Reversible and irreversible in a quantum measurement, or do quantum measurements reset initial conditions?

WHZ (2018) Quantum reversibility is relative, or does a quantum measurement reset initial conditions?
Phil. Trans. R. Soc. A 376: 20170315.

Information plays a central role in quantum (but not in classical!) physics.

Classical apparatus measures the state of a classical system

$$\mathtt{s}\mathtt{A}_0 \overset{\mathfrak{E}_{SA}}{\Longrightarrow} \mathtt{s}\mathtt{A}_\mathtt{s}.$$

Reversibility of evolution → one can (in principle) undo that measurement…

$$\mathtt{s}\mathtt{A}_\mathtt{s} \overset{\mathfrak{E}_{SA}^{-1}}{\Longrightarrow} \mathtt{s}\mathtt{A}_0$$

…so that both the apparatus and the system return to initial states.

This is true also when a copy of the outcome is made and kept:

$$\mathtt{s}\mathtt{A}_\mathtt{s}\mathtt{D}_0 \overset{\mathfrak{E}_{AD}}{\Longrightarrow} \mathtt{s}\mathtt{A}_\mathtt{s}\mathtt{D}_\mathtt{s}$$

$$\mathtt{s}\mathtt{A}_\mathtt{s}\mathtt{D}_\mathtt{s} \overset{\mathfrak{E}_{SA}^{-1}}{\Longrightarrow} \mathtt{s}\mathtt{A}_0\mathtt{D}_\mathtt{s}$$

And it remains true in measurements of classical mixtures:

$$(\mathtt{w}_\mathtt{s}\mathtt{s} + \mathtt{w}_\mathtt{r}\mathtt{r})\mathtt{A}_0 \overset{\mathfrak{E}_{SA}}{\Longrightarrow} \mathtt{w}_\mathtt{s}\mathtt{s}\,\mathtt{A}_\mathtt{s} + \mathtt{w}_\mathtt{r}\mathtt{r}\,\mathtt{A}_\mathtt{r}$$

$$(\mathtt{w}_\mathtt{s}\mathtt{s}\,\mathtt{A}_\mathtt{s} + \mathtt{w}_\mathtt{r}\mathtt{r}\,\mathtt{A}_\mathtt{r})\mathtt{D}_0 \overset{\mathfrak{E}_{AD}}{\Longrightarrow} \mathtt{w}_\mathtt{s}\mathtt{s}\mathtt{A}_\mathtt{s}\mathtt{D}_\mathtt{s} + \mathtt{w}_\mathtt{r}\mathtt{r}\,\mathtt{A}_\mathtt{r}\mathtt{D}_\mathtt{r}.$$

$$\mathtt{w}_\mathtt{s}\mathtt{s}\,\mathtt{A}_\mathtt{s}\mathtt{D}_\mathtt{s} + \mathtt{w}_\mathtt{r}\mathtt{r}\,\mathtt{A}_\mathtt{r}\mathtt{D}_\mathtt{r} \overset{\mathfrak{E}_{SA}^{-1}}{\Longrightarrow} (\mathtt{w}_\mathtt{s}\mathtt{s}\,\mathtt{D}_\mathtt{s} + \mathtt{w}_\mathtt{r}\mathtt{r}\,\mathtt{D}_\mathtt{r})\mathtt{A}_0.$$

Measurement by a quantum apparatus of a quantum system

$$\left(\sum_s \alpha_s|s\rangle\right)|A_0\rangle \overset{\mathfrak{U}_{SA}}{\Longrightarrow} \sum_s \alpha_s|s\rangle|A_s\rangle.$$

$$\sum_s \alpha_s|s\rangle|A_s\rangle \overset{\mathfrak{U}_{SA}^{\dagger}}{\Longrightarrow} \left(\sum_s \alpha_s|s\rangle\right)|A_0\rangle.$$

…both the apparatus and the system return to initial states.

This is no longer true when copy of the outcome is kept…

$$\left(\sum_s \alpha_s|s\rangle|A_s\rangle\right)|D_0\rangle \overset{\mathfrak{U}_{AD}}{\Longrightarrow} \sum_s \alpha_s|s\rangle|A_s\rangle|D_s\rangle.$$

$$\mathfrak{U}_{SA}^{\dagger}\left(\sum_s \alpha_s|s\rangle|A_s\rangle|D_s\rangle\right) = |A_0\rangle\left(\sum_s \alpha_s|s\rangle|D_s\rangle\right)$$

…since the state of the system is now mixed:

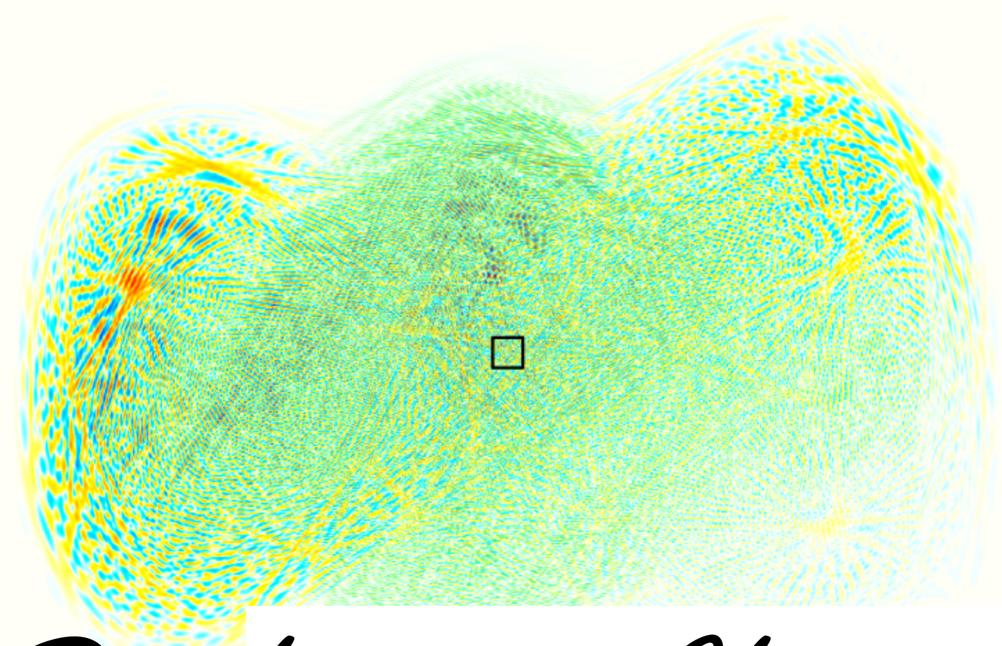
$$\varrho^S = \sum_s w_{ss}|s\rangle\langle s|, \quad w_{ss} = |\alpha_s|^2.$$

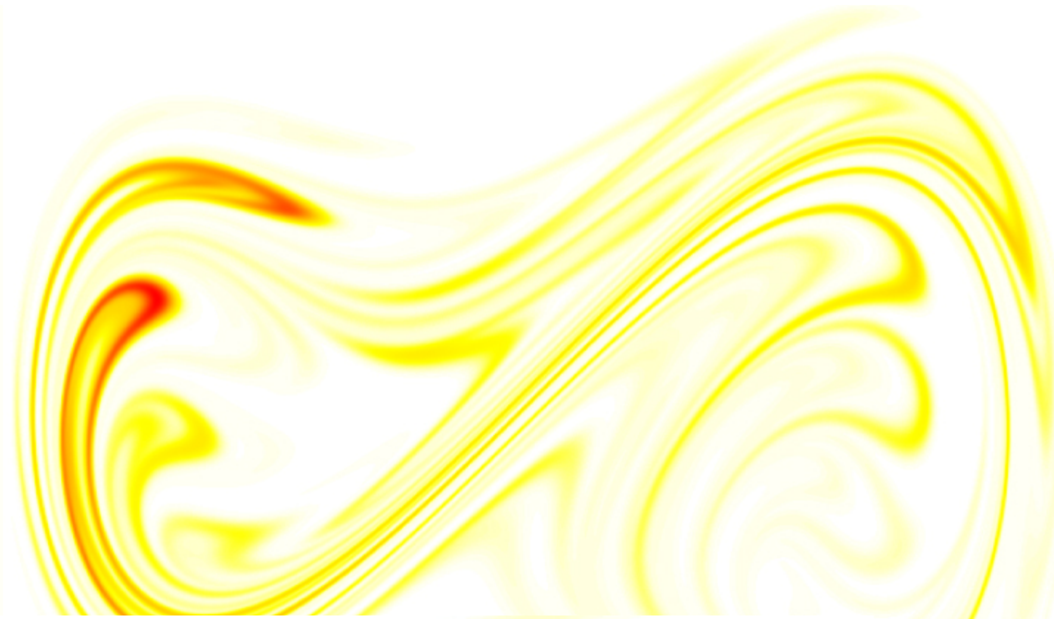
Conclusions

- Irreversibility in quantum physics can arise from the acquisition of information (not necessarily its loss – not always entropy increase).
- The records you keep define the branch of the Universe you inhabit.
- Quasiclassical case: $|s\rangle|A_0\rangle|D_0\rangle \stackrel{\mathfrak{U}_{\mathcal{SA}}}{\Longrightarrow} |s\rangle|A_s\rangle|D_0\rangle \quad |s\rangle|A_s\rangle|D_0\rangle \stackrel{\mathfrak{U}_{\mathcal{AD}}}{\Longrightarrow} |s\rangle|A_s\rangle|D_s\rangle,$

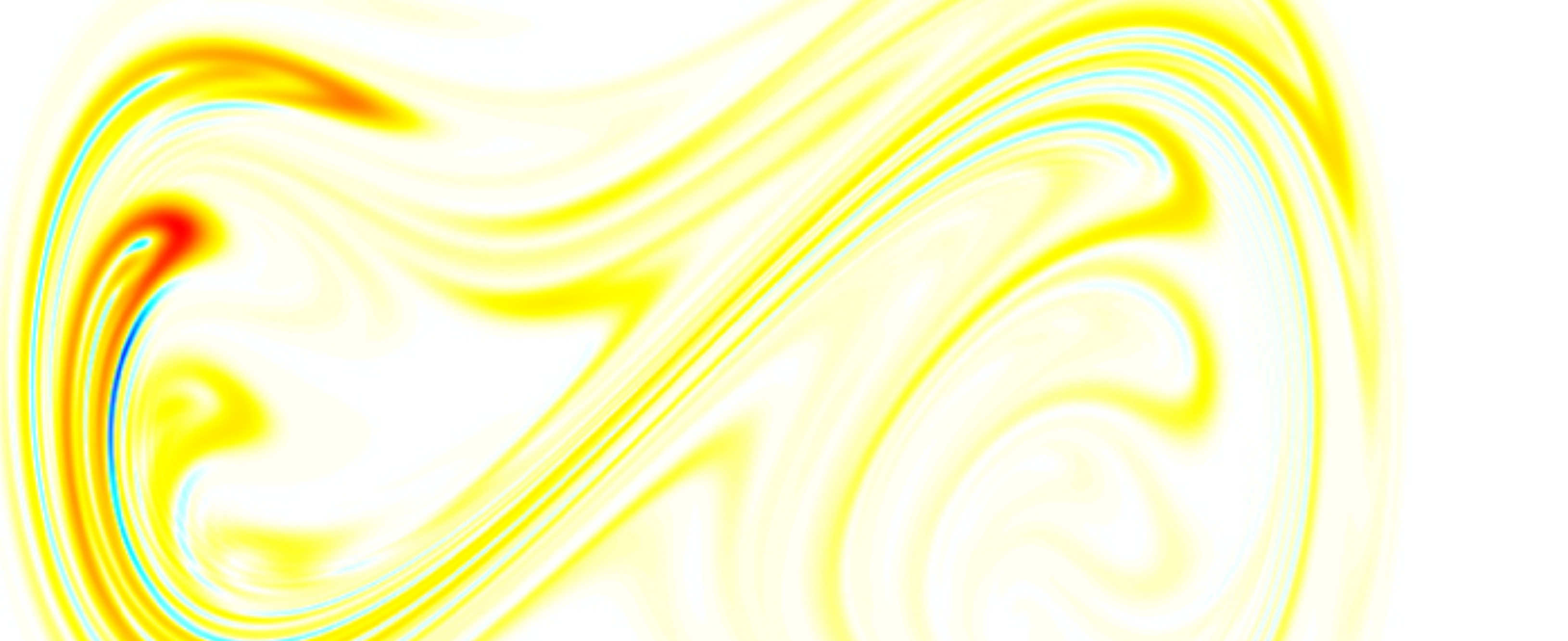
$$|s\rangle|A_s\rangle|D_s\rangle \stackrel{\mathfrak{U}^{\dagger}_{\mathcal{SA}}}{\Longrightarrow} |s\rangle|A_0\rangle|D_s\rangle.$$

- Increase of entropy after an attempted reversal is tied to discord.
- Agent's ("Wigner's") friend can confirm that agent knows the outcome and can still reverse the dynamics (providing he does not copy the outcome!).
- Information about the outcome does not preclude reversal of classical measurements (classical dynamics is independent of information about evolving systems) but quantum evolutions adjust to what is known about them.
- **Information plays a central role in quantum (but not in classical!) physics. Measurement "resets" initial conditions relevant for the observer!**



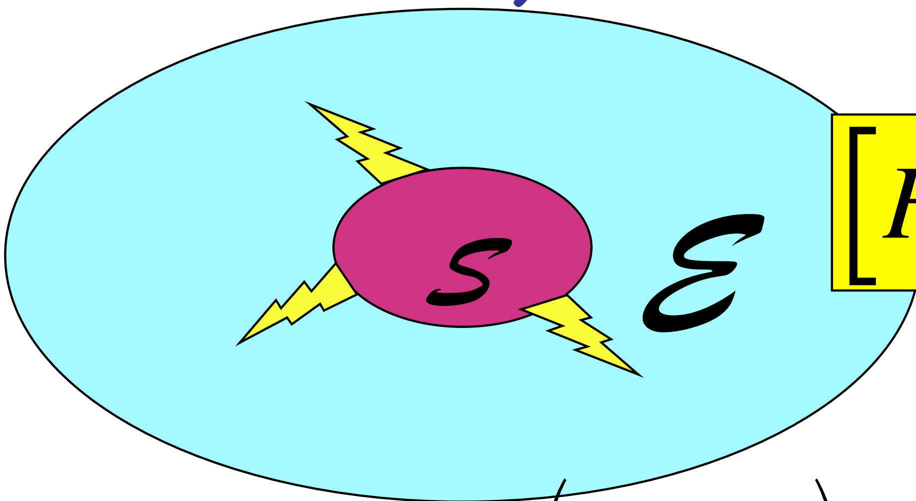


Decoherence, Chaos, and the Second Law



DECOHERENCE, CHAOS, AND THE 2ND LAW WHZ & PAZ, JP PRL **72** pp. 2508-2511 (1994); WHZ, **DECOHERENCE, CHAOS, QUANTUM-CLASSICAL….**, Physica Scr. T76, 186-198 (1998)

EINSELECTION*, POINTER BASIS, AND DECOHERENCE



$$\left[H_{\mathcal{S}\mathcal{E}}, |\sigma_i\rangle\langle\sigma_i| \right] = 0$$

$$|\Phi_{\mathcal{S}\mathcal{E}}(0)\rangle = |\psi_{\mathcal{S}}\rangle \otimes |\varepsilon_0\rangle = \left(\sum \alpha_i |\sigma_i\rangle \right) \otimes |\varepsilon_0\rangle \xrightarrow[\text{Entanglement}]{\text{Interaction}\; H_{\mathcal{S}\mathcal{E}}} \sum_i \alpha_i |\sigma_i\rangle \otimes |\varepsilon_i\rangle = |\Phi_{\mathcal{S}\mathcal{E}}(t)\rangle$$

We accept Born's rule!

REDUCED DENSITY MATRIX $\rho_{\mathcal{S}}(t) = Tr_{\mathcal{E}} |\Phi_{\mathcal{S}\mathcal{E}}(t)\rangle\langle\Phi_{\mathcal{S}\mathcal{E}}(t)| = \sum_i |\alpha_i|^2 |\sigma_i\rangle\langle\sigma_i|$

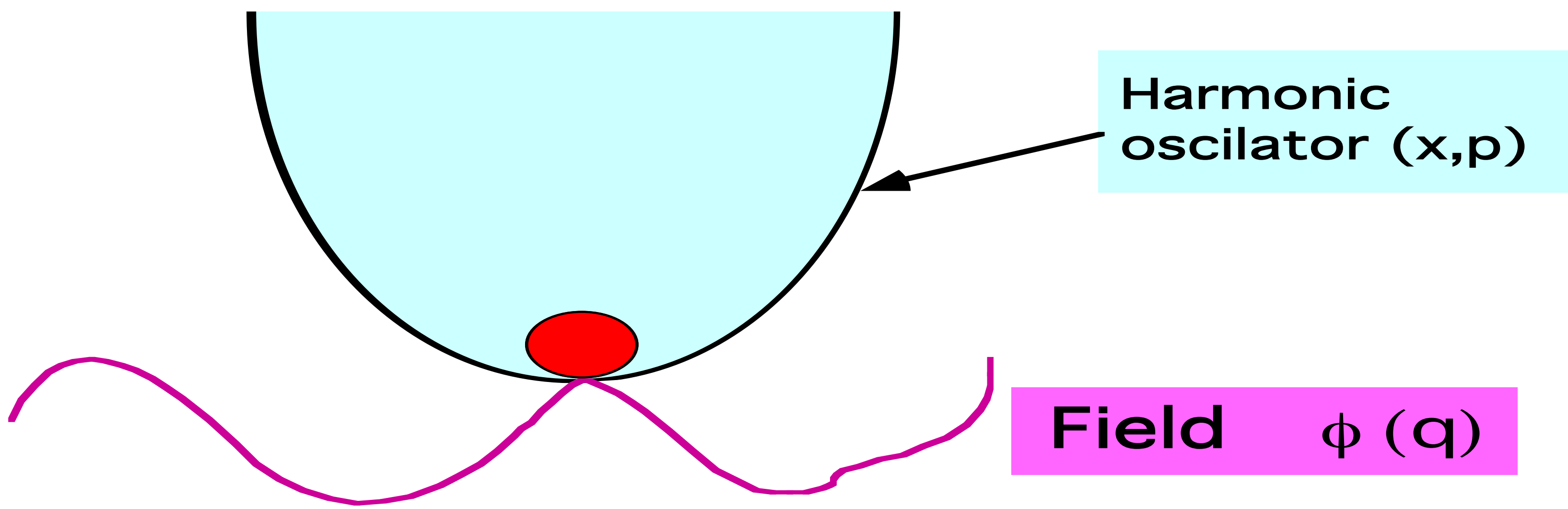
EINSELECTION* leads to POINTER STATES

Stable states, appear on the diagonal of $\rho_{\mathcal{S}}(t)$ after decoherence time; **pointer states** are effectively classical!
Pointer states **left unperturbed** by the "environmental monitoring".

***Environment INduced superSELECTION**

Reduction of the Wavepacket in Quantum Brownian Motion

Harmonic oscillator system coupled to a free field environment via H_{int}.



$$H_{int} = \varepsilon \times \partial_t \phi(q)$$

To obtain the effective equation of motion for the density matrix of the harmonic oscillator:

1. Obtain the exact solution of the whole problem.

2. Trace out the field.

MASTER EQUATION*#

(in the position representation)

Von Neumann **relaxation/damping**

$$\dot{\rho}(x,x') = -\frac{i}{\hbar}\left[H_R,\rho\right] - \gamma(x-x')(\partial_x - \partial_{x'})\rho$$

$$-\frac{2m\gamma k_B T}{\hbar^2}(x-x')^2\rho$$

DECOHERENCE

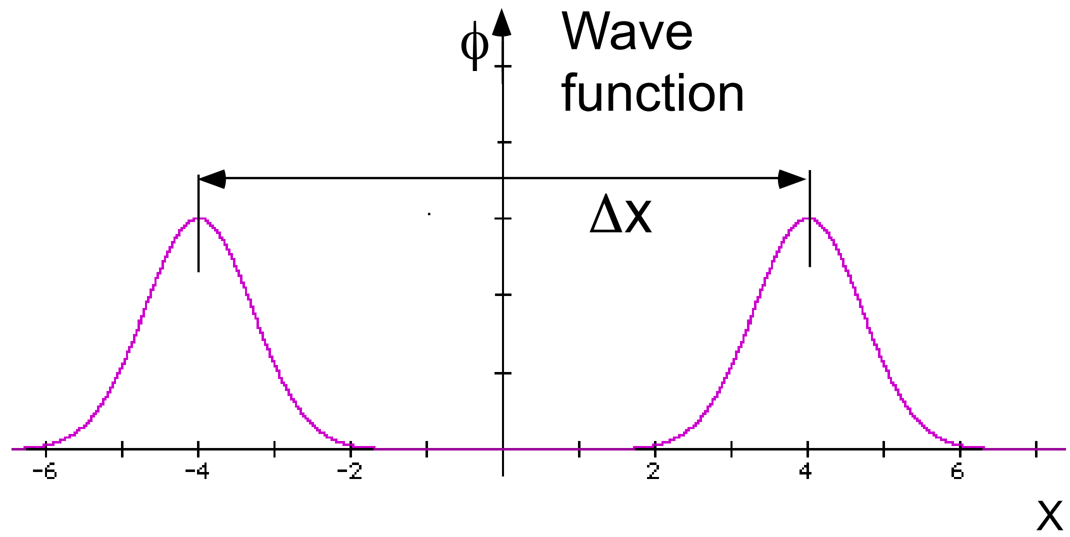
$\gamma = \dfrac{\eta}{2M}$, viscosity $\eta = \dfrac{\varepsilon^2}{2}$, where ε is the coupling constant in $H_{int} = \varepsilon\, x\dot{\varphi}$.

A solution (to a leading order "for small Planck constant").

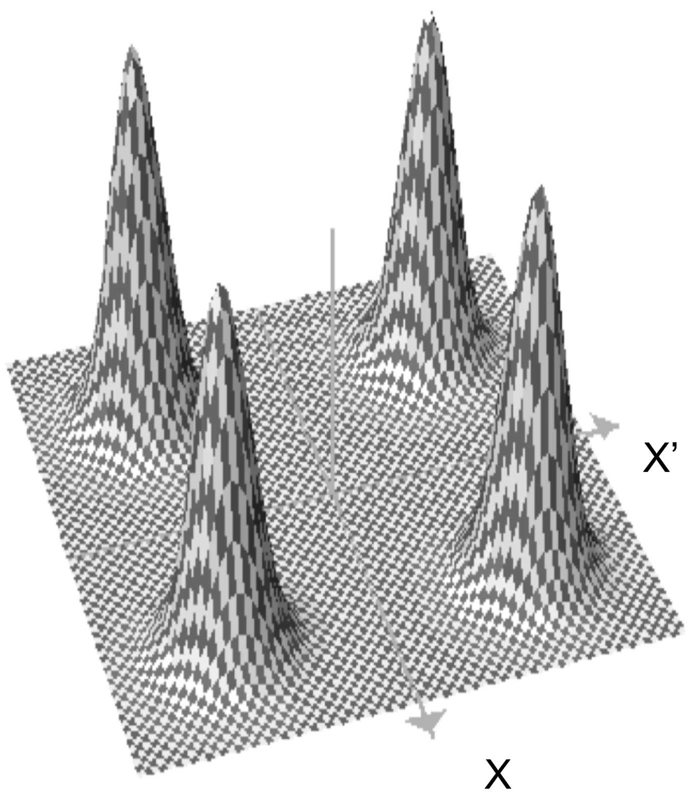
$$\rho(x,x';t) = \rho(x,x';0)\exp\left\{-\frac{\eta k_B T(x-x')^2}{\hbar^2}\,t\right\}$$

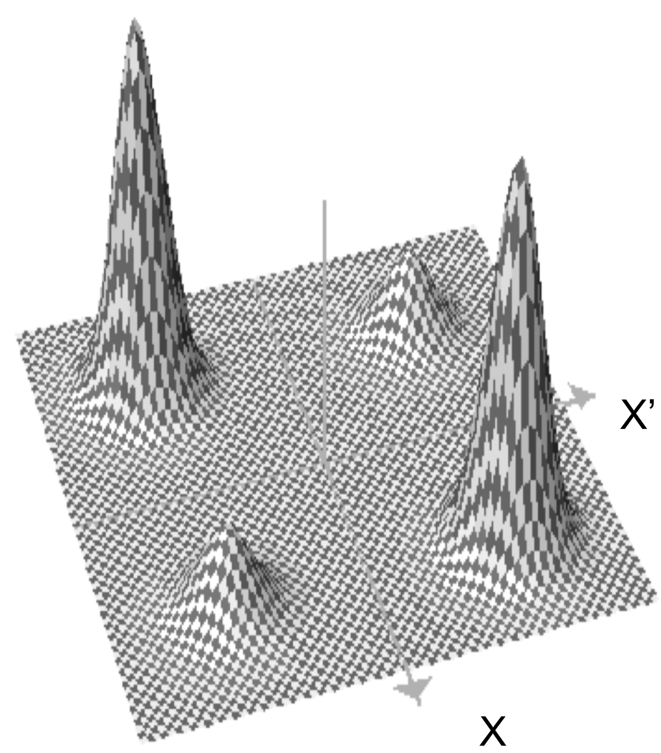
***High temperature limit. Master equations for arbitrary temperatures, non- ohmic spectral densities, *etc.* also can be derived and predict decoherence.**

 # ... Feyman and Vernon; Caldeira and Leggett; Unruh and Zurek; Hu, Paz and Zhang; Gallis; Gell-Mann and Hartle; Anglin, Paz and Zurek...

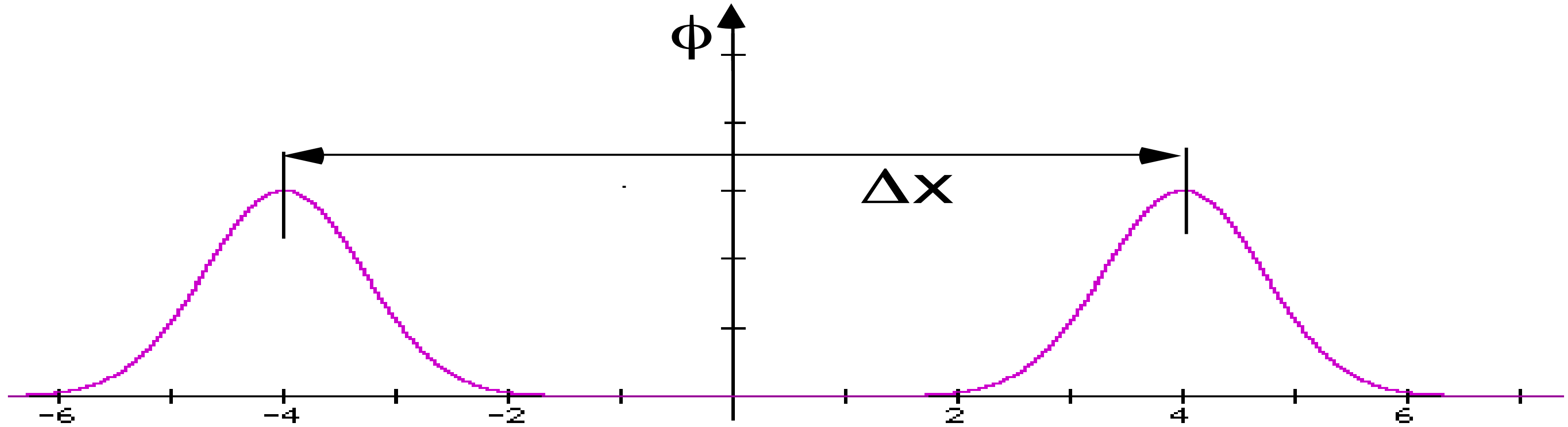


Density Matrix ρ (x, x’)





Decoherence Time Scale and the Role of Position



RELAXATION:

$$\dot{p} = -\gamma p$$

Experimental confirmation: ENS (Brune, Haroche, Raimond…); NIST (Monroe, Wineland….); Vienna (Arndt, Zeilinger…)….

DECOHERENCE RATE:

$$\delta = \tau_D^{-1} \cong \gamma \left(\frac{\Delta x}{\lambda_{dB}(T)} \right)^2$$

Where thermal de Broglie length is:

$$\lambda_{dB} = \hbar / \sqrt{2 m k_B T}$$

Zurek '84-86; Joos & Zeh'85
Paz, Habib & Zurek'93

Decoherence in the Phase Space & Wigner Distribution

$$W(x,p) = \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} e^{ipy/\hbar}\left\langle x - \frac{y}{2}\right|\rho\left| x + \frac{y}{2}\right\rangle dy$$

A Gaussian Wigner Distribution:

$$W(x,p;x_0,p_0) = \frac{1}{\pi\hbar}\times$$

$$\times\exp\left(-\frac{2(p-p_0)^2}{\hbar^2}\delta^2 - \frac{(x-x_0)^2}{2\delta^2}\right)$$

Goes into a point in the phase space in the classical limit.

For a superposition of two stationary peaks:

$$W(x,p) = \frac{1}{\sqrt{2}\pi\hbar}\exp\left(-\frac{2p^2\delta^2}{\hbar^2} - \frac{(x-\Delta x/2)^2}{2\delta^2}\right) +$$

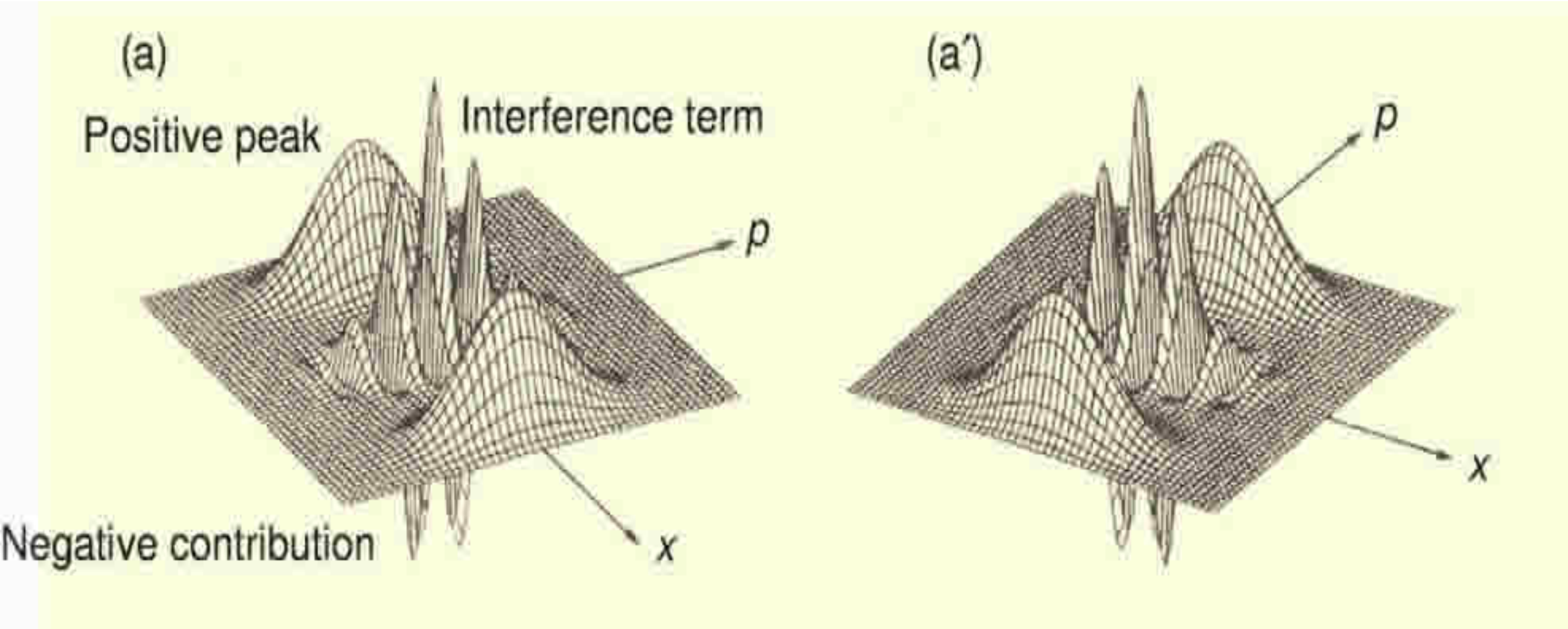
$$\frac{1}{\sqrt{2}\pi\hbar}\exp\left(-\frac{2p^2\delta^2}{\hbar^2} - \frac{(x+\Delta x/2)^2}{2\delta^2}\right) +$$

two peaks

$$\frac{1}{\sqrt{2}\pi\hbar}\exp\left(-\frac{2p^2\delta^2}{\hbar^2} - \frac{x^2}{2\delta^2}\right)\cos\left(\frac{\Delta x}{\hbar}p\right)$$

interference!

NOT EVEN POSITIVE -cannnot be classical!!



(a)
Positive peak
Interference term
p
x
Negative contribution
(a′)
p
x

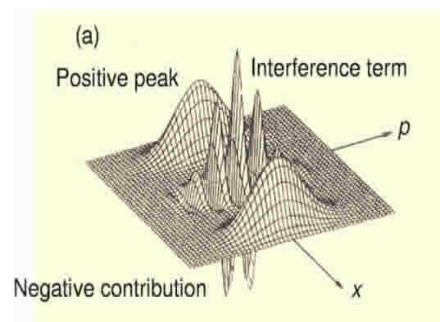
Classical Limit of Quantum Dynamics & the Fokker - Planck Equation

$\frac{\partial W}{\partial t} = -\frac{P}{m}\frac{\partial W}{\partial x} + \frac{\partial V_R}{\partial x}\frac{\partial W}{\partial P}$	$\{H,\rho\}$ –Poisson Bracket Liouville Dynamics
$+2\gamma \frac{\partial}{\partial p} pW$	relaxation & dissipation
$+2m\gamma k_B T \frac{\partial^2 W}{\partial p^2}$	Here diffusion in momentum: Decoherence

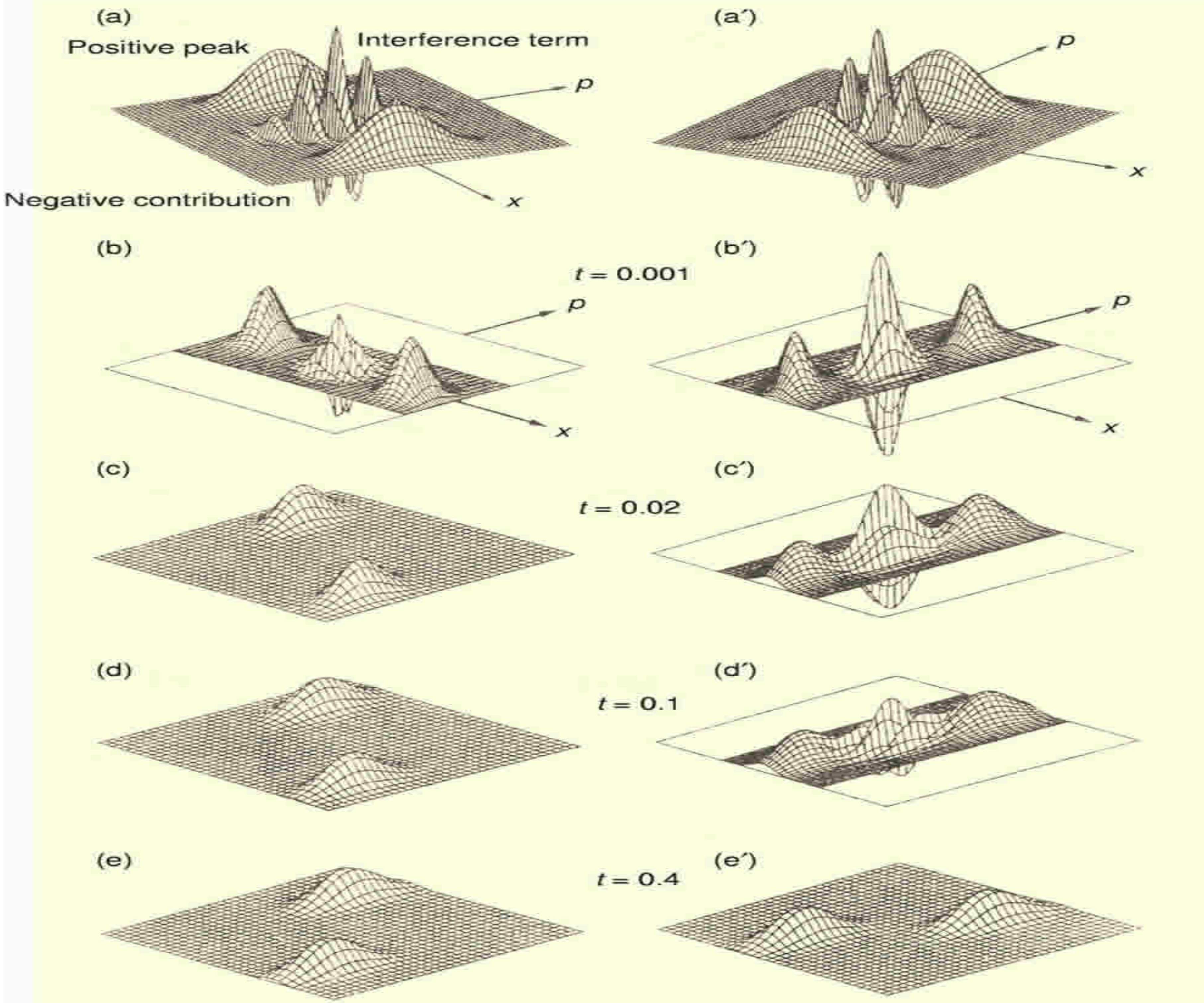
Estimate of the Decoherence Timescale:

$$\tau_D^{-1} = -\frac{\dot{W}}{W}\bigg|_{Decoh} \cong 2m\gamma k_B T \frac{\partial^2}{\partial p^2} \cos\frac{\Delta x}{\hbar} p$$

$$\tau_D = \frac{2m\gamma k_B T(\Delta x)^2}{\hbar^2}$$



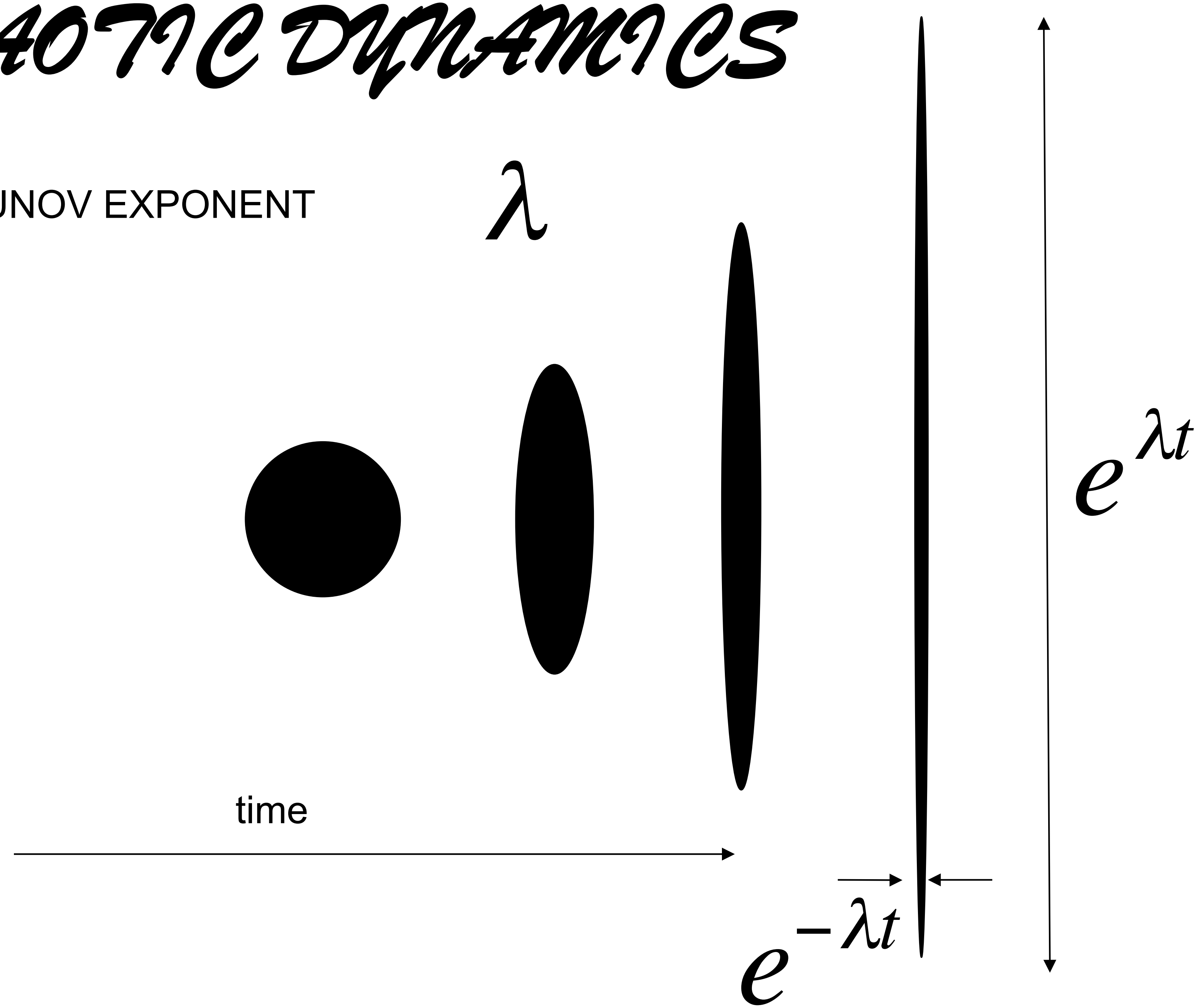
Oscilations in W disappear on timescale τ_D: W becomes nonnegative and can be regarded as "CLASSICAL"!



(a)
Positive peak
Interference term
Negative contribution
p
x
(a′)
p
x
(b)
t = 0.001
p
x
(b′)
p
x
(c)
t = 0.02
(c′)
(d)
t = 0.1
(d′)
(e)
t = 0.4
(e′)

CHAOTIC DYNAMICS

LYAPUNOV EXPONENT λ



Quantum-Classical Correspondence and Chaotic Dynamics

Von Neumann equation for density operator:

$$i\hbar\dot{\rho} = \left[\hat{H}, \rho\right]$$

Von Neumann bracket

Evolution in phase space:

$$\dot{W} = \{H, W\}_{Moyal} =$$

$$\{H, W\}_{Poisson} + \left(-\frac{\hbar^2}{24}\partial_x^3 V \partial_p^3 W + ..\right)$$

Classical | **Quantum Corrections**

$V_x W_p$ | $\frac{\hbar^2}{24} V_{xxx} W_{ppp}$

$O\left(\hbar^n\right)$ | $O(e^{\lambda nt})$

small | **LARGE**

Breakdown of Quantum-Classical Correspondence

In quantum chaotic systems correction to Poisson bracket grow as ~exp(λt) and become comparable with it after:

$$t_{\hbar} \cong \lambda^{-1} \ln \frac{\Delta p_0 \chi}{\hbar} .$$

The scale of nonlinearities:

$$\chi \cong \sqrt{V_x / V_{xxx}} .$$

A simple (over)estimate of the correspondence breakdown time:

$$t_r \cong \lambda^{-1} \ln \frac{A}{\hbar} ,$$

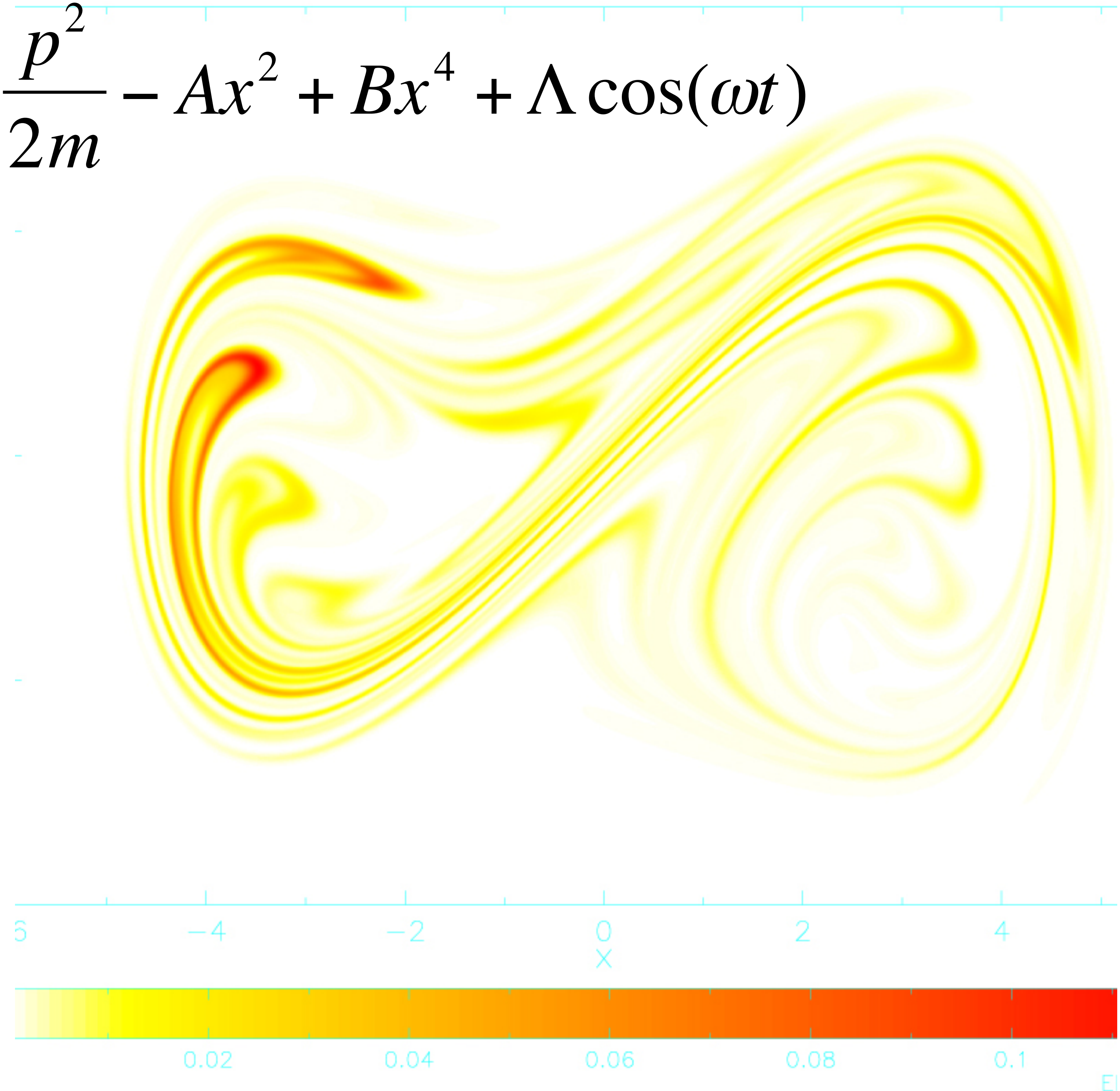
where A is the characteristic action of the system.

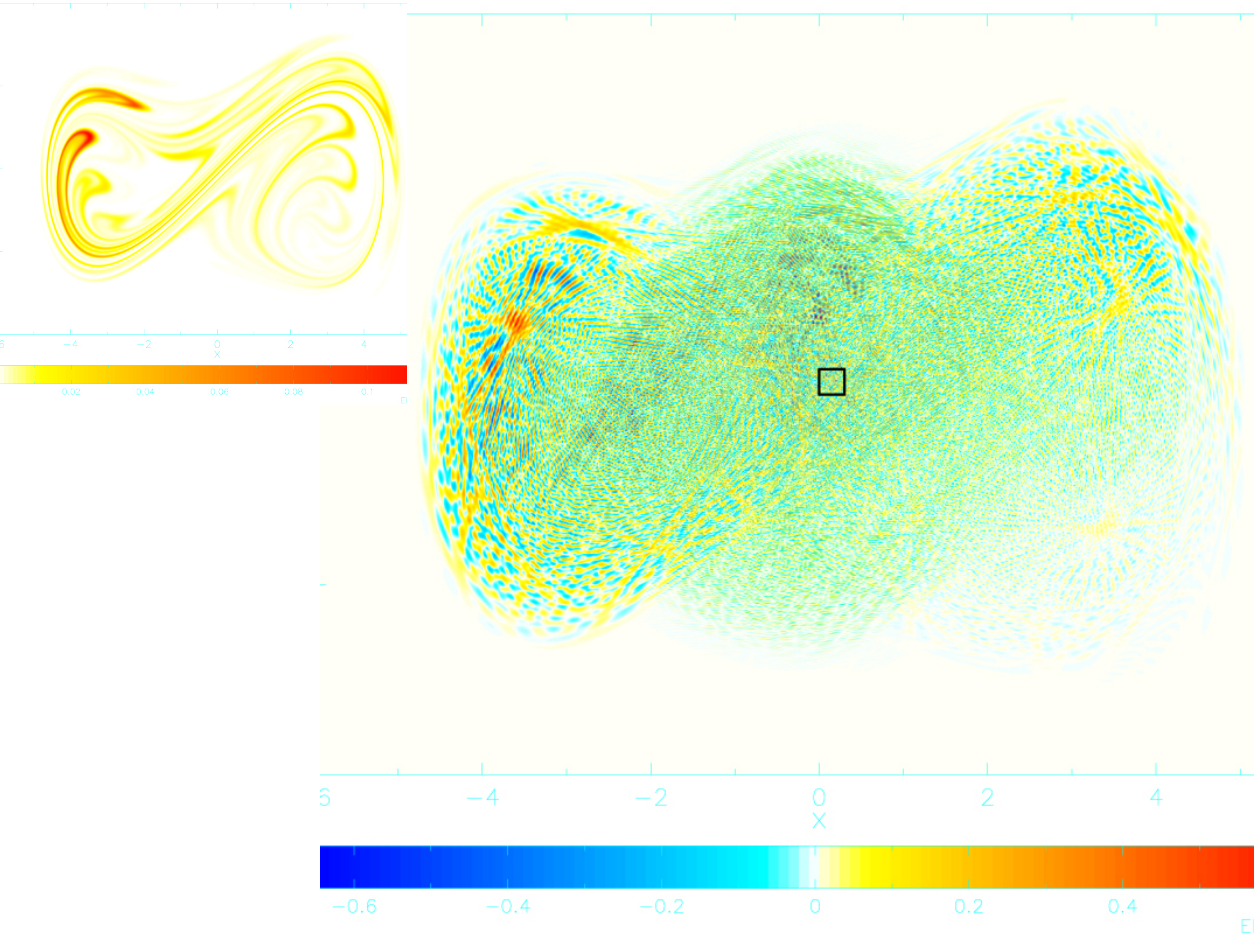
HYPERION

YEAR: 2006
MISSION: CASSINI
TARGET: SATURN / HYPERION

Assumed to be a "captured" asteroid, Hyper
scientists when it revealed an almost sponge
and mysterious dark bottoms to many crater

$$H = \frac{p^2}{2m} - Ax^2 + Bx^4 + \Lambda\cos(\omega t)$$



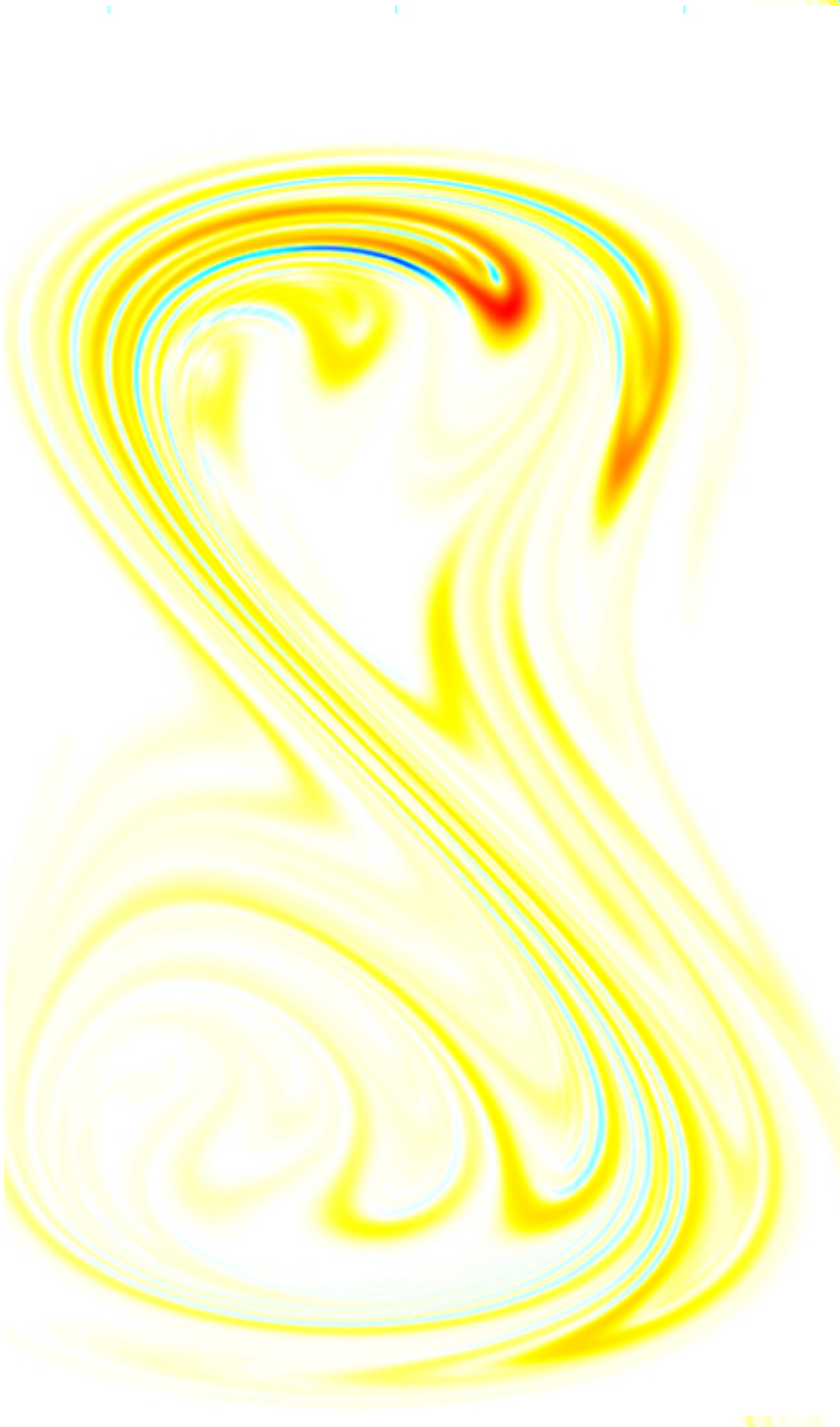
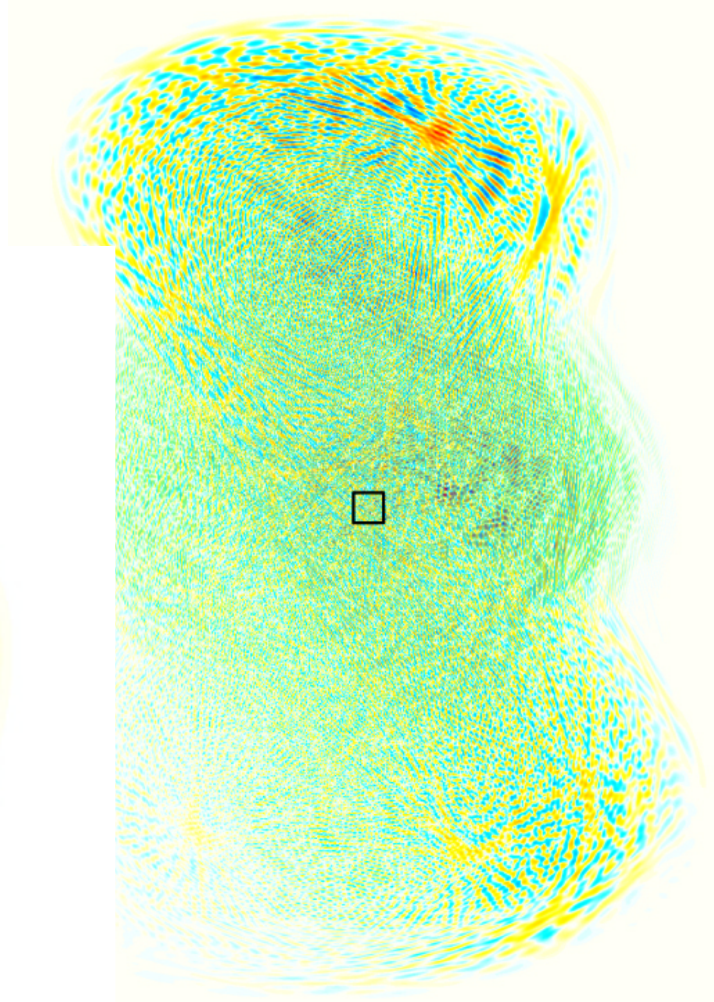


−4
−2
0
2
4
X
0.02
0.04
0.06
0.08
0.1
−4
−2
0
2
4
X
−0.6
−0.4
−0.2
0
0.2
0.4

Quantum Chaos + Decoherence

$$\dot{W} = \{H, W\}_{M.B.} + D\frac{\partial^2 W}{\partial p^2}$$

$$= \{H, W\}_{P.B.} + \sum_{n\geq 1} \frac{\hbar^{2n}(-)^n}{2^{2n}(2n+1)!} \partial_x^{2n+1} V \partial_p^{2n+1} W + D\frac{\partial^2 W}{\partial p^2}$$

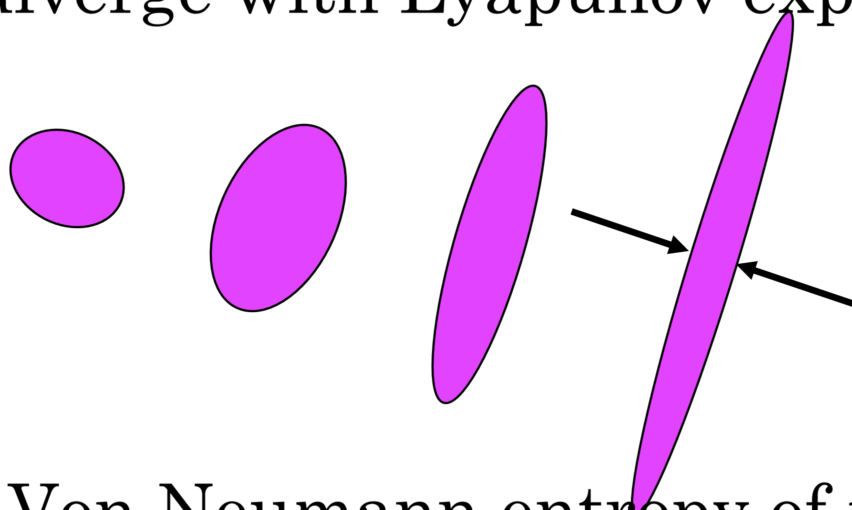




DECOHERENCE, CHAOS, & THE SECOND LAW

Paz & WHZ, PRL 1994…Monteoliva and Paz, 2001

INGREDIENTS:

- Open quantum system with classically chaotic counterpart -- i.e., trajectories that diverge with Lyapunov exponents λ_k^+

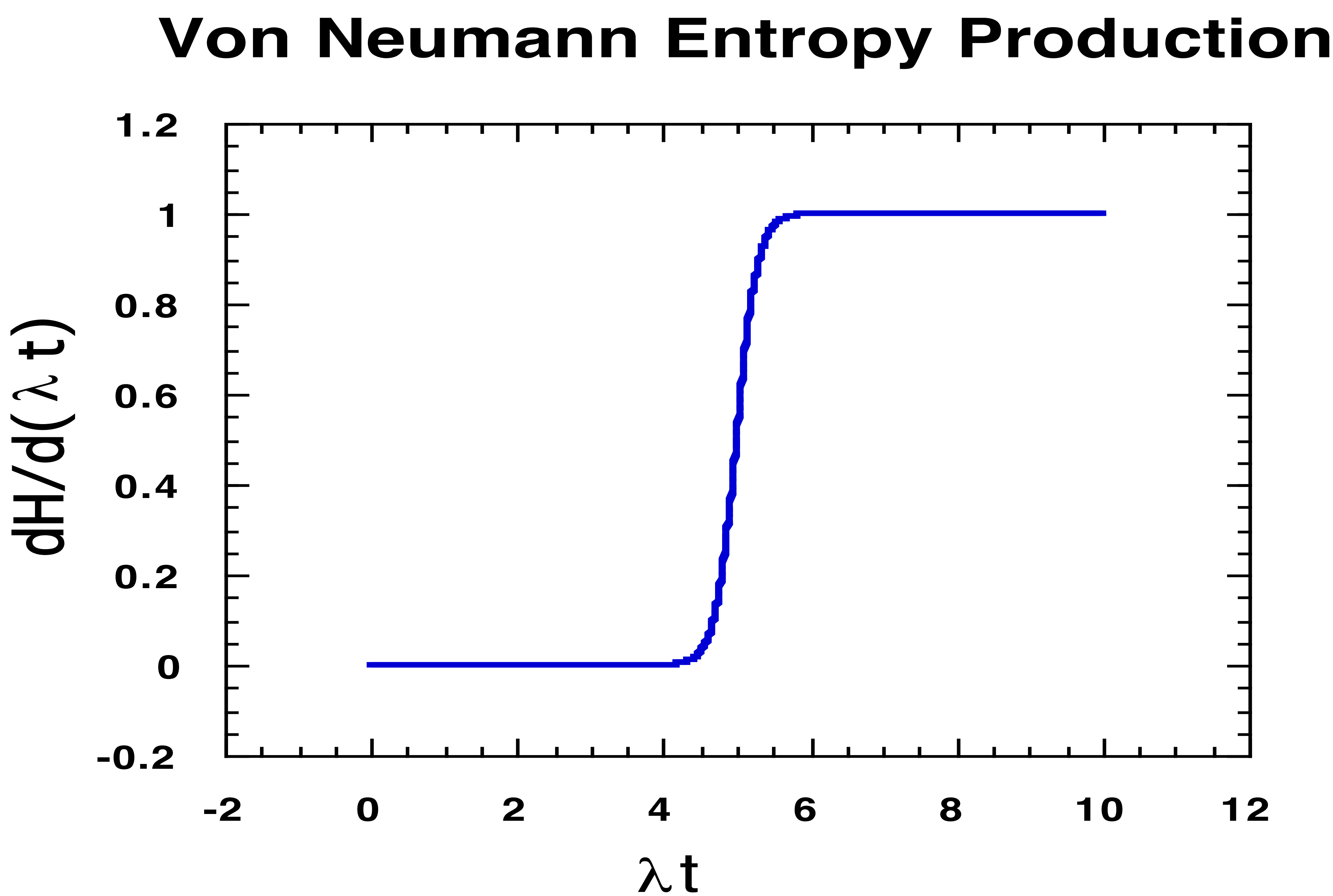


$$\sigma_{c_k} = \sqrt{\frac{2D}{\lambda_k^+}}$$

- Von Neumann entropy of the reduced density matrix of the system:

$$H = -Tr \rho_S(t) \ln \rho_S(t)$$

$$\therefore \dot{H} \approx \sum_k \lambda_k^+$$


Von Neumann Entropy Production
dH/d(λ t)
λ t
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-2
0
2
4
6
8
10
12

Decoherence, Chaos, & the 2nd Law

1. $$\begin{pmatrix} \text{Quantum Dynamics} \\ + \text{Decoherence} \end{pmatrix} \Rightarrow \begin{pmatrix} \text{Classical Dynamics} \\ + \text{Irreversibility} \end{pmatrix}$$

(i) Equations of motion; (ii) Structure of phase space;
(iii) "Limited" reversibility (for smooth W)

Classicality Condition for Chaotic Systems:

$$\chi \sigma_c >> \hbar$$

or

$$\chi >> \ell_c = \hbar / \sigma_c$$

where $\chi \cong \sqrt{V' / V'''}$, and $\sigma_c = \sqrt{2D/\lambda}$

2. Entropy Production (Dynamical 2nd Law!)

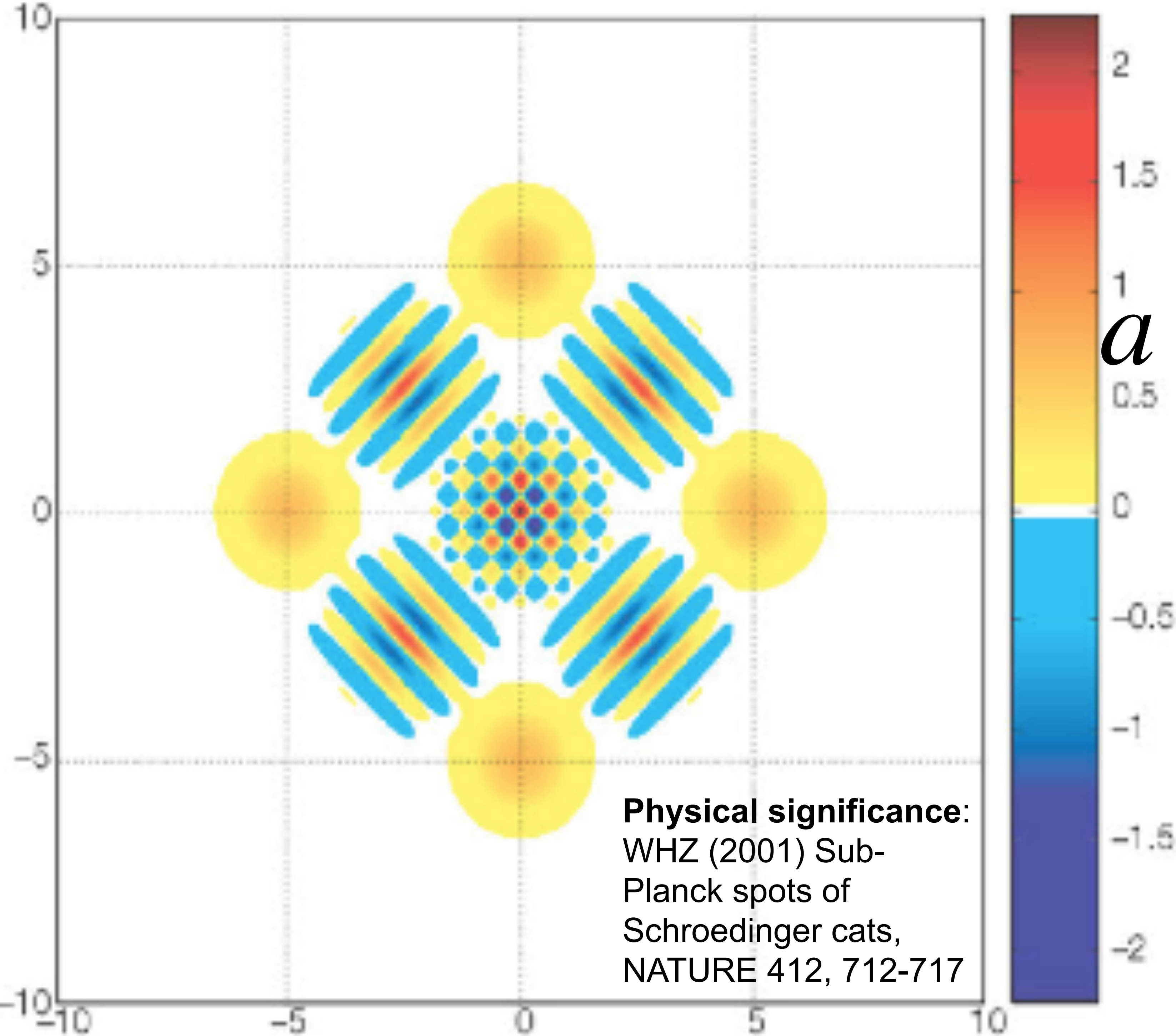
$$\dot{H} = \frac{d}{dt} Tr \rho \ln \rho^{-1} \cong \sum_i \lambda_i^+$$

Independent of the strength of the coupling to the environment.......

DECOHERENCE, CHAOS, AND THE 2ND LAW ZUREK, WH; PAZ, JP
PHYSICAL REVIEW LETTERS **72** Pages: 2508-2511 (1994)

Foundations of statistical mechanics from symmetries of entanglement
S. Deffner, W. H. Zurek N. J. Phys. 18 (2016) 063013 arXiv:1504.02797

Sub-Planck structures in phase space



$$a \approx \hbar^2 / A$$

Physical significance: WHZ (2001) Sub-Planck spots of Schroedinger cats, NATURE 412, 712-717

Foundations of Statistical Physics from Symmetries of Entanglement

Case Study: Szilard Engine

Wojciech H. Zurek
Los Alamos

S. Deffner & WHZ (2016), Foundations of statistical mechanics from symmetries of entanglement, N. J. Phys. Volume 18, 063013

WHZ (2018), "Eliminating Ensembles from Equilibrium Statistical Physics: Maxwell's Demon, Szilard's Engine, and Thermodynamics via Entanglement", Physics Reports, in press, arxiv:1806.03532

Summary and conclusions:

EQUILIBRIUM ← → "NOTHING HAPPENS"

Without ensembles there is no possibility of equilibrium in Newtonian physics!!!

- Thermodynamics is organized logically around the equilibrium states – states in which nothing happens.
- Yet, in states of individual classical systems "something always happens" – Newtonian dynamics precludes equilibrium.
- Ensembles were introduced to reconcile Newtonian dynamics with thermodynamics.
- In quantum physics states of individual systems which do not evolve can exist – entanglement makes this possible.
- Ensembles were also needed to motivate probabilities (needed for statistical mechanics!).
- We can however get quantum states that embody equilibrium / deduce quantum probabilities w/o ensembles, using symmetries of entanglement.
- Our world is quantum. Therefore, in our quantum world, we can practice statistical physics with individual systems (that embedded in / entangled with their environments).

Statistical ensembles

- In mathematical physics, especially as introduced into statistical mechanics and thermodynamics by J. Willard Gibbs in 1902, an **ensemble** (also **statistical ensemble**) is an idealization consisting of a large number of virtual copies (sometimes infinitely many) of a system, considered all at once, each of which represents a possible state that the real system might be in. **In other words, a statistical ensemble is a probability distribution for the state of the system.**[1]
- A **thermodynamic ensemble** is a specific variety of statistical ensemble that, among other properties, **is in equilibrium**, and **is used to derive the properties of thermodynamic systems from the laws of classical or quantum mechanics.**

EQUILIBRIUM ← → "NOTHING HAPPENS"

Without ensembles there is no possibility of equilibrium in classical Newtonian mechanics!!!

[1] *Gibbs, Josiah Willard (1902). Elementary Principles in Statistical Mechanics. New York: Charles Scribner's Sons.*

Microcanonical Equilibrium via Entanglement

Entangled "even" state of the system $\mathcal{S}$ and the environment $\mathcal{E}$ (the system $\mathcal{S}$ alone in microcanonical equilibrium):

$$|\bar{\Psi}_{\mathcal{SE}}\rangle \propto \sum_{k=1}^{K} e^{i\phi_k}|s_k\rangle|\varepsilon_k\rangle$$

The state of the system alone is NOT evolving dynamically – it is invariant under EVERY unitary…

$$\mathbf{U}_{\mathcal{S}}(\{\tilde{s}_k\} \rightleftharpoons \{s_k\}) = \sum_{|s_k\rangle \in \mathcal{H}_{\mathcal{S}}} |\tilde{s}_k\rangle\langle s_k|$$

…because evolution under any unitary can be undone by acting on the environment -- i.e., w/o acting on the system – with:

$$\mathbf{U}_{\mathcal{E}}(\{\tilde{\varepsilon}_k\} \rightleftharpoons \{\varepsilon_k\}) = \sum_{|\varepsilon_k\rangle \in \mathcal{H}_{\mathcal{E}}} |\tilde{\varepsilon}_k\rangle\langle \varepsilon_k|$$

Proof: It is enough to specify the set of orthogonal states that define the evolution of the environment. In other words, decomposition on RHS of:

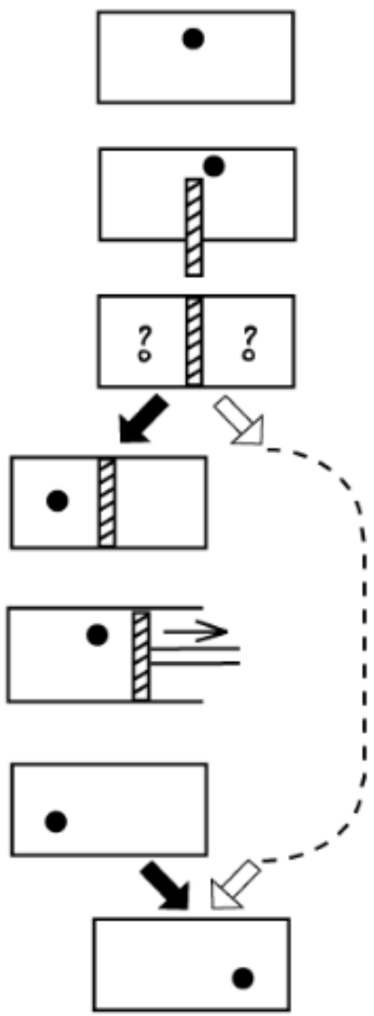
$$|\bar{\Psi}_{\mathcal{SE}}\rangle \propto \sum_{l=1}^{K} |\tilde{s}_l\rangle (\sum_{k=1}^{K} e^{i\phi_k}\langle \tilde{s}_l|s_k\rangle|\varepsilon_k\rangle) = \sum_{l=1}^{K} |\tilde{s}_l\rangle|\tilde{\varepsilon}_l\rangle$$

should be also Schmidt. That is, the set of states:

$$\{|\tilde{\varepsilon}_l\rangle = (\textstyle\sum_{k=1}^{K} e^{i\phi_k}\langle \tilde{s}_l|s_k\rangle|\varepsilon_k\rangle)\}$$

should be an orthogonal basis in the Hilbert space of the environment. This is true, as can be verified. QED.

Szilard engine: Classical version



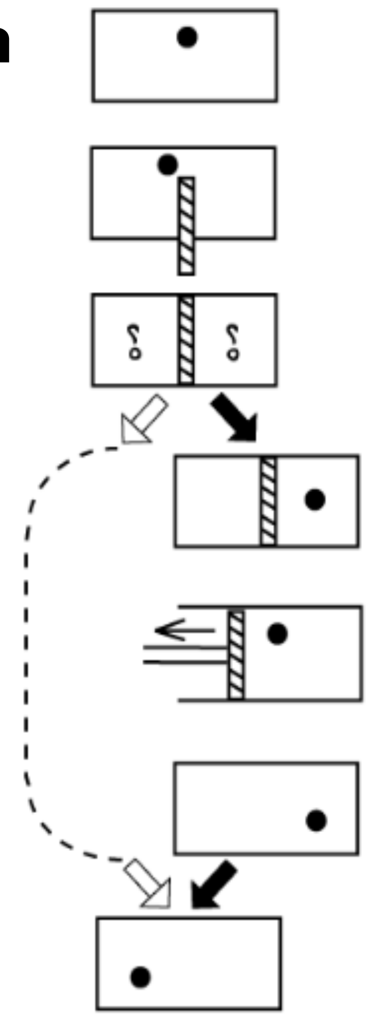
$$\Delta W = \int_{V/2}^{V} p(v)dv = k_B T \int_{V/2}^{V} dv/v = k_B T \ln 2$$

NOTE THE ROLE OF OF THE PARTITION in compressing the one-molecule gas.

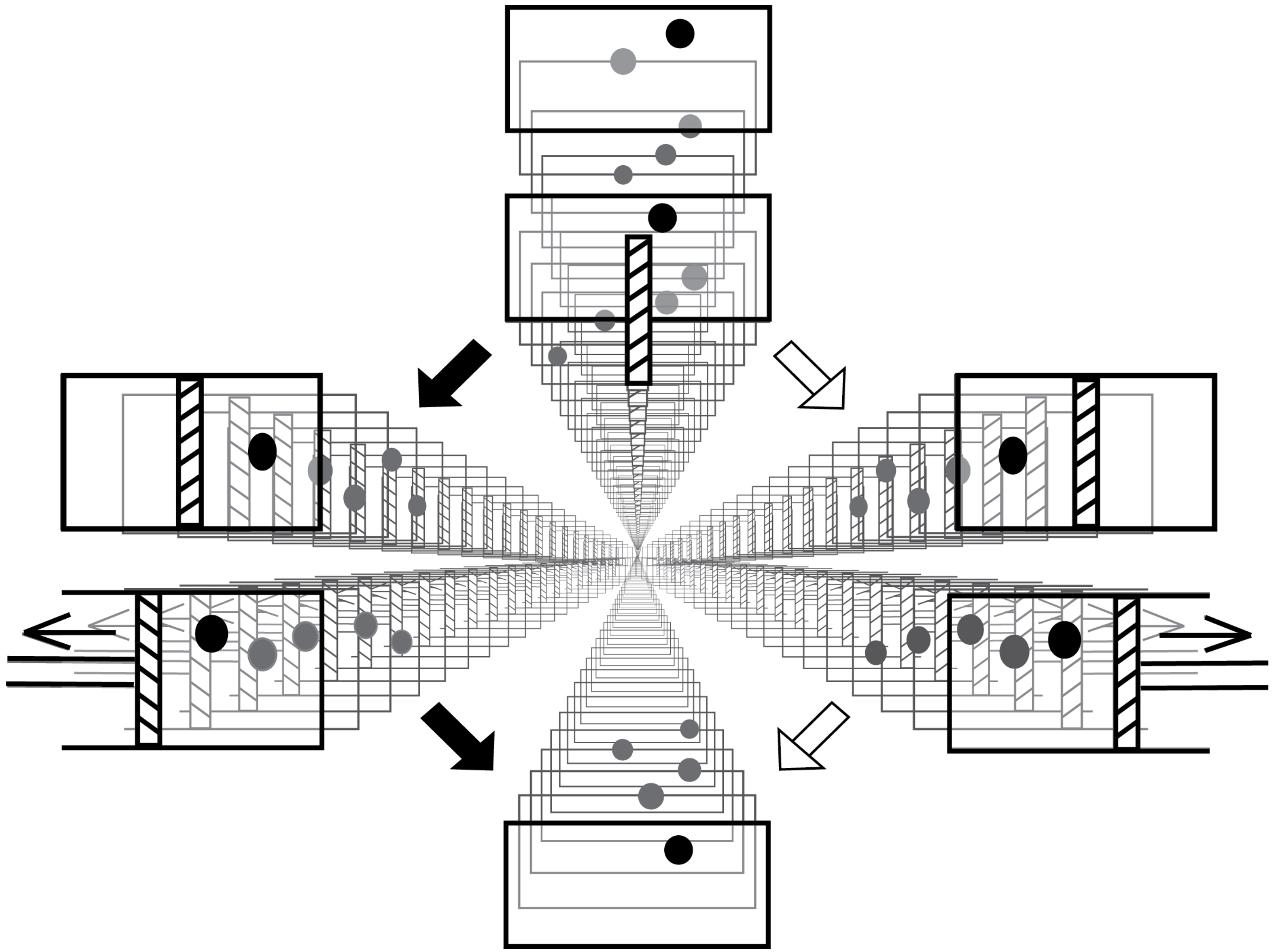
Above, we have used the law of Gay-Lussac, $p = kT/V$, for one-molecule gas.

$$\Delta F = k_B T \ln 2$$

ENSEMBLE ACCOMMODATES BOTH OF THESE ALTERNATIVES



Szilard engine: Classical version



Frequentist view of probability (*relative frequencies* in an ensemble)

- …it is assumed that as the length of a series of trials increases, the fraction of experiments in which a given event occurs will approach a fixed value … the **limiting relative frequency**.
- This interpretation is often contrasted with Bayesian probability…the term 'frequentist' was first used by M. G. Kendall*, to contrast with Bayesians, whom he called "non-frequentists".
- ....we may broadly distinguish two main attitudes. One takes probability as 'a degree of rational belief', or some similar idea...the second defines probability in terms of frequencies of occurrence of events, or by relative proportions in '**populations**' or '**collectives**'
- …. It might be thought that the differences between the frequentists and the non-frequentists … are largely due to the differences of the domains which they purport to cover. …***I assert that this is not so …*** **The essential distinction between the frequentists and the non-frequentists is, I think, that the former, in an effort to avoid anything savouring of matters of opinion, seek to define probability in terms of the objective properties of a population, real or hypothetical, whereas the latter do not. [SUBJECTIVE vs OBJECTIVE]**

IN QUANTUM THEORY ONE CAN DERIVE OBJECTIVE PROBABILITY W/O APPEAL TO RELATIVE FREQUENCIES – W/O APPEAL TO ENSEMBLES

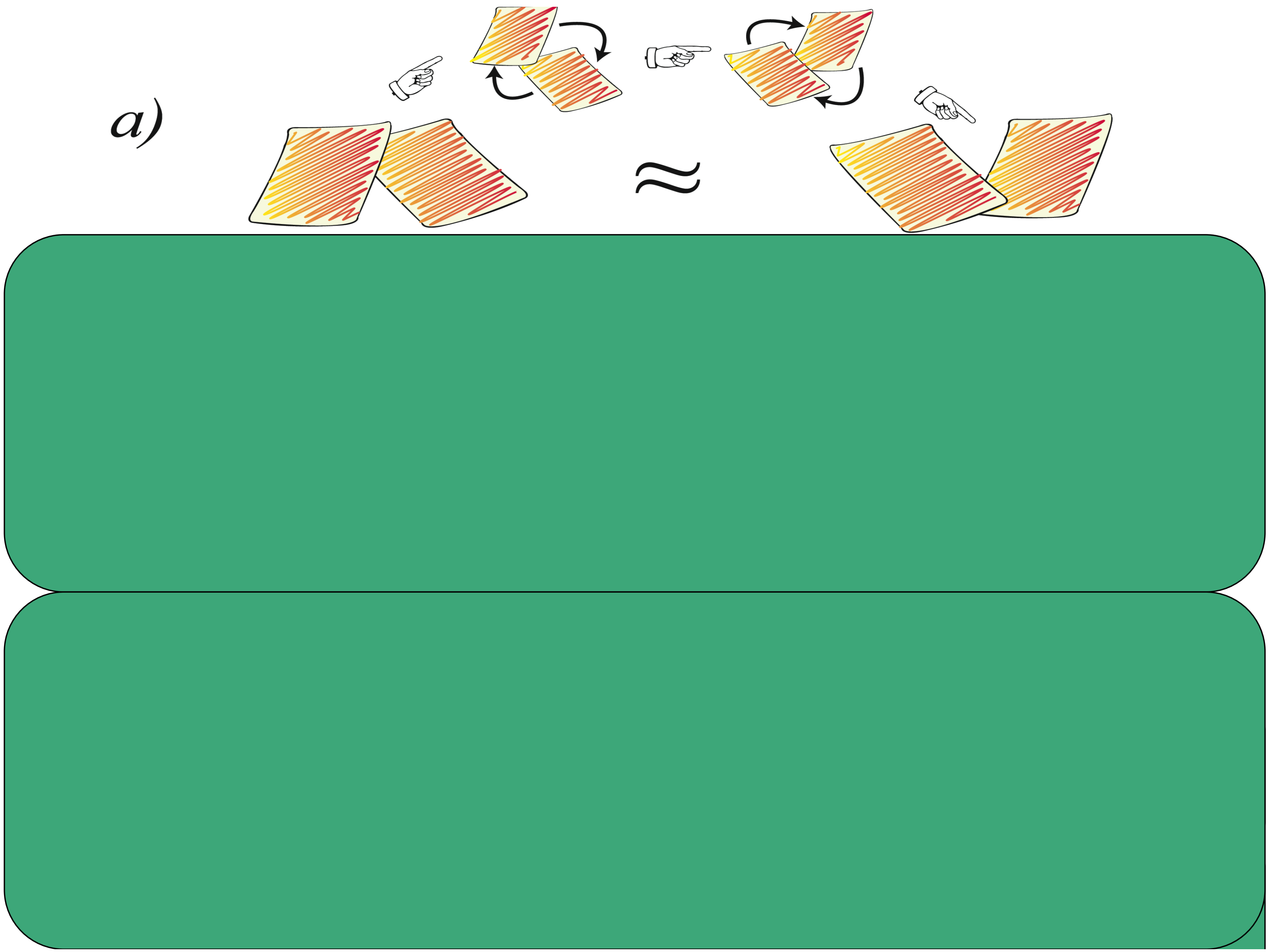
*Kendall, Maurice George (1949). "On the Reconciliation of Theories of Probability". *Biometrika*. Biometrika Trust. **36** (1/2): 101–116

Born’s rule: $p_k = |\psi_k|^2$

- Born’s rule is one of the “textbook axioms” of quantum theory.
- It relates the probability of an outcome of a measurement to the state vector.
- Probability of finding an outcome $|k\rangle$ given $|\Psi\rangle = \sum_k \psi_k |k\rangle$, quantum state vector of the system.
- Born’s rule is essential in connecting mathematics of quantum theory with physics – with the experiments.
- Without it we would just have “quantum mathematics”, not quantum physics.

We first derive Born’s rule to get objective probabilities then use similar ideas to show how one can practice statistical physics in our quantum Universe without ensembles.

a)



ENVARIANCE

(Entanglement-Assisted Invariance)

DEFINITION:

Consider a composite quantum object consisting of system $\mathcal{S}$ and environment $\mathcal{E}$. **When the combined state $\psi_{\mathcal{S}\mathcal{E}}$ is transformed by:**

$$U_{\mathcal{S}} = u_{\mathcal{S}} \otimes \mathbf{1}_{\mathcal{E}}$$

but can be "untransformed" by acting solely on $\mathcal{E}$, that is, if there exists:

$$U_{\mathcal{E}} = \mathbf{1}_{\mathcal{S}} \otimes u_{\mathcal{E}}$$

then $\psi_{\mathcal{S}\mathcal{E}}$ is ENVARIANT with respect to $u_{\mathcal{S}}$.

$$U_{\mathcal{E}}(U_{\mathcal{S}}|\psi_{\mathcal{S}\mathcal{E}}\rangle) = U_{\mathcal{E}}|\varphi_{\mathcal{S}\mathcal{E}}\rangle = |\psi_{\mathcal{S}\mathcal{E}}\rangle$$

Envariance is a property of U_S and the joint state $\psi_{\mathcal{S}\mathcal{E}}$ of two systems, $\mathcal{S}$ & $\mathcal{E}$.

ENTANGLED STATE AS AN EXAMPLE OF ENVARIANCE:

DECOHERENCE AS A LOCAL ENTANGLEMENT-ASSISTED SYMMETRY: DECOHERENCE IS DUE TO ENVARIANCE

Schmidt decomposition:

$$|\psi_{\mathcal{SE}}\rangle = \sum_{k=1}^{N} \alpha_k |s_k\rangle|\varepsilon_k\rangle$$

Above Schmidt states $|s_k\rangle, |\varepsilon_k\rangle$ are orthonormal and α_k complex.

Lemma: Unitary transformations with Schmidt eigenstates:

$$u_{\mathcal{S}}(s_k) = \sum_{k=1} \exp(i\phi_k)|s_k\rangle\langle s_k|$$

leave $\psi_{\mathcal{SE}}$ envariant.

Proof: $u_{\mathcal{S}}(s_k)|\psi_{\mathcal{SE}}\rangle = \sum_{k=1} \alpha_k \exp(i\phi_k)|s_k\rangle|\varepsilon_k\rangle$, $u_{\mathcal{E}}(\varepsilon_k) = \sum_{k=1} \exp\{i(-\phi_k + 2\pi\, l_k)\}|\varepsilon_k\rangle\langle\varepsilon_k|$

$$u_{\mathcal{E}}(\varepsilon_k)\{u_{\mathcal{S}}(s_k)|\psi_{\mathcal{SE}}\rangle\} = \sum_{k=1} \alpha_k \exp\{i(\phi_k - \phi_k + 2\pi\, l_k)|s_k\rangle|\varepsilon_k\rangle = \sum_{k=1} \alpha_k |s_k\rangle|\varepsilon_k\rangle = |\psi_{\mathcal{SE}}\rangle$$

LOCALLY, SCHMIDT PHASES DO NOT MATTER: DECOHERENCE!!!

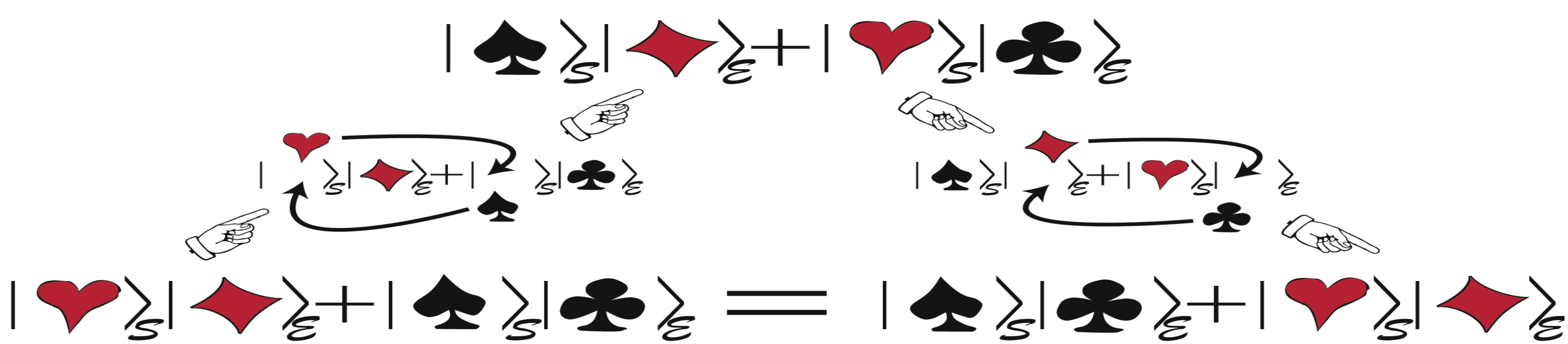
Envariance of entangled states: the case of **equal coefficients**

$$|\psi_{\mathcal{SE}}\rangle \propto \sum_{k=1}^{N} \exp(i\phi_k)|s_k\rangle|\varepsilon_k\rangle$$

In this case ANY orthonormal basis is Schmidt. In particular, in the Hilbert subspace spanned by any two $\{|s_k\rangle, |s_l\rangle\}$ one can define a Hadamard basis;

$$|\pm\rangle = (|s_k\rangle \pm |s_l\rangle)/\sqrt{2}$$

c)



Probability of envariantly swappable states

$$|\psi_{\mathcal{SE}}\rangle \propto \sum_{k=1}^{N} \exp(i\phi_k)|s_k\rangle|\varepsilon_k\rangle$$

By the Phase Envariance Theorem the set of pairs $|\alpha_k|, |s_k\rangle$ provides a complete description of $\mathcal{S}$. But all $|\alpha_k|$ are equal.

With additional assumption about probabilities (e.g., perfect correlation as on the previous transparency), one can prove

THEOREM: Probabilities of envariantly swappable states are equal.

(a) "Pedantic assumption"; when states get swapped, so do probabilities;

(b) When the state of the system does not change under any unitary in a part of its Hilbert space, probabilities of any set of basis states are equal.

(c) Because there is one-to-one correlation between $|s_k\rangle, |\varepsilon_k\rangle$

Therefore, by normalization: $p_k = \frac{1}{N} \quad \forall_k$

Microcanonical Equilibrium via Entanglement

Entangled "even" state of the system and the environment (the system alone in microcanonical equilibrium):

$$|\bar{\Psi}_{\mathcal{S}\mathcal{E}}\rangle \propto \sum_{k=1}^{K} e^{i\phi_k} |s_k\rangle |\varepsilon_k\rangle$$

The state of the system alone is NOT evolving dynamically – it is invariant under EVERY unitary…

$$\mathbf{U}_{\mathcal{S}}(\{\tilde{s}_k\} \rightleftharpoons \{s_k\}) = \sum_{|s_k\rangle \in \mathcal{H}_{\mathcal{S}}} |\tilde{s}_k\rangle\langle s_k|$$

…because evolution under any unitary can be undone by acting on the environment -- i.e., w/o acting on the system – with:

$$\mathbf{U}_{\mathcal{E}}(\{\tilde{\varepsilon}_k\} \rightleftharpoons \{\varepsilon_k\}) = \sum_{|\varepsilon_k\rangle \in \mathcal{H}_{\mathcal{E}}} |\tilde{\varepsilon}_k\rangle\langle \varepsilon_k|$$

Proof: It is enough to specify the set of orthogonal states that define the evolution of the environment. In other words, decomposition on RHS of:

$$|\bar{\Psi}_{\mathcal{S}\mathcal{E}}\rangle \propto \sum_{l=1}^{K} |\tilde{s}_l\rangle \Big(\sum_{k=1}^{K} e^{i\phi_k} \langle \tilde{s}_l | s_k\rangle |\varepsilon_k\rangle\Big) = \sum_{l=1}^{K} |\tilde{s}_l\rangle |\tilde{\varepsilon}_l\rangle$$

should be also Schmidt. That is, the set of states:

$$\{|\tilde{\varepsilon}_l\rangle = (\textstyle\sum_{k=1}^{K} e^{i\phi_k} \langle \tilde{s}_l | s_k\rangle |\varepsilon_k\rangle)\}$$

should be an orthogonal basis in the Hilbert space of the environment. This is true, as can be verified. QED.

Special case with **unequal** coefficients

Consider system $\mathcal{S}$ with two states $\{|0\rangle, |2\rangle\}$

The environment $\mathcal{E}$ has three states $\{|0\rangle, |1\rangle, |2\rangle\}$ and $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$

$$|\psi_{\mathcal{SE}}\rangle = \sqrt{\frac{2}{3}}|0\rangle|+\rangle + \sqrt{\frac{1}{3}}|2\rangle|2\rangle$$

An auxilliary environment $\mathcal{E}'$ interacts with $\mathcal{E}$ so that:

$$|\psi_{\mathcal{SE}}\rangle|\mathcal{E}'_0\rangle = \left(\sqrt{\frac{2}{3}}|0\rangle|+\rangle + \sqrt{\frac{1}{3}}|2\rangle|2\rangle\right)|0\rangle \Rightarrow \sqrt{\frac{2}{3}}|0\rangle(|0\rangle|0\rangle + |1\rangle|1\rangle)/\sqrt{2} + \sqrt{\frac{1}{3}}|2\rangle|2\rangle|2\rangle =$$

$$= (|0\rangle|0\rangle|0\rangle + |0\rangle|1\rangle|1\rangle + |2\rangle|2\rangle|2\rangle)/\sqrt{3}$$

States $|0\rangle|0\rangle$, $|0\rangle|1\rangle$, $|2\rangle|2\rangle$ have equal coefficients. Therefore, Each of them has probability of 1/3. Consequently:

p(0) = p(0,0)+p(0,1) = 2/3, and p(2) = 1/3.

….. BORN's RULE!!! **no need to assume additivity!** (p(0)=1-p(2))!

Probabilities from Envariance

The case of commensurate probabilities: $|\psi_{\mathcal{SE}}\rangle = \sum_{k=1}^{N} \underbrace{\sqrt{m_k/M}}_{\alpha_k} |s_k\rangle |\varepsilon_k\rangle$

Attach the auxiliary "counter" environment $\mathcal{C}$:

$$|\psi_{\mathcal{SE}}\rangle |e'_0\rangle = \left(\sum_{k=1}^{N} \underbrace{\sqrt{m_k/M}}_{\alpha_k} |s_k\rangle \left(\underbrace{\sum_{j_k=1}^{m_k} \frac{1}{\sqrt{m_k}} |e_{j_k}\rangle}_{|\varepsilon_k\rangle} \right) \right) |c_0\rangle \Rightarrow$$

$$\Rightarrow \frac{1}{\sqrt{M}} \sum_{j=1}^{M} |s_{k(j)}\rangle |e_j\rangle |c_j\rangle$$

THEOREM 3: The case with commensurate probabilities can be reduced to the case with equal probabilities. **BORN's RULE follows:**

$$p_j = \frac{1}{M}, \qquad p_k = \sum_{j_k=1}^{m_k} p_{j_k} = \frac{m_k}{M} = |\alpha_k|^2$$

General case -- by continuity. QED.

ENVARIANCE* -- SUMMARY

1. New symmetry - ENVARIANCE - of joint states of quantum systems. It is related to causality.

2. In quantum physics perfect knowledge of the whole may imply complete ignorance of a part.

3. **BORN's RULE** follows as a consequence of envariance – objective symmetry of entangled quantum states yields objective probabilities.

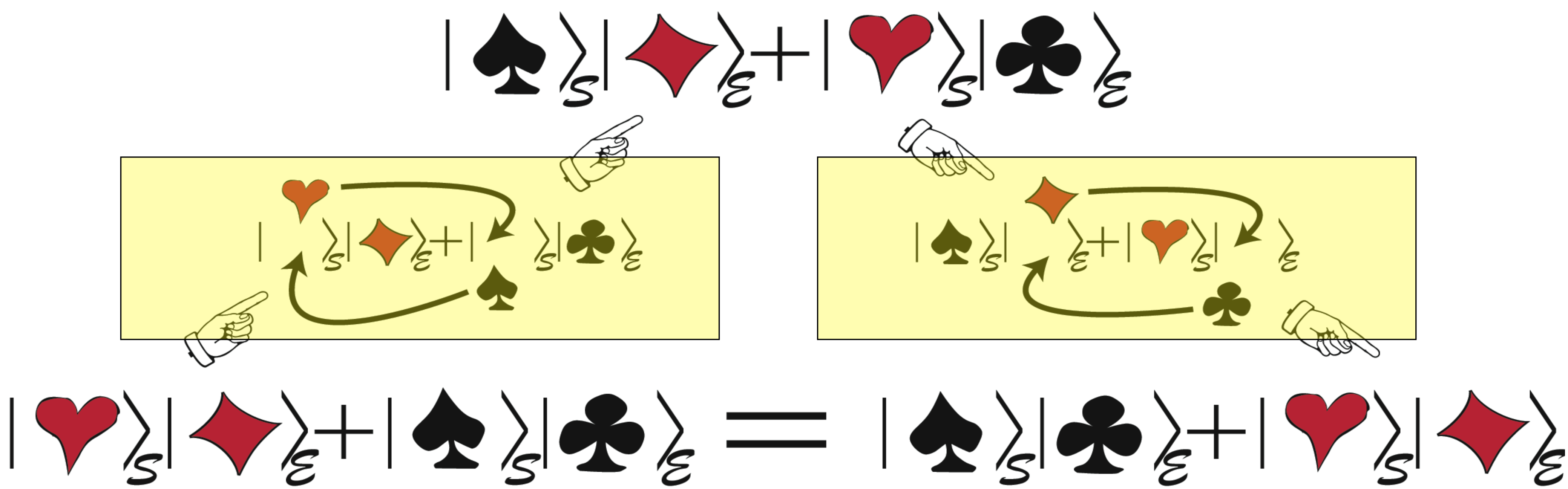
4. Relative frequency interpretation of probabilities naturally follows.

5. Envariance supplies a new foundation for environment - induced superselection, decoherence, quantum statistical physics, etc., by justifying the form and interpretation of reduced density matrices.

We have derived objective probabilities without employing ensembles -- without relative frequencies. Now we show ensembles are not needed for statistical physics…

*WHZ, PRL 90, 120404; RMP 75, 715 (2003); PRA 71, 052105 (2005); PRL 106, 250402 (2011), Physics Today Oct. (2014).

Microcanonical equilibrium state of a single system – envariant definition



The microcanonical equilibrium of a system $\mathcal{S}$ with Hamiltonian $H_{\mathcal{S}}$ is an energetically degenerate quantum state envariant under all unitaries.

Canonical equilibrium state of a single system from envariance

Let us now imagine that we can separate the total system $\mathcal{S}$ into a smaller subsystem of interest $\mathfrak{S}$ and its complement, which we call heat bath $\mathfrak{B}$. The Hamiltonian of $\mathcal{S}$ can then be written as

$$H_{\mathcal{S}} = H_{\mathfrak{S}} \otimes \mathbb{I}_{\mathfrak{B}} + \mathbb{I}_{\mathfrak{S}} \otimes H_{\mathfrak{B}} + h_{\mathfrak{S},\mathfrak{B}}, \tag{9}$$

where $h_{\mathfrak{S},\mathfrak{B}}$ denotes an interaction term. Physically this term is necessary to facilitate exchange of energy between the $\mathfrak{S}$ and the heat bath $\mathfrak{B}$. In the following, however, we will assume that $h_{\mathfrak{S},\mathfrak{B}}$ is sufficiently small so that we can neglect its contribution to the total energy, $E_{\mathcal{S}} = E_{\mathfrak{S}} + E_{\mathfrak{B}}$, and its effect on the composite equilibrium state $|\psi_{\mathcal{S}\mathcal{E}}\rangle$. These assumptions are in complete analogy to the ones of classical statistical mechanics [12, 13]. They will, however, be relaxed in a final part of the analysis.

Under these assumptions every composite energy eigenstate $|s_k\rangle$ can be written as a product

$$|s_k\rangle = |\mathfrak{s}_k\rangle \otimes |\mathfrak{b}_k\rangle, \tag{10}$$

where the states $|\mathfrak{s}_k\rangle$ and $|\mathfrak{b}_k\rangle$ are energy eigenstates in $\mathfrak{S}$ and $\mathfrak{B}$, respectively. At this point envariance is crucial in our treatment: all orthonormal bases are equivalent under envariance (see footnote 5). Therefore, we can choose $|s_k\rangle$ as energy eigenstates of $H_{\mathcal{S}}$.

Boltzmann-Gibbs formula for thermal equilibrium state

We are now ready to derive the Boltzmann–Gibbs formula. To this end consider that in the limit of very large, $N \gg 1$, $\mathfrak{N}(\mathfrak{e}_k)$ (17) can be approximated with Stirling's formula. We have

$$\ln(\mathfrak{N}(\mathfrak{e}_k)) = N \ln(N) - \sum_{j=1}^{m} n_j \ln(n_j). \tag{18}$$

As pointed out earlier, thermodynamic equilibrium states are characterized by a maximum of symmetry or maximal number of 'involved energy states', which corresponds classically to a maximal volume in phase space. In the case of the microcanonical equilibrium this condition was met by the state that is maximally envariant, namely envariant under all unitary maps. Now, following Boltzmann's line of thought we identify the canonical equilibrium by the configuration of the heat reservoir $\mathfrak{B}$ for which the maximal number of energy eigenvalues are occupied. Under the constraints

$$\sum_{j=1}^{m} n_j = N \quad \text{and} \quad E_{\mathcal{S}} - \mathfrak{e}_k = \sum_{j=1}^{m} n_j \, e_j^{\mathfrak{B}} \tag{19}$$

this problem can be solved by variational calculus. One obtains

Single quantum system – no ensemble!

$$n_j = \mu \exp(\lambda \, e_j^{\mathfrak{B}}), \tag{20}$$

which is the celebrated Boltzmann–Gibbs formula. Notice that equation (20) is the number of states in the heat reservoir $\mathfrak{B}$ with energy $e_j^{\mathfrak{B}}$ for $\mathfrak{S}$ and $\mathfrak{B}$ being in thermodynamic, canonical equilibrium. In this treatment temperature merely enters through the Lagrangian multiplier λ.

Szilard engine: Classical & quantum versions

$$\Delta W = \int_{V/2}^{V} p(v)dv = k_B T \int_{V/2}^{V} dv/v = k_B T \ln 2$$

Above, we have used the law of Gay-Lussac,

$p = kT/V$, for one-molecule gas. $\Delta F = k_B T \ln 2$

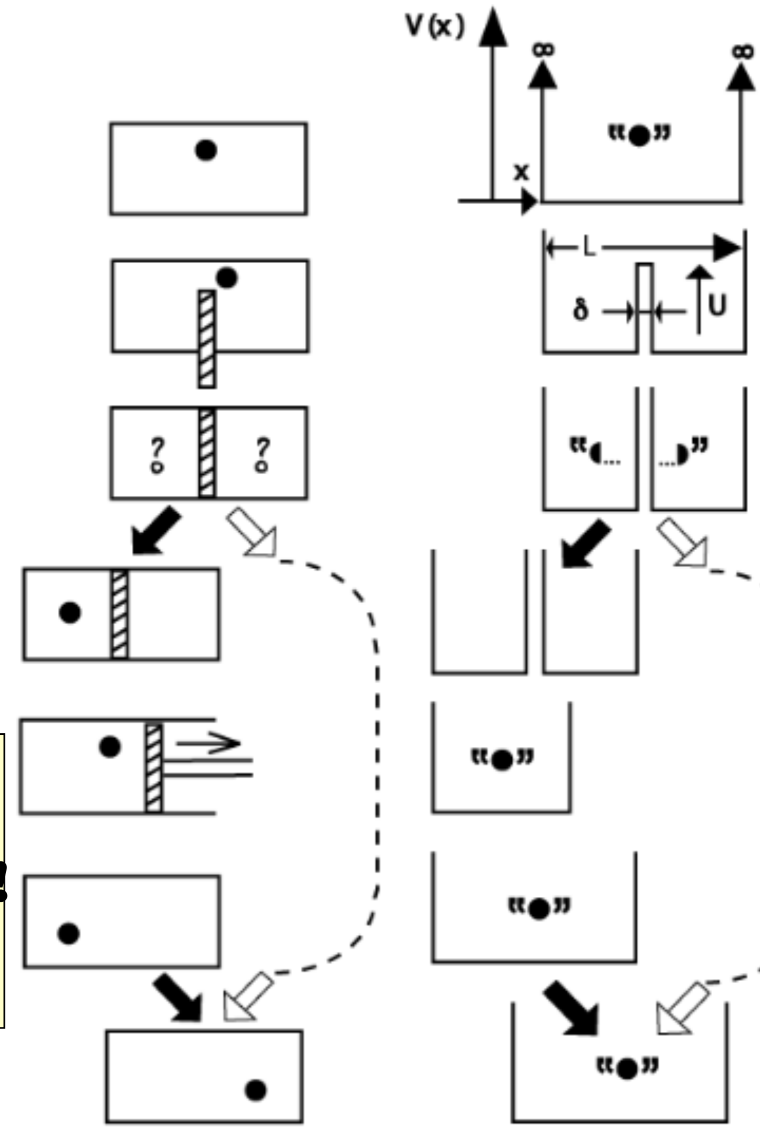
$$\rho = Z^{-1} \sum \exp(-\beta E_n)|\psi_n\rangle\langle\psi_n| \ .$$

Single quantum system - no ensemble!

$$\tilde{\rho} = \tilde{Z}^{-1} \sum_{k=1}^{\infty} \exp(\beta E_k) \{\exp(-\beta\Delta_k)|\psi_k^+\rangle\langle\psi_k^+|\} + \{\exp(\beta\Delta_k)|\psi_k^-\rangle\langle\psi_k^-|\}$$

Single quantum system - no ensemble!

$$\tilde{\rho} = \tilde{Z}^{-1} \sum_{k=1}^{\infty} \exp(-\beta E_k) \{\cosh(\beta\Delta_k)(|L_k\rangle\langle L_k| + |R_k\rangle\langle R_k|) + \sinh(\beta\Delta_k)(|L_k\rangle\langle R_k| + |R_k\rangle\langle L_k|)\}$$



Consider a measuring apparatus which, when inserted into Szilard's engine, determines on which side the molecule is. Formally, this can be accomplished by the measurement of the observable

$$\hat{\Pi} = \lambda(|L\rangle\langle L| - |R\rangle\langle R|) \quad .$$

Here λ is an arbitrary eigenvalue while;

$$|L\rangle\langle L| = \sum_{k=1}^{N} |L_k\rangle\langle L_k| \ , \qquad |R\rangle\langle R| = \sum_{k=1}^{N} |R_k\rangle\langle R_k| \ ,$$

The density matrix before the measurement, but after piston is inserted, is

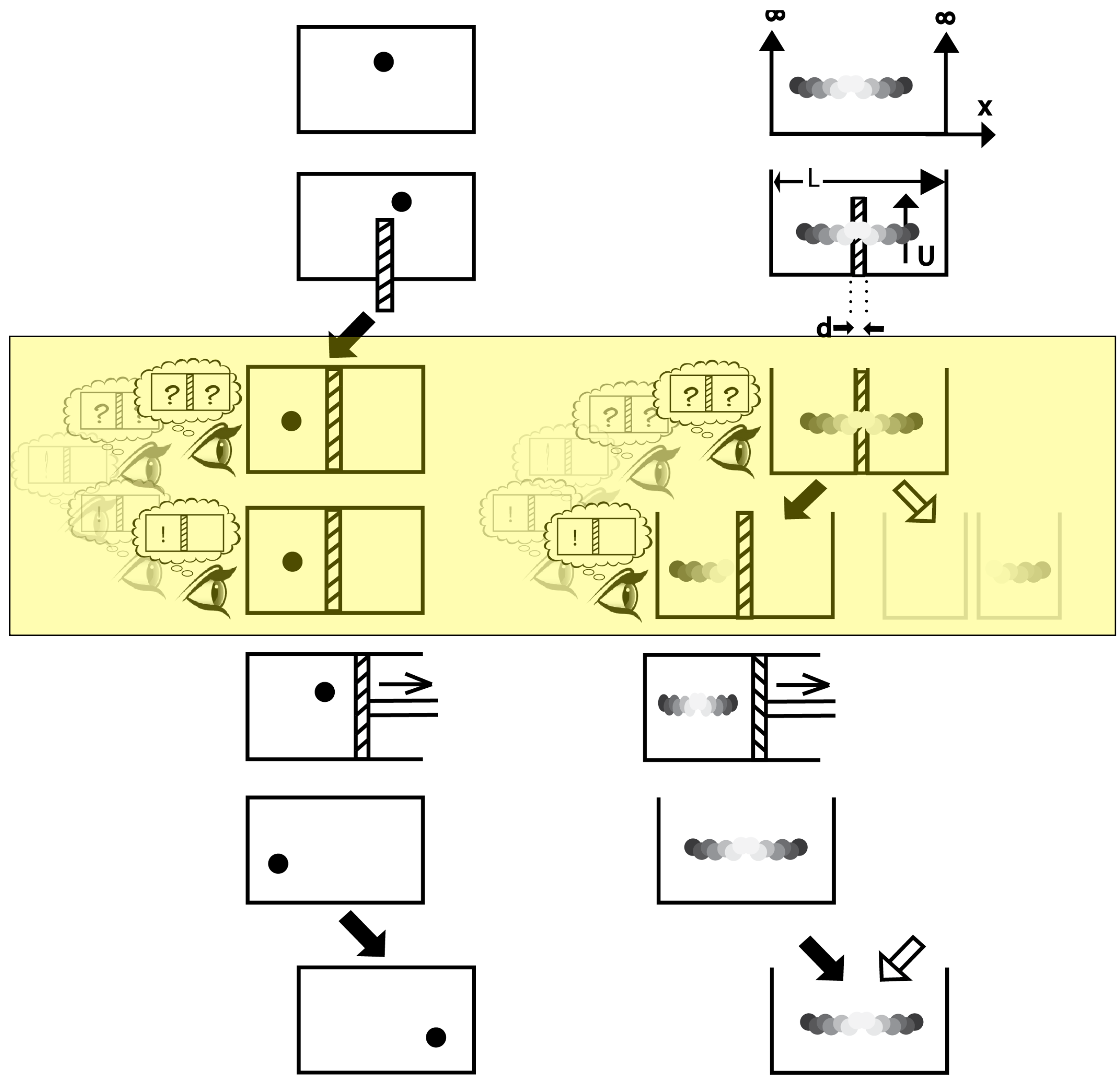
$$\tilde{\rho} = \tilde{Z}^{-1} \sum_{k=1}^{\infty} \exp(\beta E_k) \left\{ \exp(-\beta\Delta_k)|\psi_k^+\rangle\langle\psi_k^+| \right\} + \left\{ \exp(\beta\Delta_k)|\psi_k^-\rangle\langle\psi_k^-| \right\}$$

$$\tilde{\rho} = \tilde{Z}^{-1} \sum_{k=1}^{\infty} \exp(-\beta E_k) \left\{ \cosh(\beta\Delta_k)(|L_k\rangle\langle L_k| + |R_k\rangle\langle R_k|) + \sinh(\beta\Delta_k)(|L_k\rangle\langle R_k| + |R_k\rangle\langle L_k|) \right\}$$

Depending on the outcome of the observation, the density matrix becomes either ρ_L or ρ_L where;

Single quantum system - no ensemble!

$$\rho_L = Z_L^{-1} \sum_{k=1}^{\infty} \exp(-\beta E_k) \cosh \beta\Delta_k |L_k\rangle\langle L_k| \quad , \quad \rho_R = Z_R^{-1} \sum_{k=1}^{\infty} \exp(-\beta E_k) \cosh \beta\Delta_k |R_k\rangle\langle R_k| \quad .$$



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Summary and conclusions:

EQUILIBRIUM ← → "NOTHING HAPPENS"

Without ensembles there is no possibility of equilibrium in Newtonian physics!!!

- Thermodynamics is organized logically around the equilibrium states – states in which nothing happens.
- Yet, in states of individual classical systems "something always happens" – Newtonian dynamics precludes equilibrium.
- Ensembles were introduced to reconcile Newtonian dynamics with thermodynamics.
- In quantum physics states of individual systems which do not evolve can exist – entanglement makes this possible.
- Ensembles were also needed to motivate probabilities (needed for statistical mechanics!).
- We can however get quantum states that embody equilibrium / deduce quantum probabilities w/o ensembles, using symmetries of entanglement.
- Our world is quantum. Therefore, in our quantum world, we can practice statistical physics with individual systems (that embedded in / entangled with their environments).

S. Deffner & WHZ (2016), Foundations of statistical mechanics from symmetries of entanglement, N. J. Phys. Volume 18, 063013

WHZ (2018), "Eliminating Ensembles from Equilibrium Statistical Physics: Maxwell's Demon, Szilard's Engine, and Thermodynamics via Entanglement", Physics Reports, in press, arxiv:1806.03532