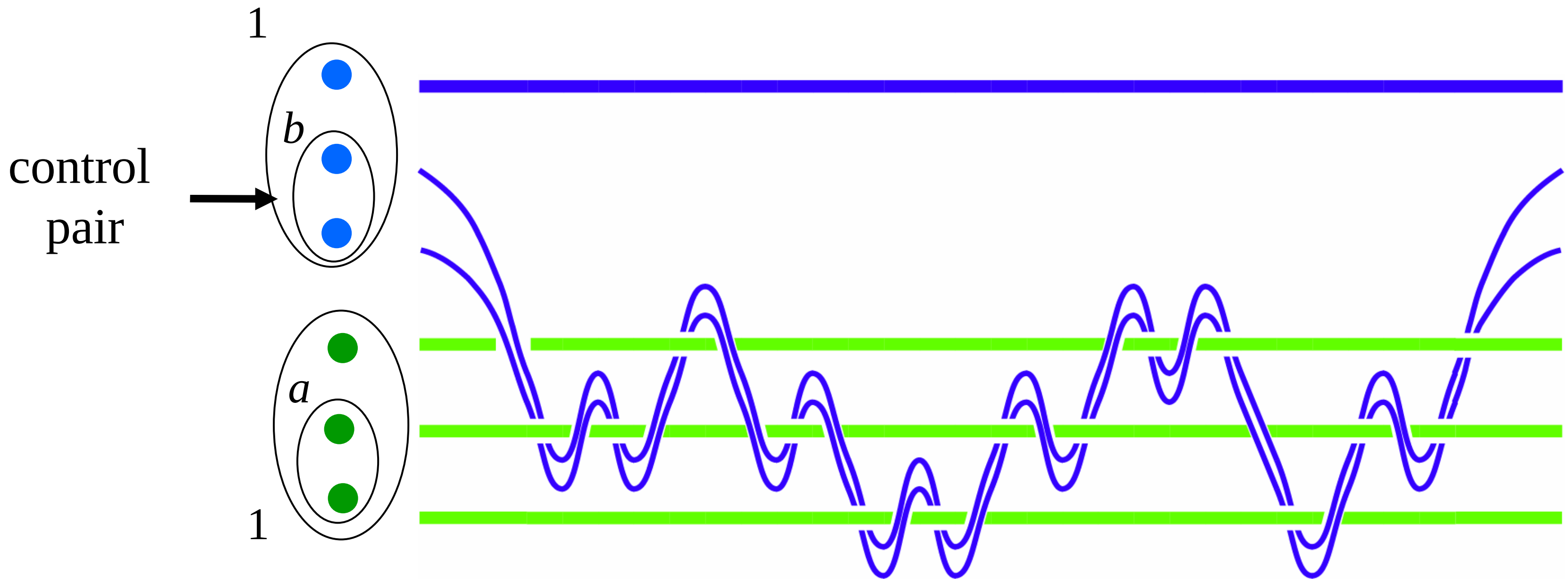
Two Qubit Controlled Gates



Goal: Find a braid in which some rotation is performed on the target qubit only if the control qubit is in the state 1. (***b*=*1***)

Constructing Two Qubit Gates by "Weaving"

Weave a *pair* of anyons from the control qubit between anyons in the target qubit.



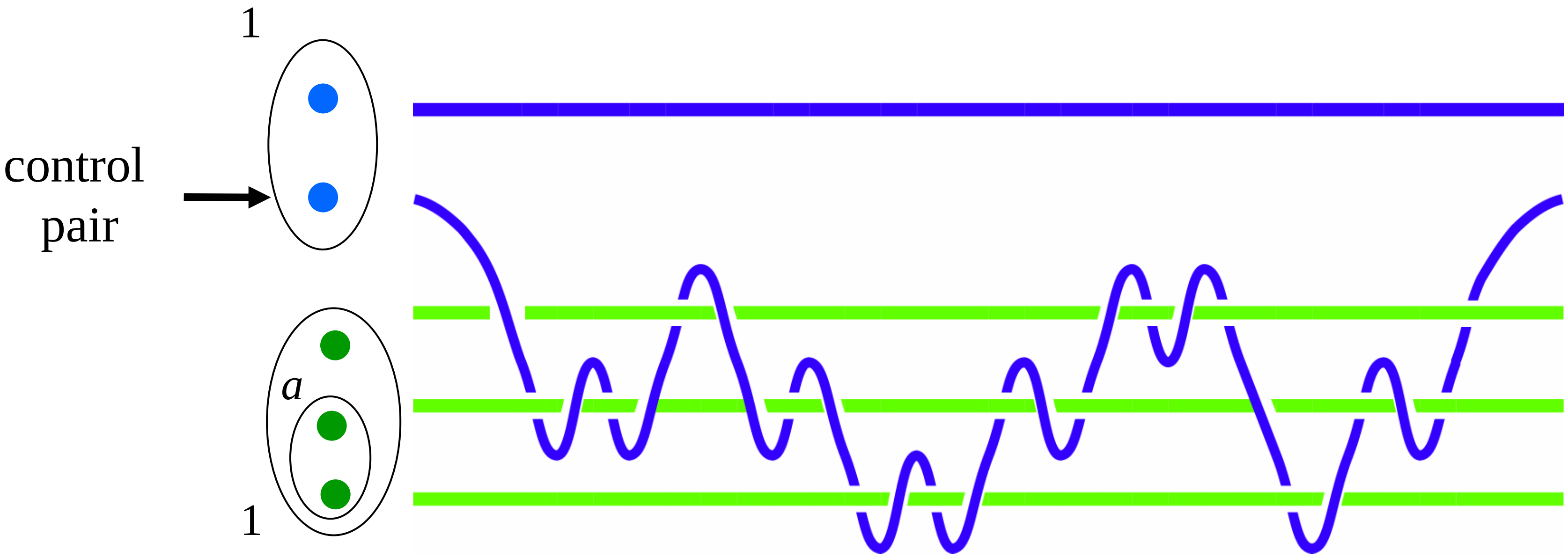
Important Rule: Braiding a q-spin 0 object does not induce transitions.

→ Target qubit is only affected if control qubit is in state $|1\rangle$

(b = 1)

Constructing Two Qubit Gates by “Weaving”

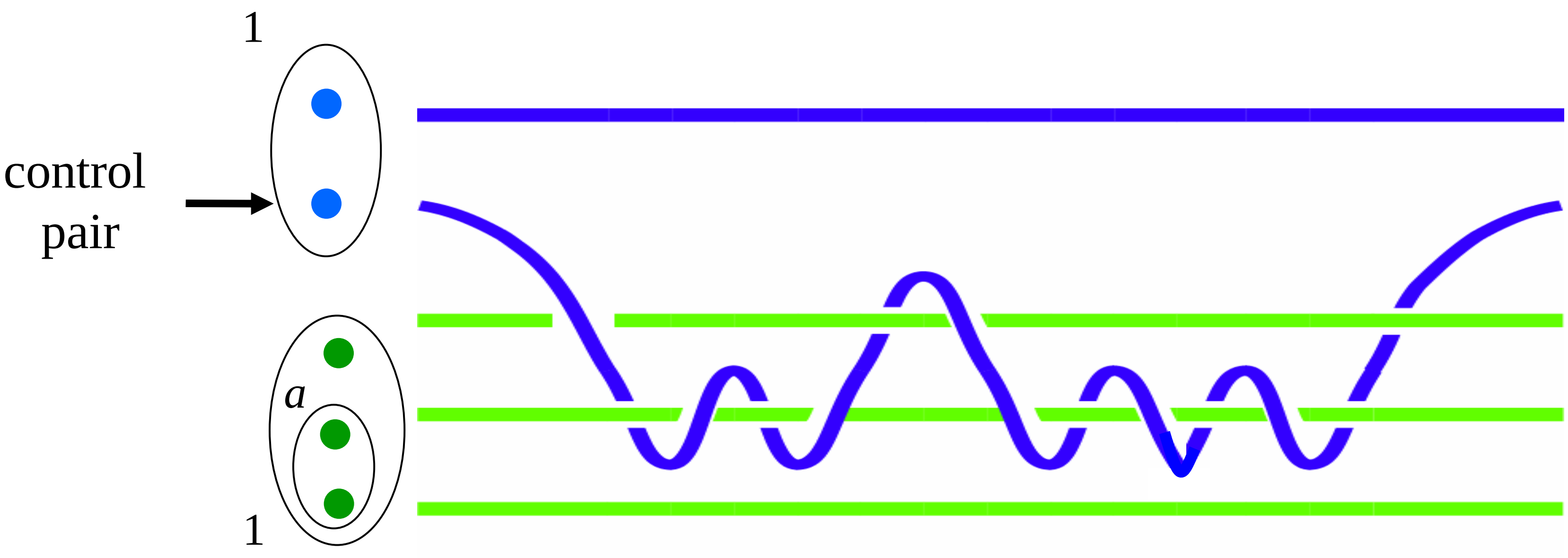
Only nontrivial case is when the control pair has q-spin 1.



We’ve reduced the problem to weaving one anyon around three others.   **Still too hard for brute force approach!**

OK, Try Weaving Through Only Two Particles

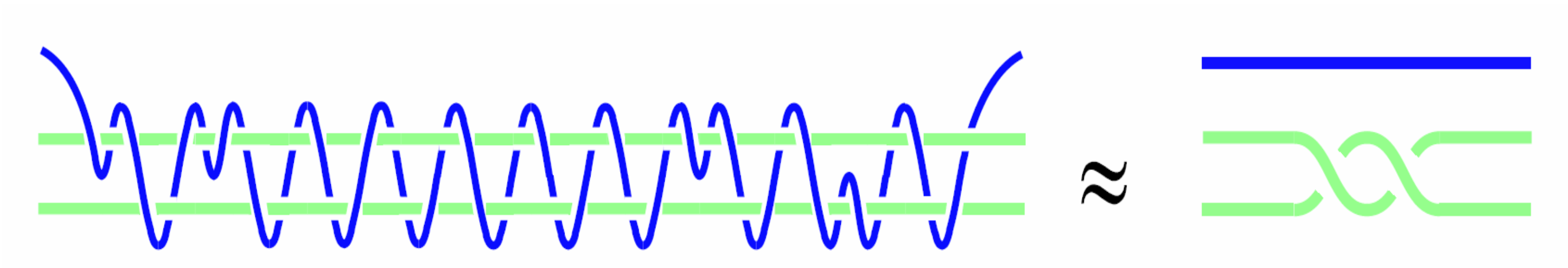
We're back to *SU(2)*, so this is numerically feasible.



Question: Can we find a weave which does not lead to **leakage errors**?

A Trick: Effective Braiding

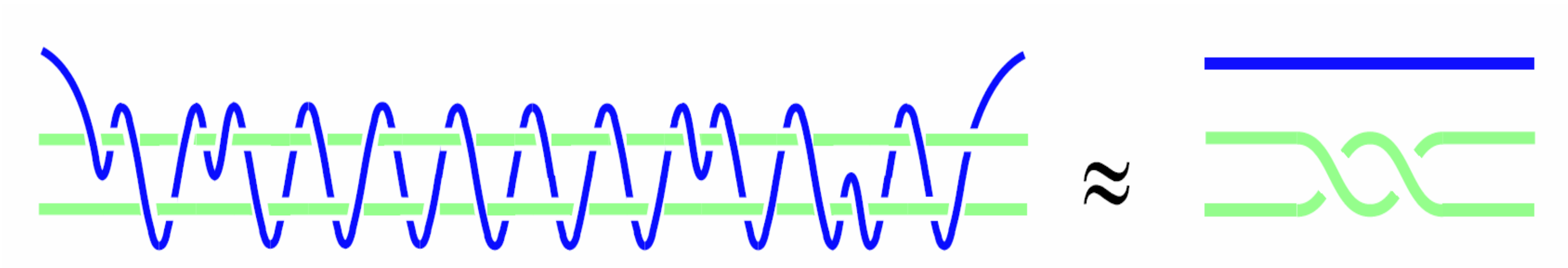
Actual Weaving — *Effective* Braiding

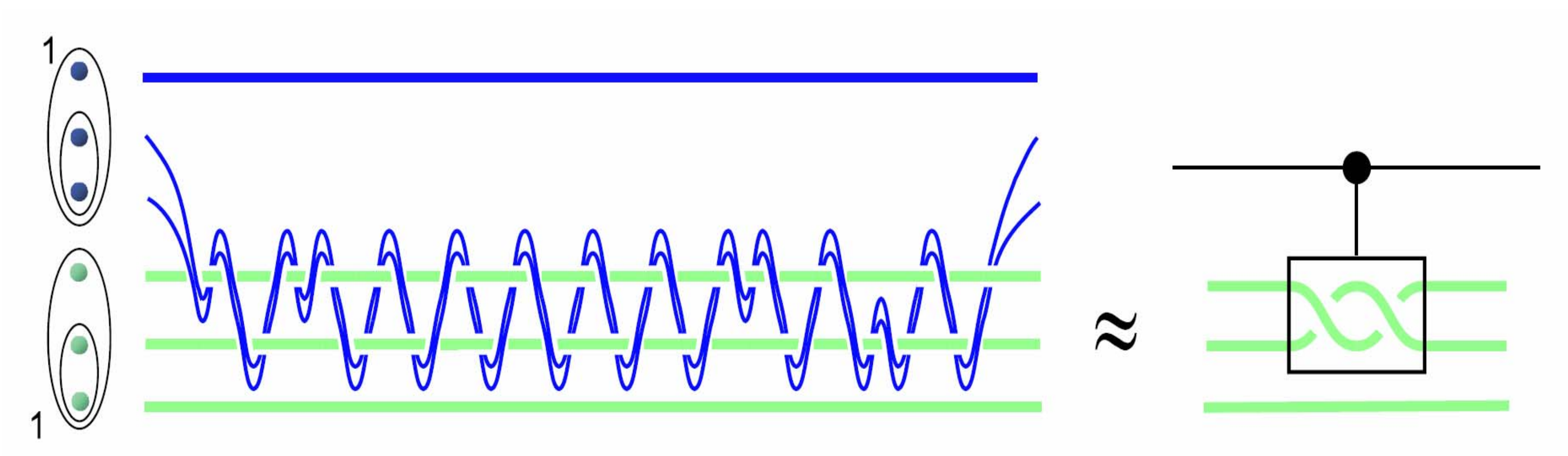


$$\sigma_2^{3}\sigma_1^{2}\sigma_2^{-4}\sigma_1^{2}\sigma_2^{2}\sigma_1^{-2}\sigma_2^{-2}\sigma_1^{-2}\sigma_2^{2}\sigma_1^{2}\sigma_2^{2}\sigma_1^{-2}\sigma_2^{2}\sigma_1^{-2}\sigma_2^{4}\sigma_1^{-2}\sigma_2^{2}\sigma_1^{4}\sigma_2^{2}\sigma_1^{-2}\sigma_2 \approx \sigma_1^{2}$$

The effect of weaving the **blue anyon** through the two **green anyons** has approximately the same effect as braiding the two **green anyons** twice.

Controlled–"Knot" Gate

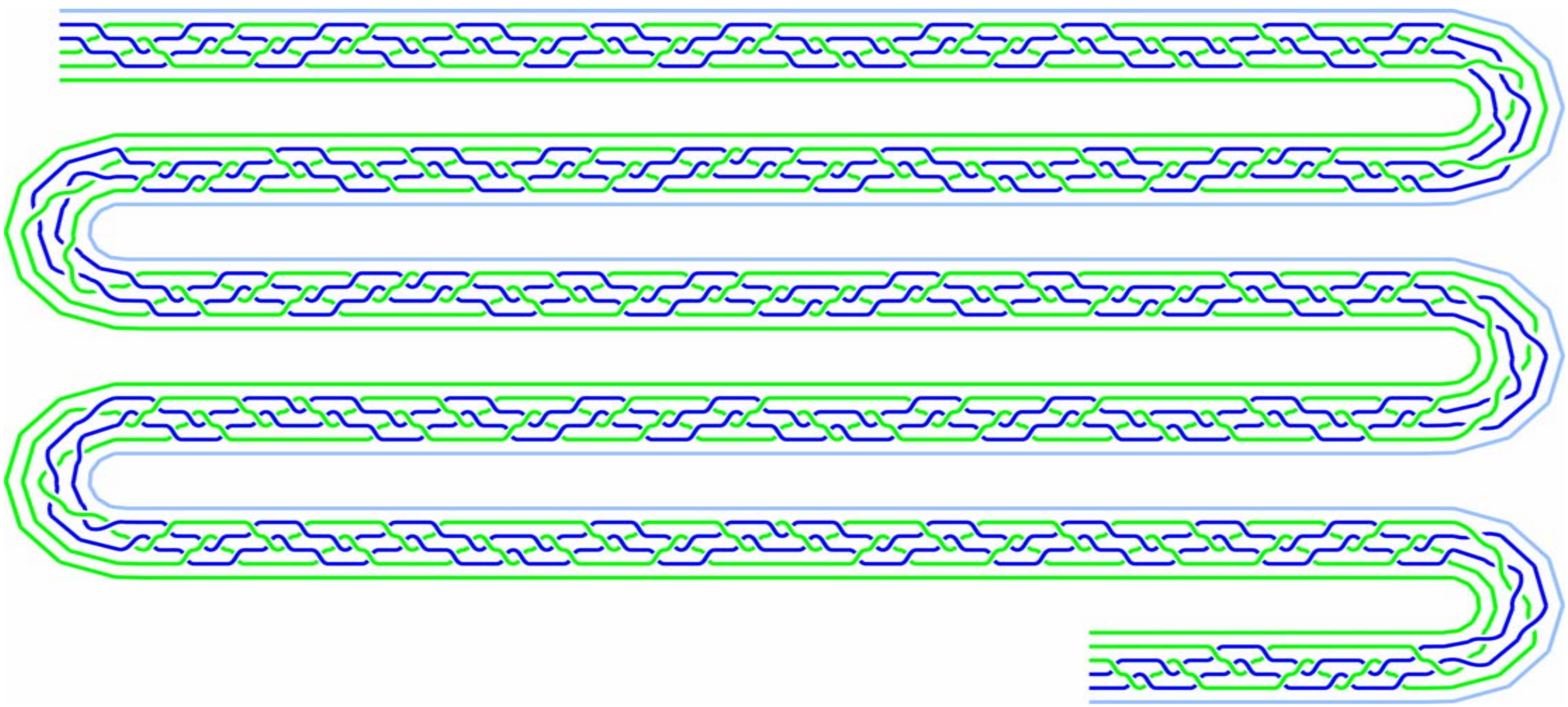




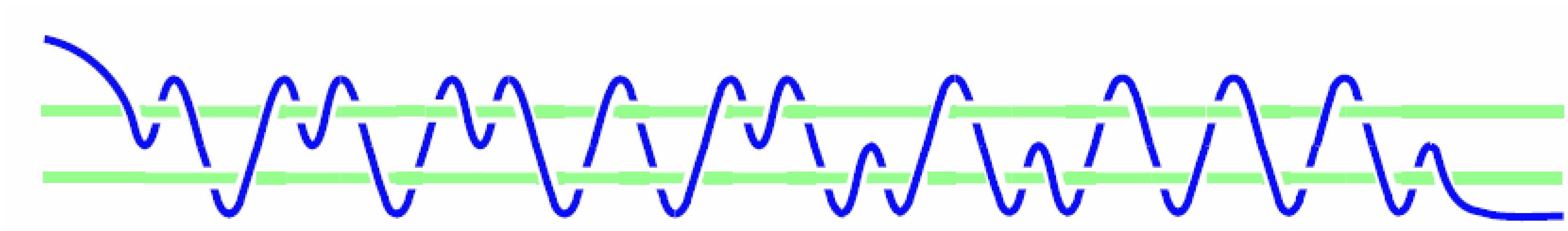
Effective braiding is all within the target qubit → No leakage!

Not a CNOT, but sufficient for universal quantum computation.

Solovay-Kitaev Improved Controlled–“Knot” Gate



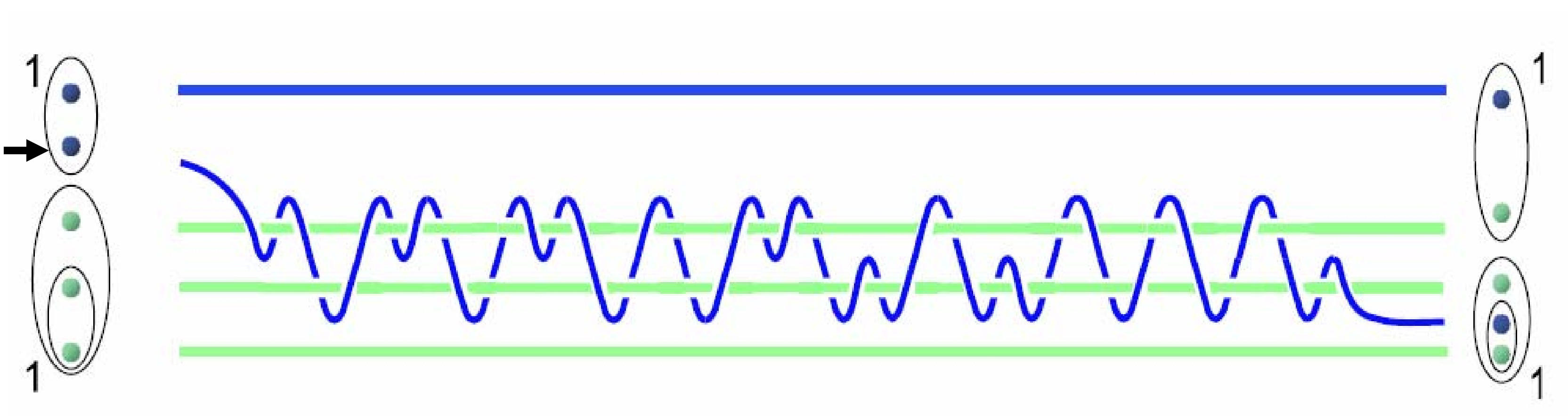
Another Trick: Injection Weaving

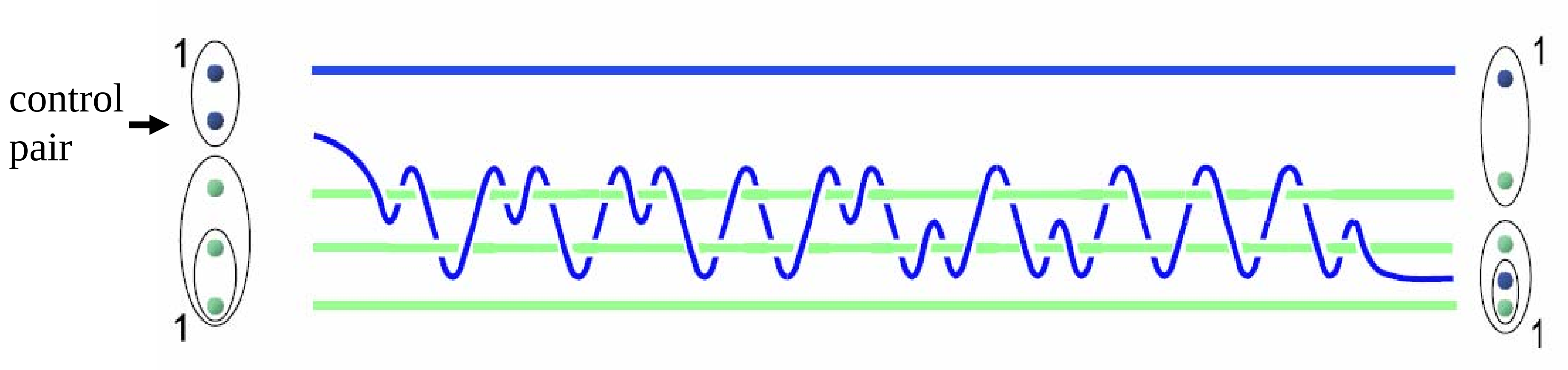


$$\sigma_2^{3}\sigma_1^{-2}\sigma_2^{-4}\sigma_1^{2}\sigma_2^{4}\sigma_1^{2}\sigma_2^{-2}\sigma_1^{-2}\sigma_2^{-4}\sigma_1^{-4}\sigma_2^{-2}\sigma_1^{4}\sigma_2^{2}\sigma_1^{-2}\sigma_2^{2}\sigma_1^{2}\sigma_2^{-2}\sigma_1^{3} \approx \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

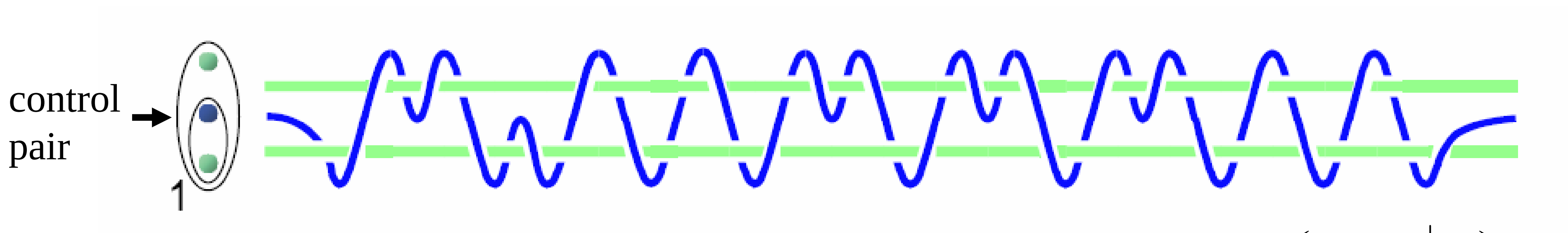
Step 1: Inject the control pair into the target qubit.

control pair



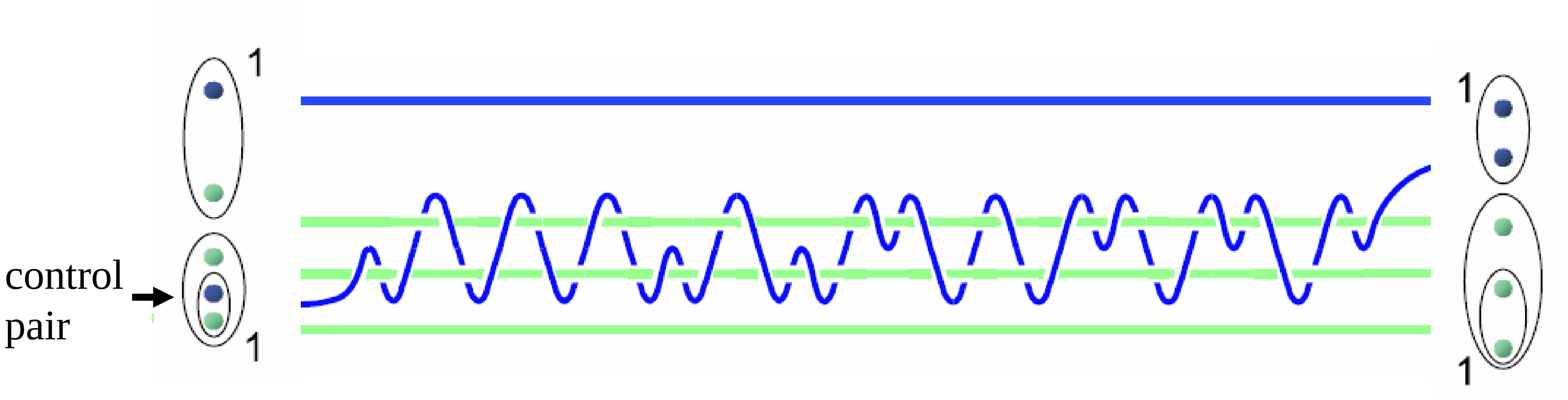


Step 2: Weave the control pair inside the injected target qubit.

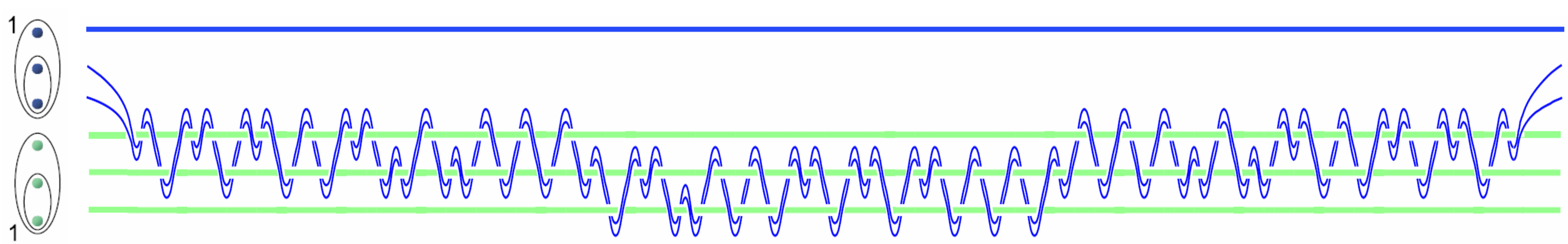


$$\sigma_1^{-2}\sigma_2^{-4}\sigma_1^{4}\sigma_2^{-2}\sigma_1^{2}\sigma_2^{2}\sigma_1^{-2}\sigma_2^{4}\sigma_1^{-2}\sigma_2^{4}\sigma_1^{2}\sigma_2^{-4}\sigma_1^{2}\sigma_2^{-2}\sigma_1^{2}\sigma_2^{-2}\sigma_1^{-2} \approx \left(\begin{array}{cc|c} 0 & i & 0 \\ i & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

Step 3: Extract the control pair from the target using the inverse of the injection weave.

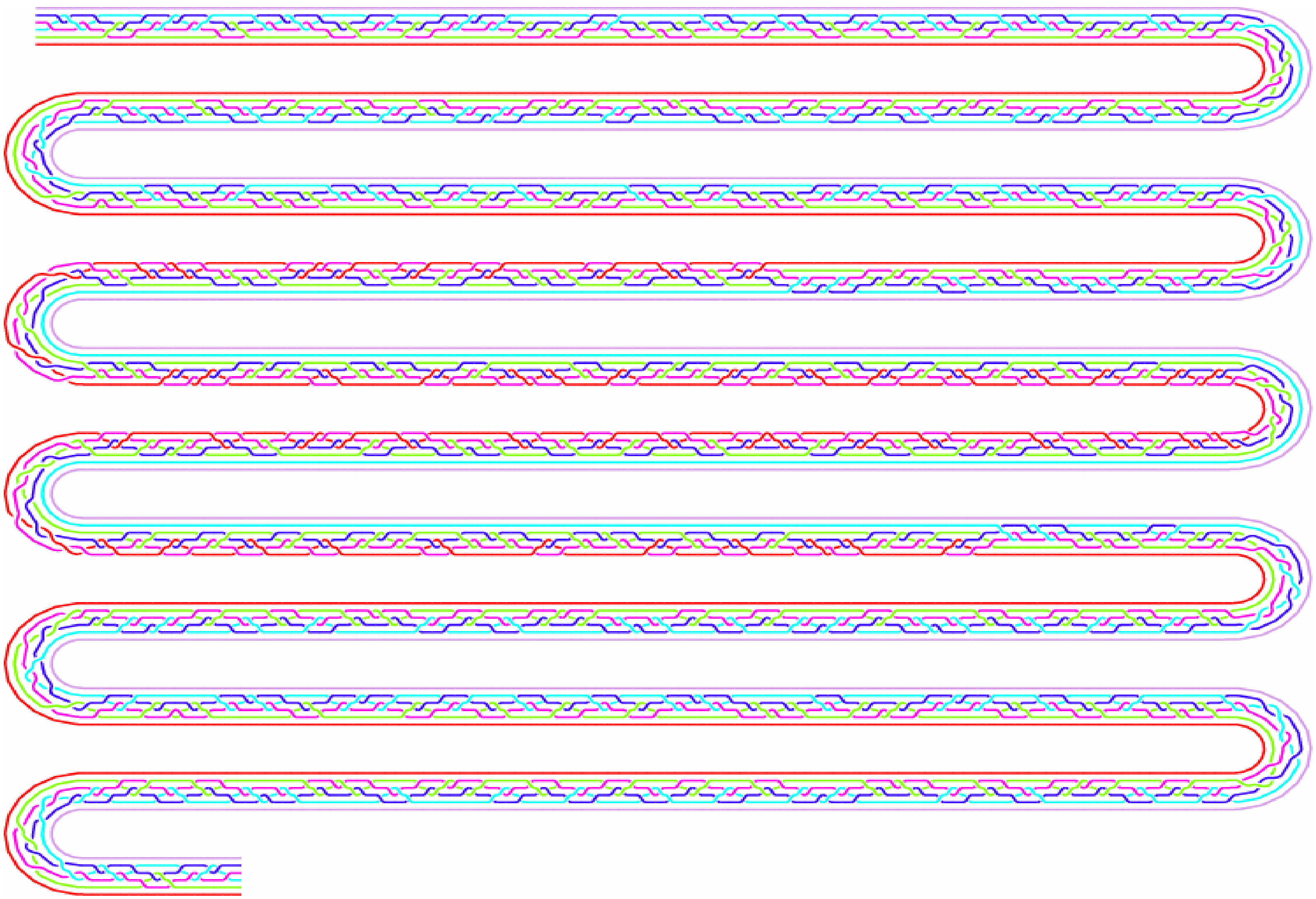


Putting it all together we have a CNOT gate:



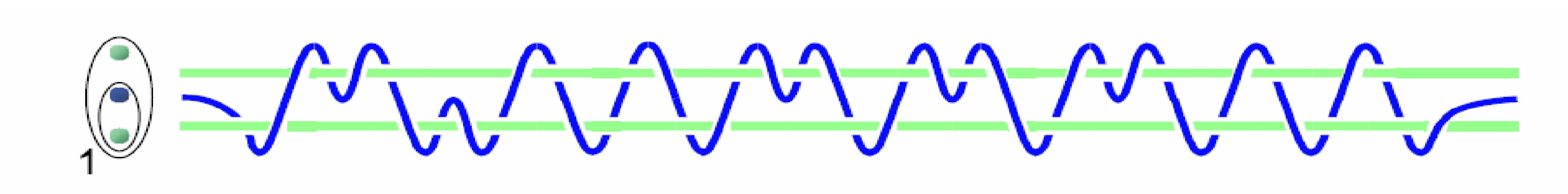
Injection | Rotation | Extraction

Solovay-Kitaev Improved CNOT

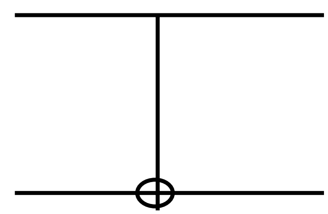


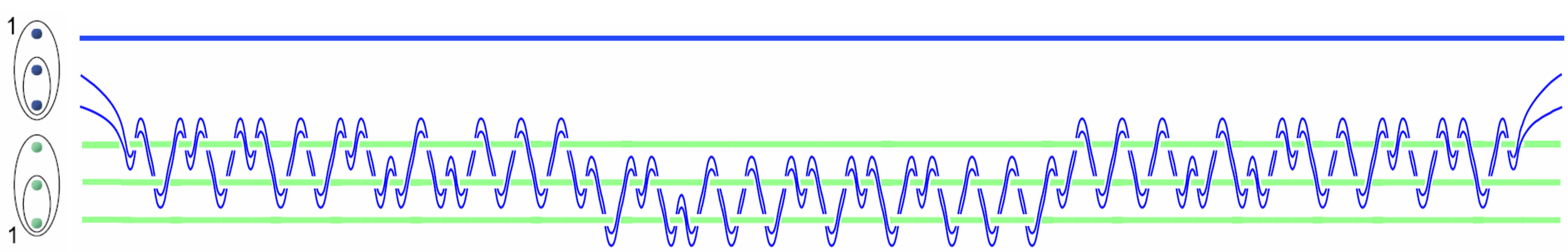
Universal Set of Fault Tolerant Gates

Single qubit rotations: $|\psi\rangle \; - \boxed{U_{\vec{\phi}}} - \; U_{\vec{\phi}}\,|\psi\rangle$

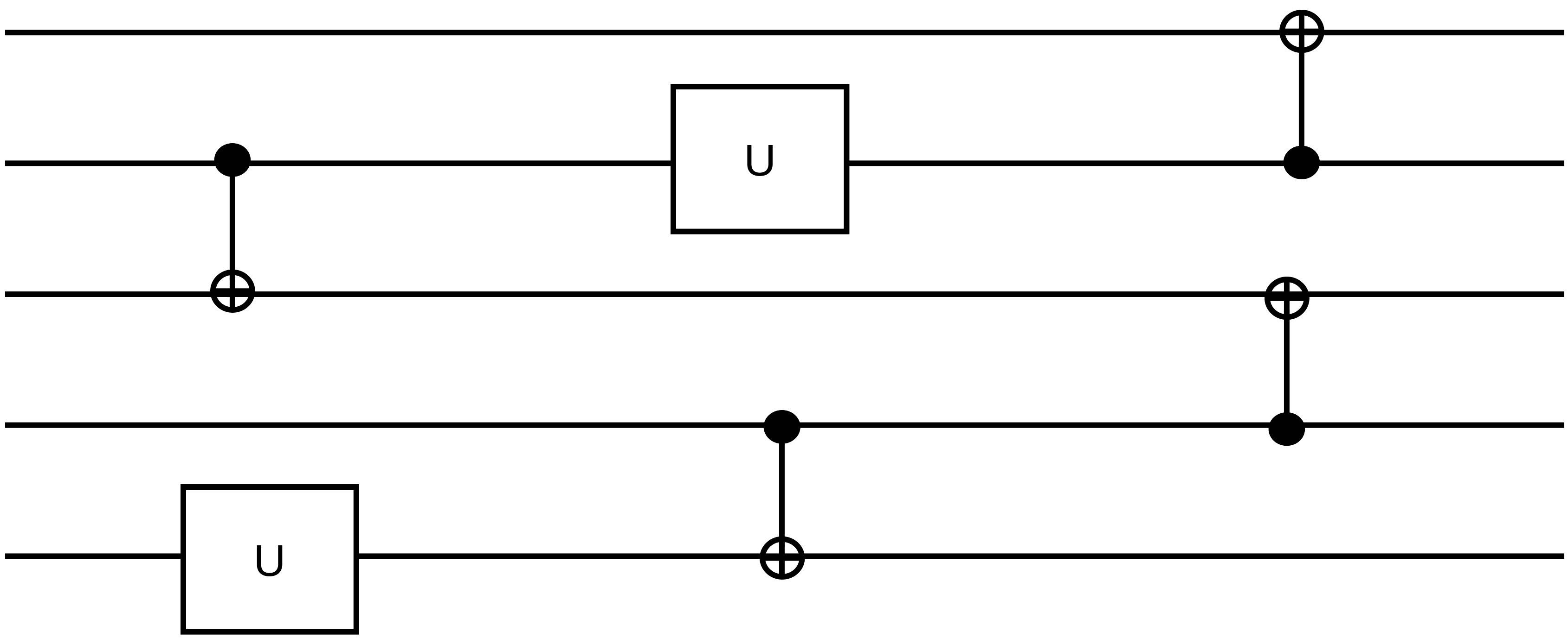


Controlled NOT:

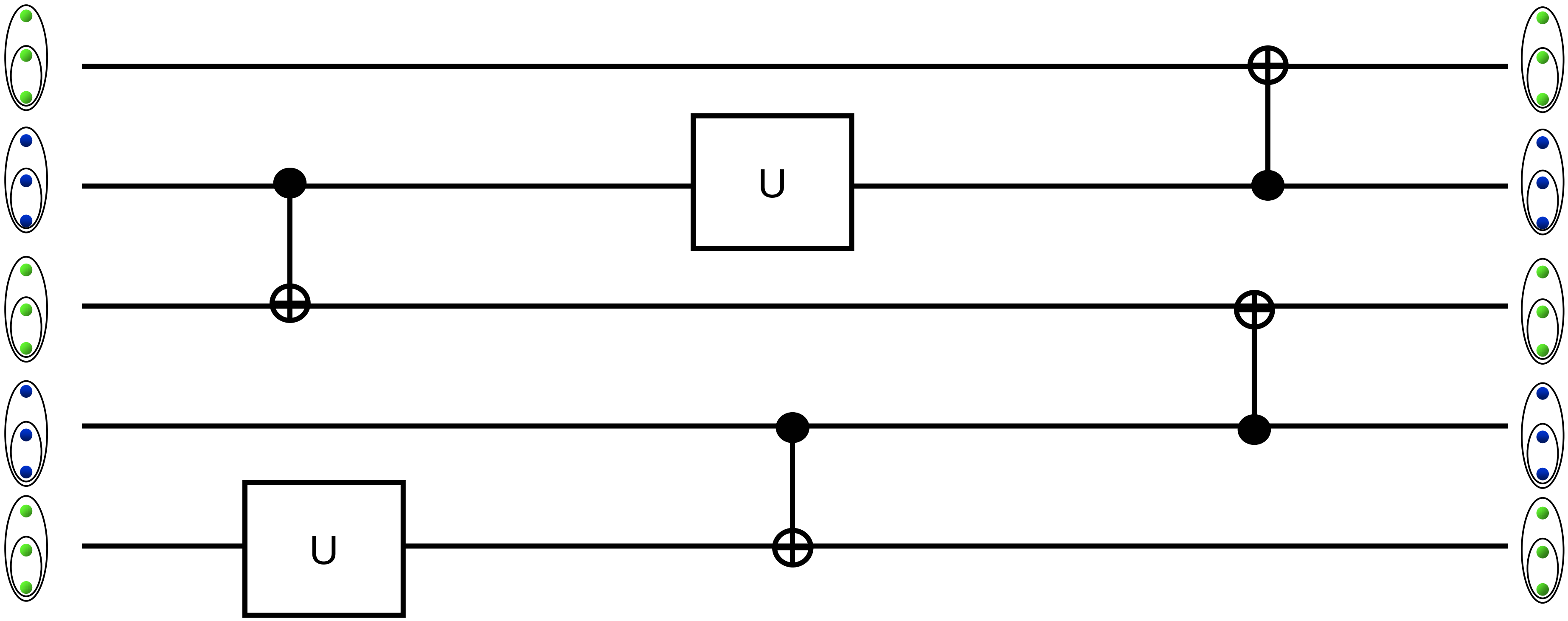




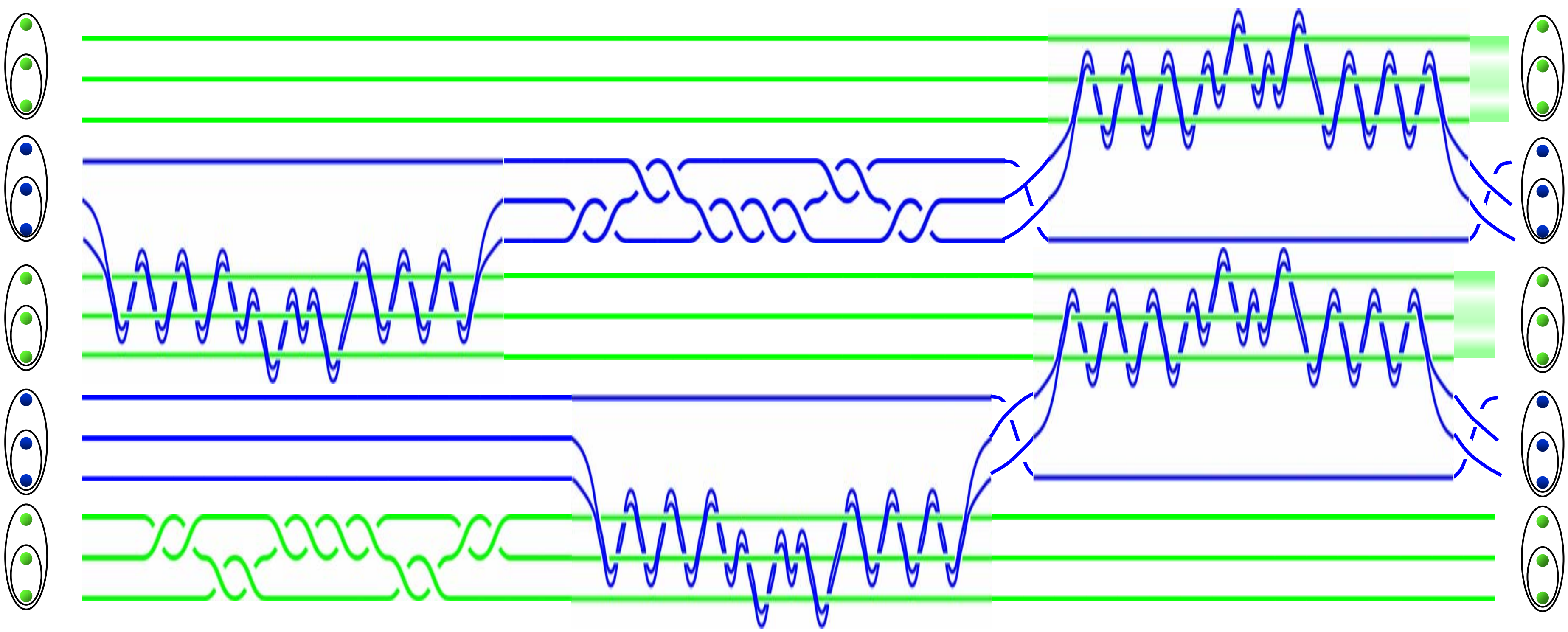
Quantum Circuit



Quantum Circuit



Braid



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Topological Quantum Computing with Only One Mobile Quasiparticle

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[2]*Department of Physics and NHMFL, Florida State University, Tallahassee, Florida 32310, USA*
[3]*Microsoft Research, One Microsoft Way, Redmond, Washington 98052, USA*

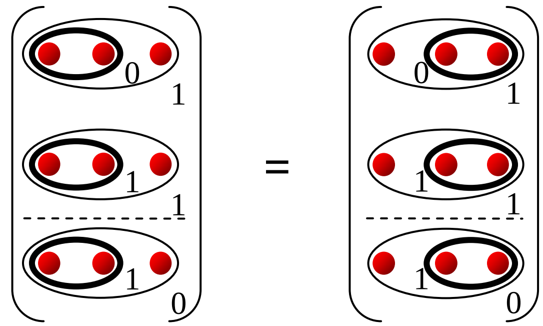
We know it is possible to carry out universal quantum computation by moving only a *single* quasiparticle.

Can we find an efficient CNOT construction in which only a single particle is woven through the other particles?

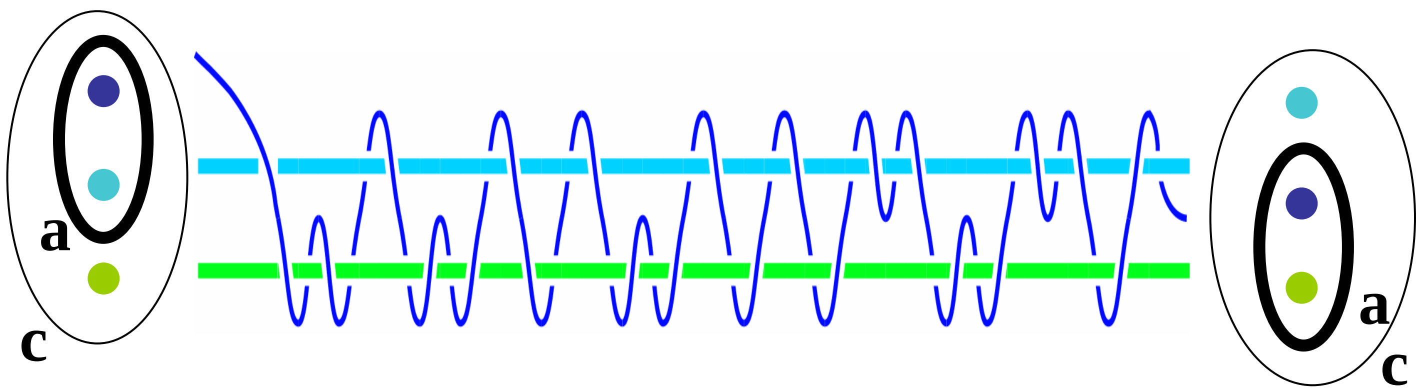
Another Useful Braid: The F-braid

F-Matrix:

$$\left(\begin{array}{cc|c} -\tau & \sqrt{\tau} & 0 \\ \sqrt{\tau} & \tau & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

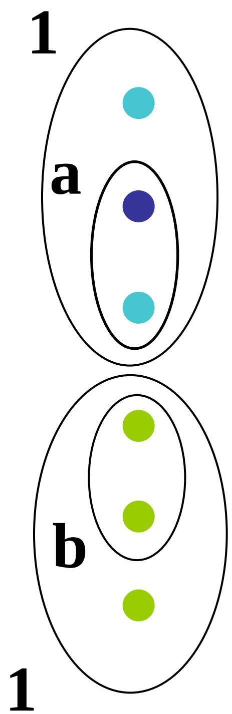


F-Braid:

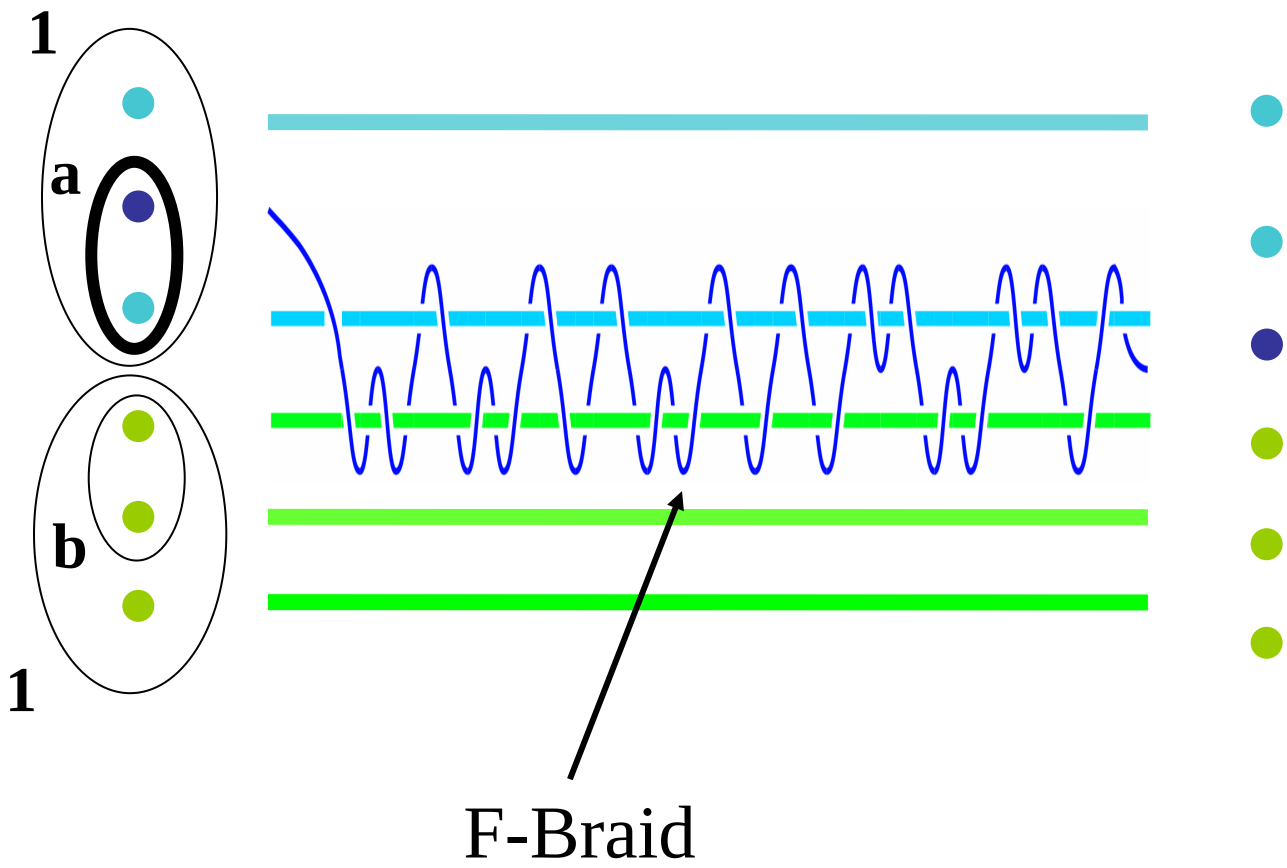


$$\sigma_2^1\sigma_1^4\sigma_2^2\sigma_1^{-4}\sigma_2^2\sigma_1^2\sigma_2^2\sigma_1^{-4}\sigma_2^2\sigma_1^{-2}\sigma_2^2\sigma_1^{-2}\sigma_2^4\sigma_1^{-4}\sigma_2^4\sigma_1^{-2}\sigma_2^{-2} \approx i \left(\begin{array}{cc|c} -\tau & \sqrt{\tau} & 0 \\ \sqrt{\tau} & \tau & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

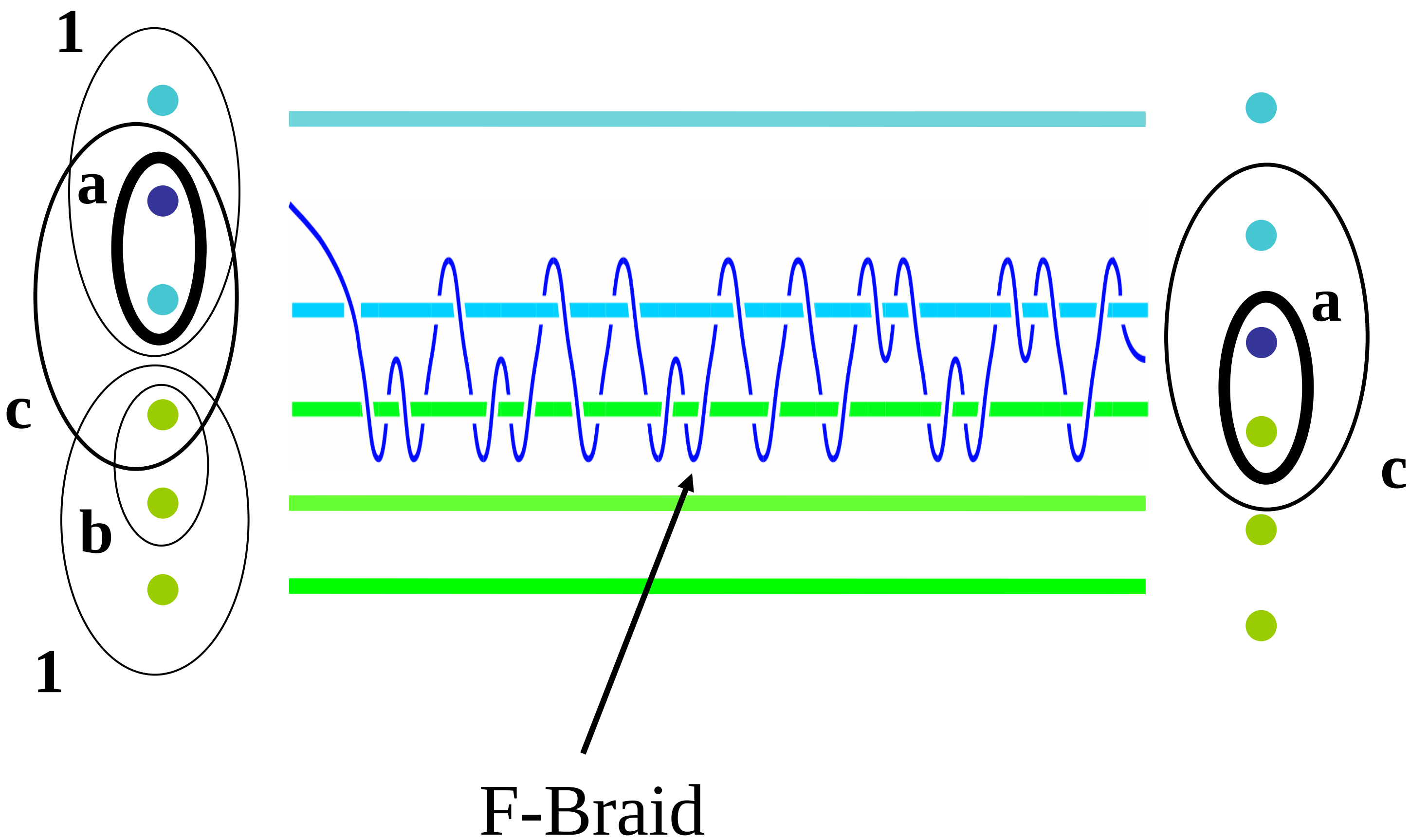
Single Particle Weave Gate: Part 1



Single Particle Weave Gate: Part 1

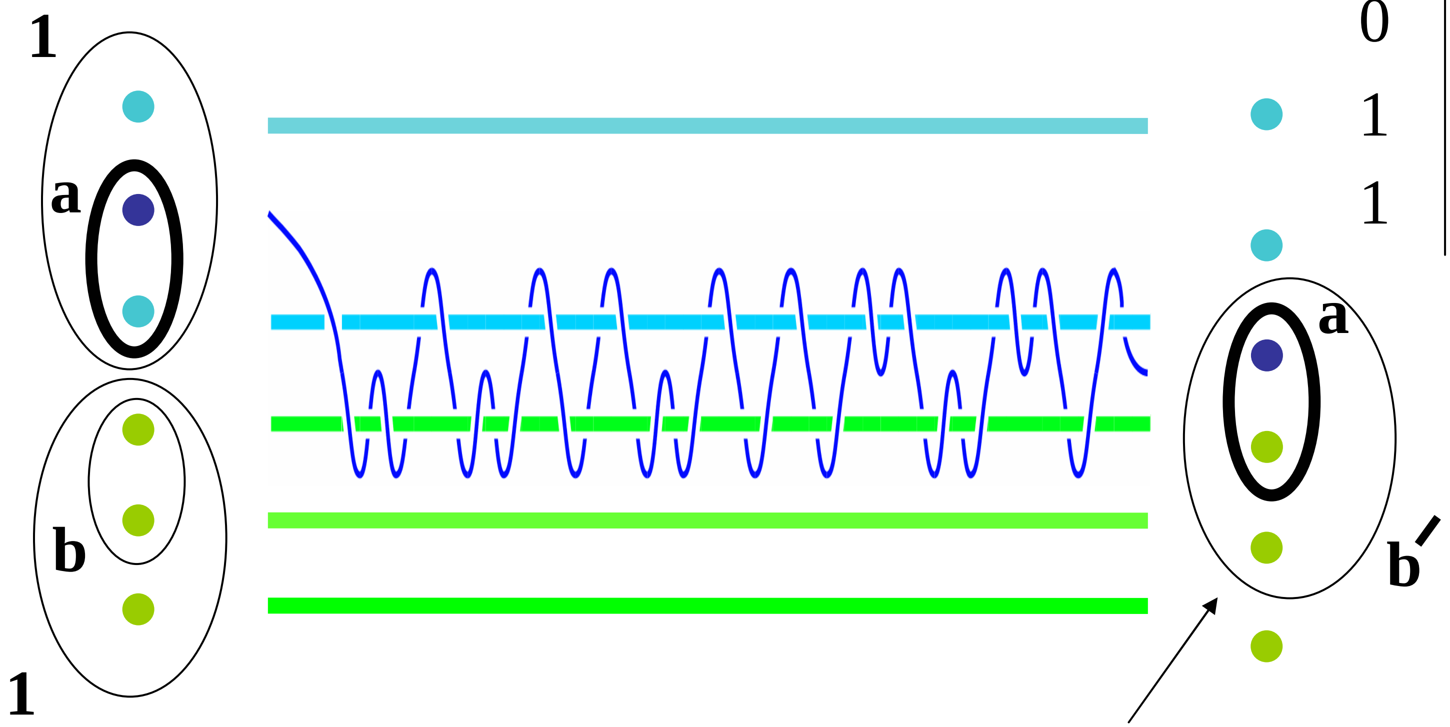


Single Particle Weave Gate: Part 1



Single Particle Weave Gate: Part 1

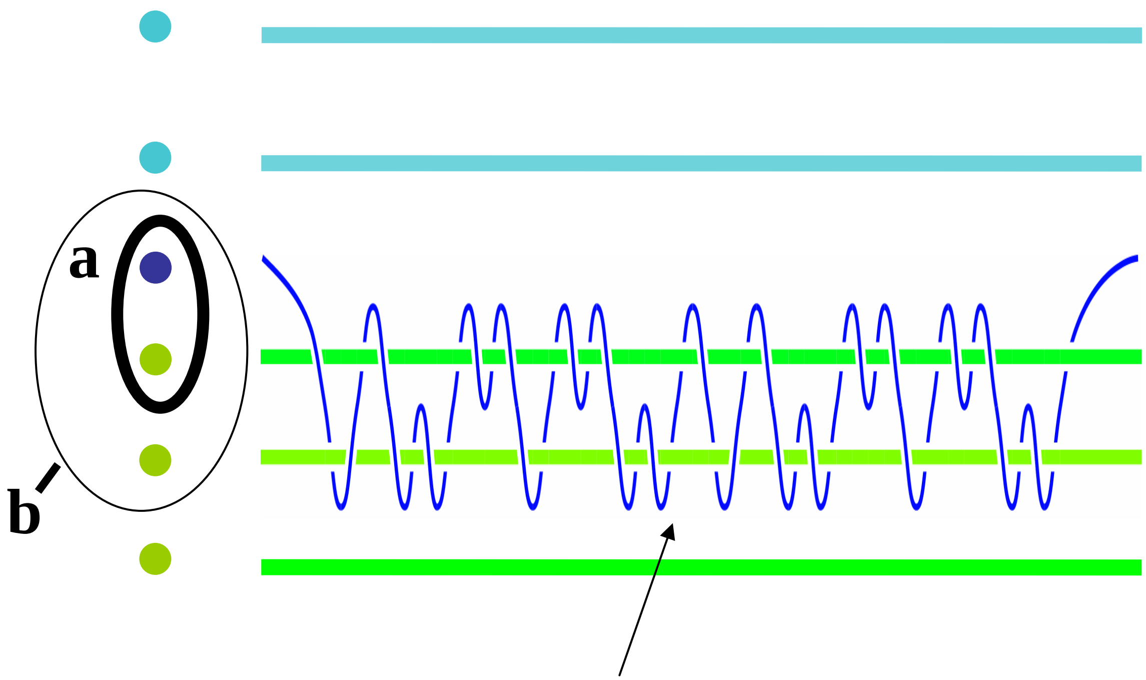
a	b	b'
0	0	1
0	1	1
1	0	0
1	1	1



Intermediate State

Single Particle Weave Gate: Part 2

a	b	b′	Phase
0	0	1	-1
0	1	1	-1
1	0	0	+1
1	1	1	-1

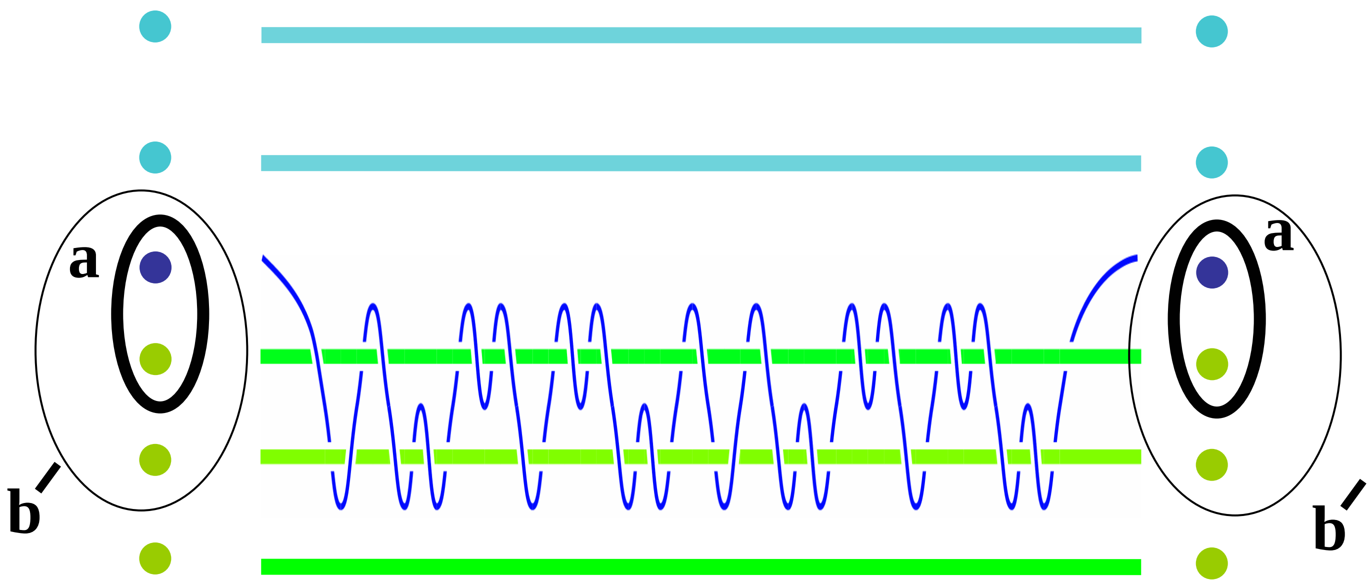


$$\approx \begin{array}{c} \overbrace{\quad\quad\quad}^{b'=1}\ \overbrace{\quad}^{b'=0} \\ \left(\begin{array}{cc:c} -1 & 0 & 0 \\ 0 & -1 & 0 \\ \hdashline 0 & 0 & 1 \end{array}\right) \end{array}$$

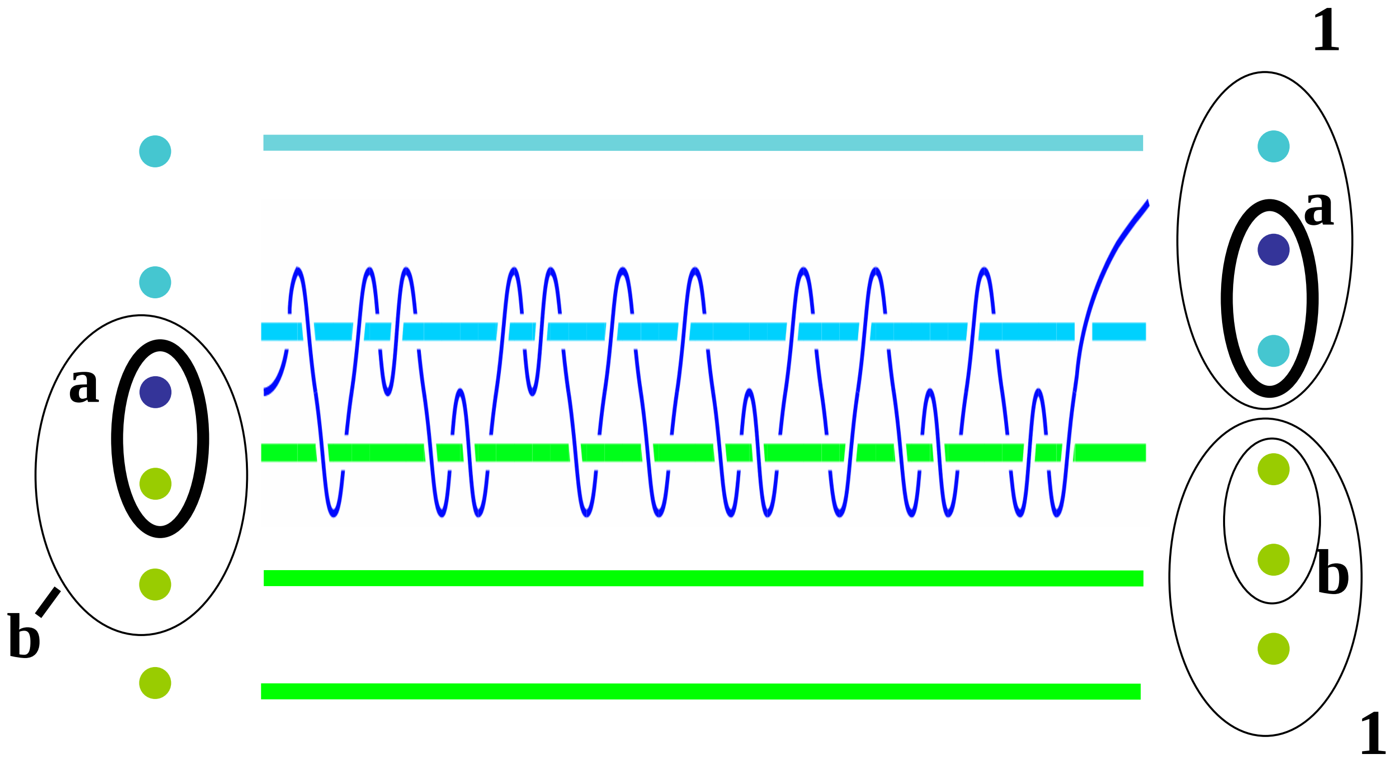
Phase Braid

Single Particle Weave Gate: Part 2

a	b	b′	Phase
0	0	1	-1
0	1	1	-1
1	0	0	+1
1	1	1	-1

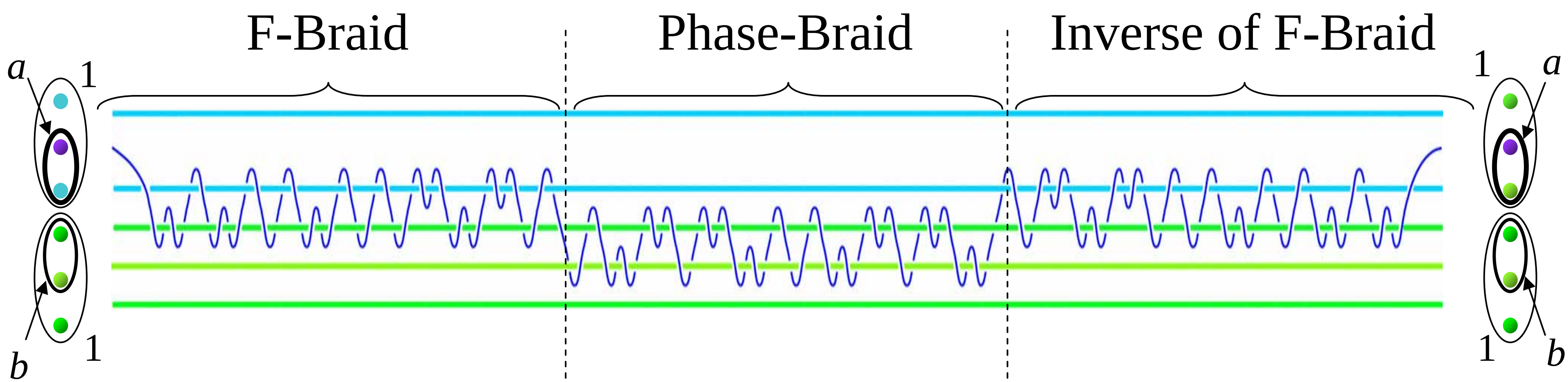


Single Particle Weave Gate: Part 3

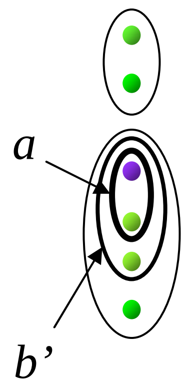


a	b	b'	Phase
0	0	1	-1
0	1	1	-1
1	0	0	+1
1	1	1	-1

Controlled-Phase Gate



Intermediate state



Final result

$$U = -\left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) + O(10^{-3})$$

Solovay-Kitaev-Improved Controlled-Phase Gate