Complete positivity in presence of initial correlations: A pathway to complete theory for non-Markovian quantum processes

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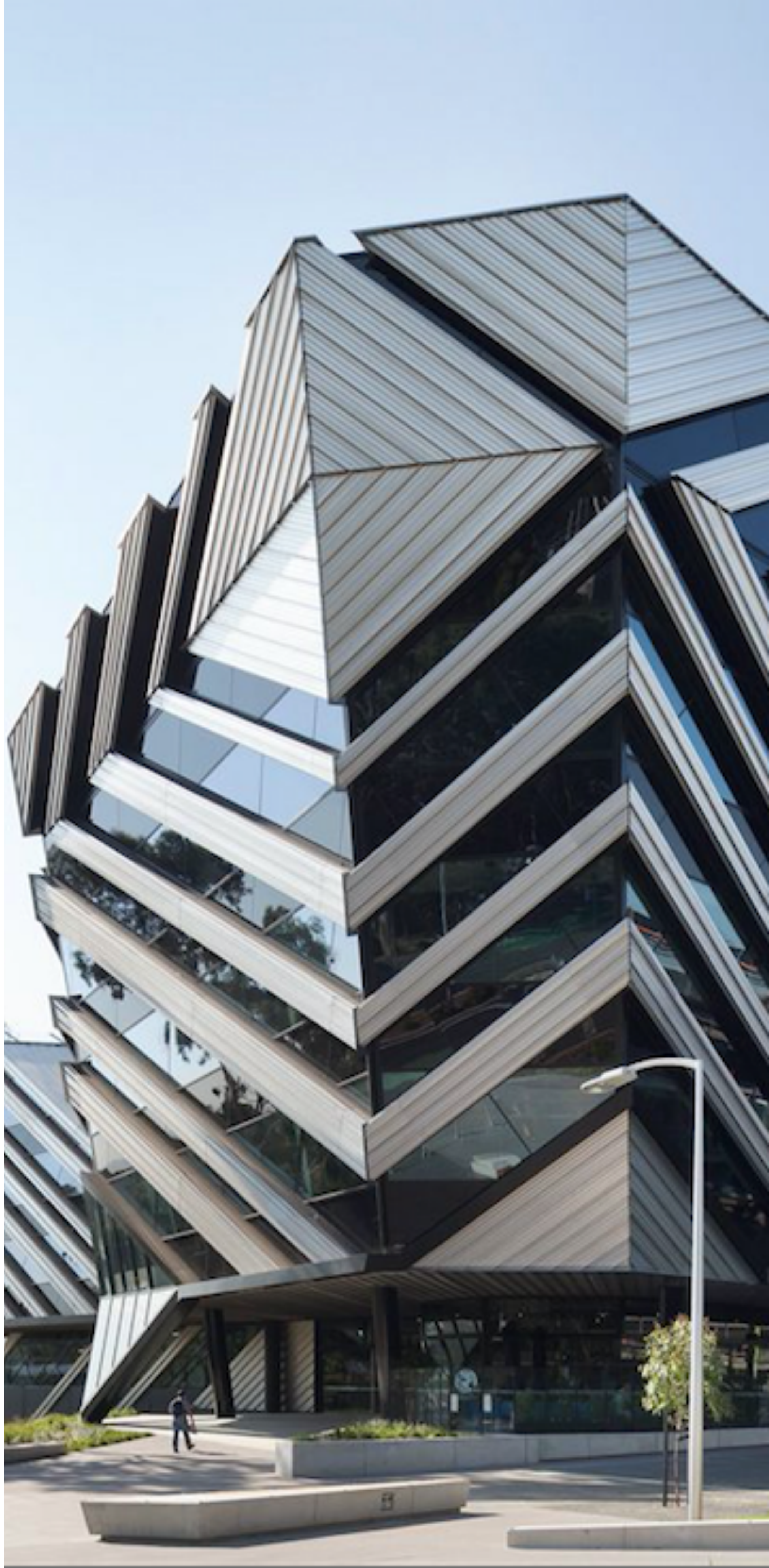

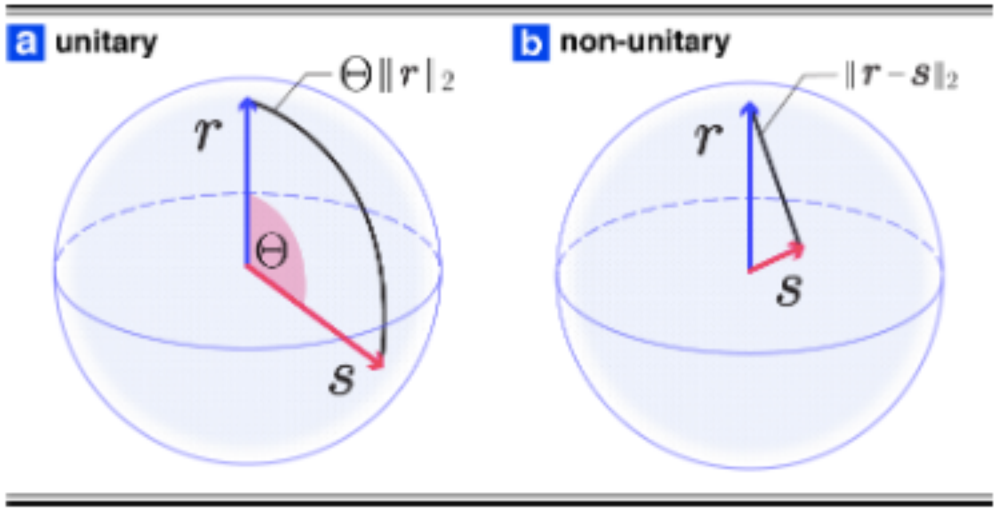

Tightening Quantum Speed Limits for Almost All States

Francesco Campaioli, Felix A. Pollock, Felix C. Binder, and Kavan Modi
Phys. Rev. Lett. **120**, 060409 – Published 9 February 2018

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Featured in Physics

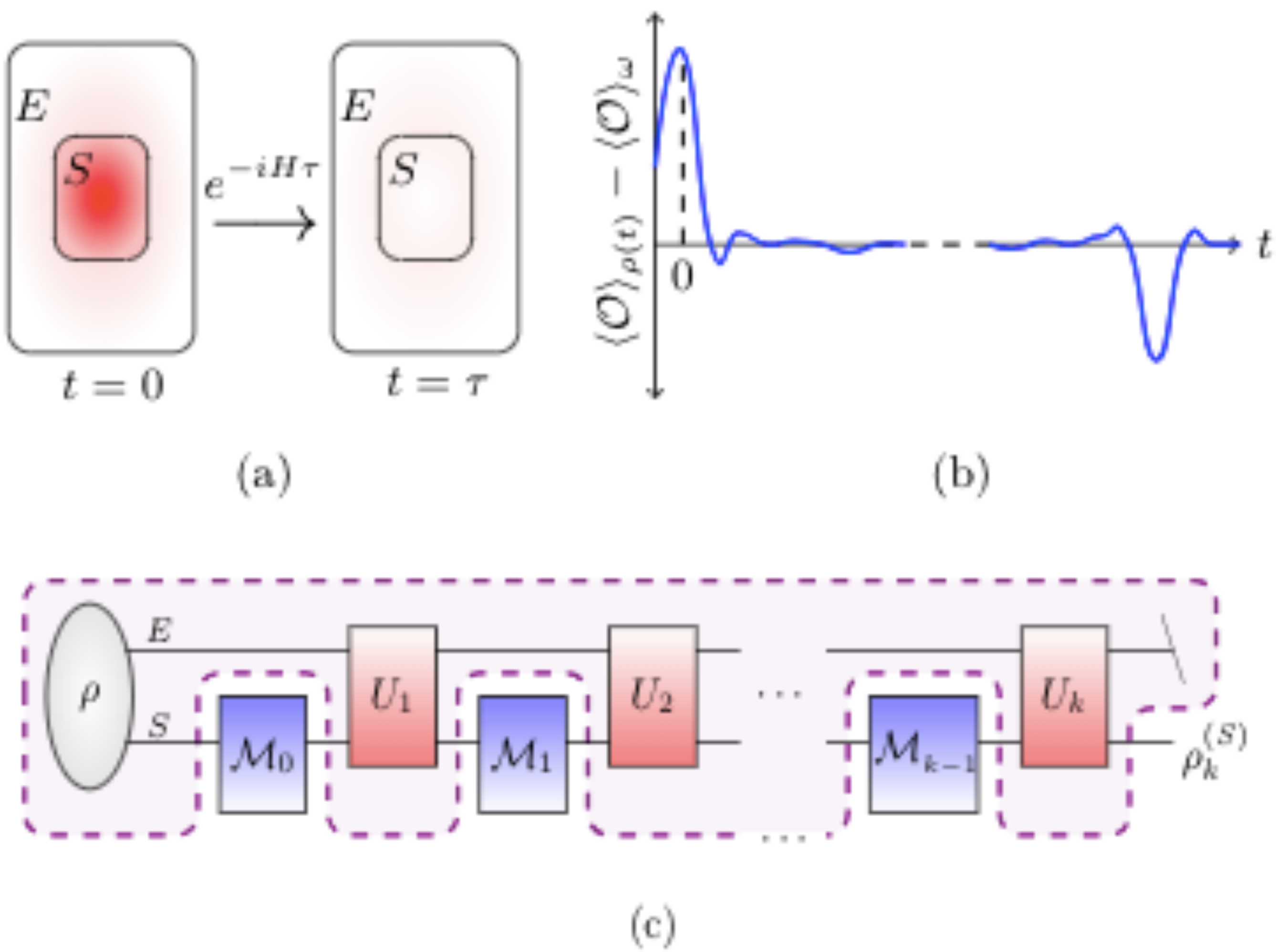
Enhancing the Charging Power of Quantum Batteries

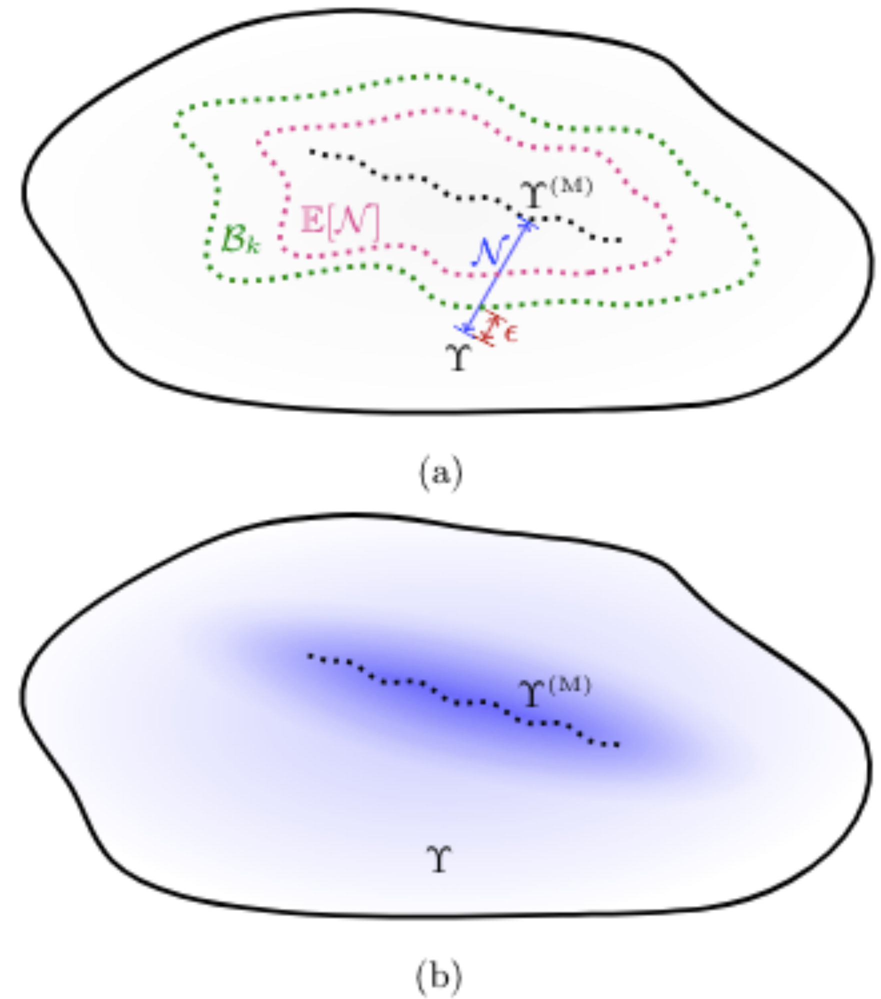
Francesco Campaioli, Felix A. Pollock, Felix C. Binder, Lucas Céleri, John Goold, Sai Vinjanampathy, and Kavan Modi
Phys. Rev. Lett. **118**, 150601 – Published 12 April 2017

1 Dynamics with initial correlations

2 Non-markovian quantum process (complete theory)

Almost Markovian processes from closed dynamics





P. Figueroa-Romero, K. Modi, F. Pollock arXiv:1802.10344 (2018)

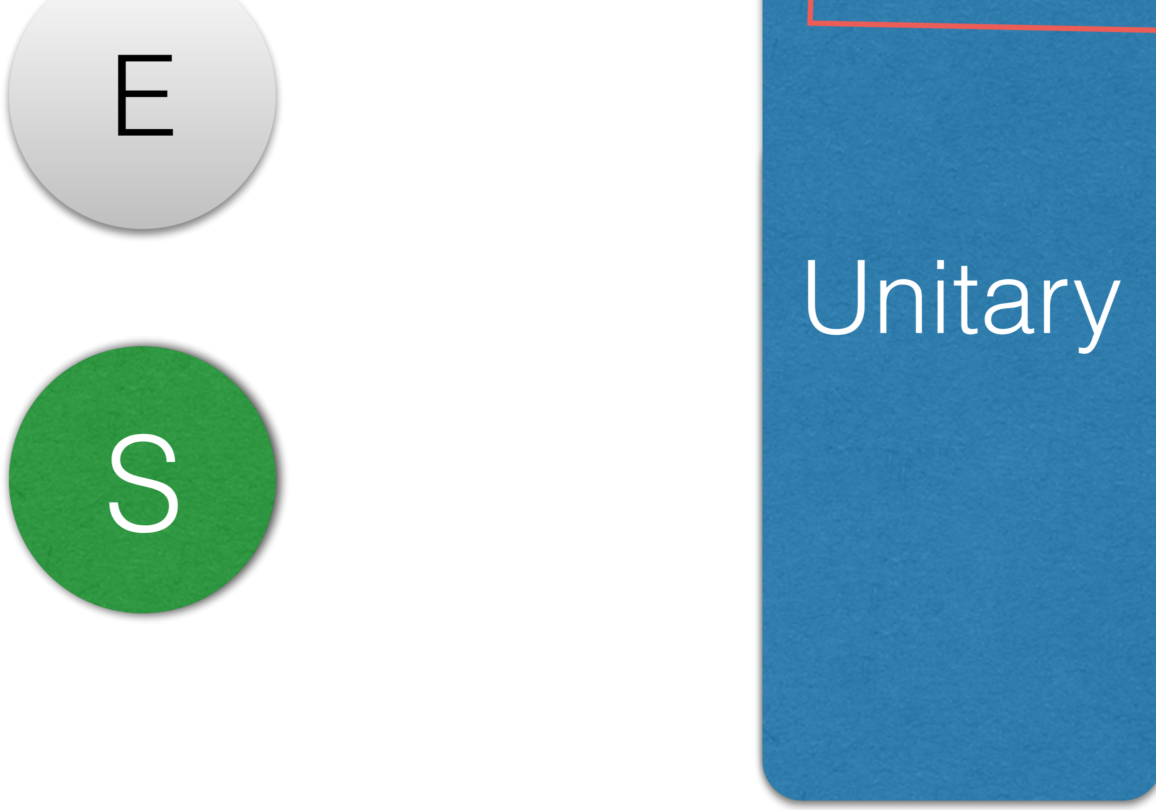
Completely positive maps

Contracted unitary dynamics

$$\mathrm{Tr}_{\mathrm{env}}[U\rho_i \otimes \rho^{\mathrm{env}}\ U^\dagger] = \rho_f$$

Dynamical map

$$\Lambda(\rho_i) = \rho_f \quad \Leftrightarrow \quad \mathrm{Tr}_{\mathrm{env}}[U \rho_i \otimes \tau \, U^\dagger] = \rho_f$$



We can use the system to characterise the Blackbox.

Properties of map

- Dynamical maps = Reduced unitary dynamics

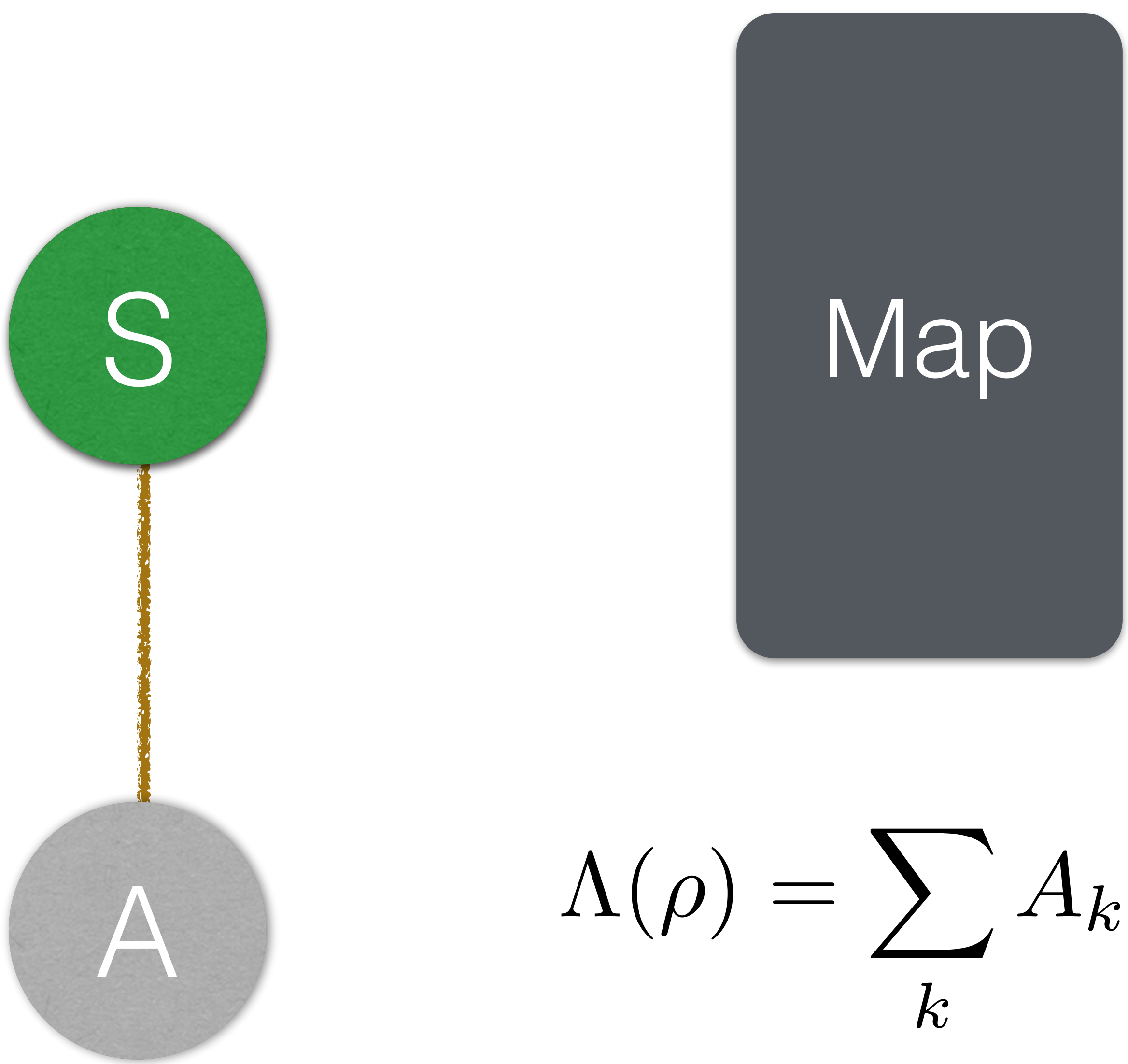
- Linear $$\Lambda(a\,\rho + b\,\sigma) = a\,\Lambda(\rho) + b\,\Lambda(\sigma)$$

- Completely positive $$\Lambda(\rho) = \sum_k A_k\,\rho A_k^\dagger$$

- Contractivity $$D(\rho,\sigma) \geq D(\Lambda[\rho],\Lambda[\sigma])$$

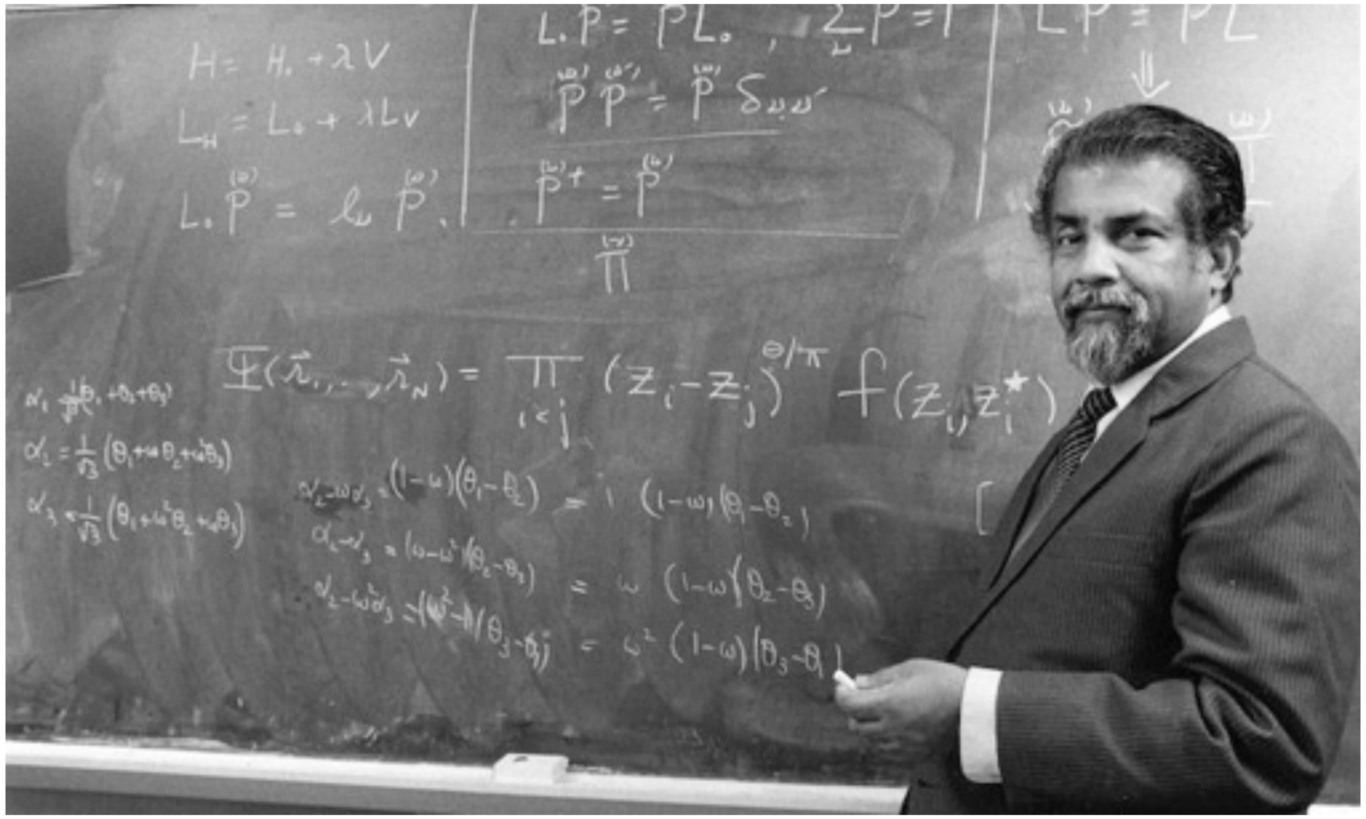
We can derive *any* master equation from a family of maps

Complete positivity



$$\Lambda(\rho) = \sum_k A_k \, \rho A_k^\dagger$$

Theory of weak force (V-A theory),
Optical theory of coherence,
The Spin statistic theorem,
Quantum Zeno effect,
and so on…



1931 - 2018

PHYSICAL REVIEW VOLUME 121, NUMBER 3 FEBRUARY 1, 1961

Stochastic Dynamics of Quantum-Mechanical Systems

E. C. G. SUDARSHAN*
Department of Physics and Astronomy, University of Rochester, New York

P. M. MATHEWS
Department of Physics, University of Madras, Madras, India

AND

JAYASEETHA RAU†
Department of Physics, Brandeis University, Waltham, Massachusetts

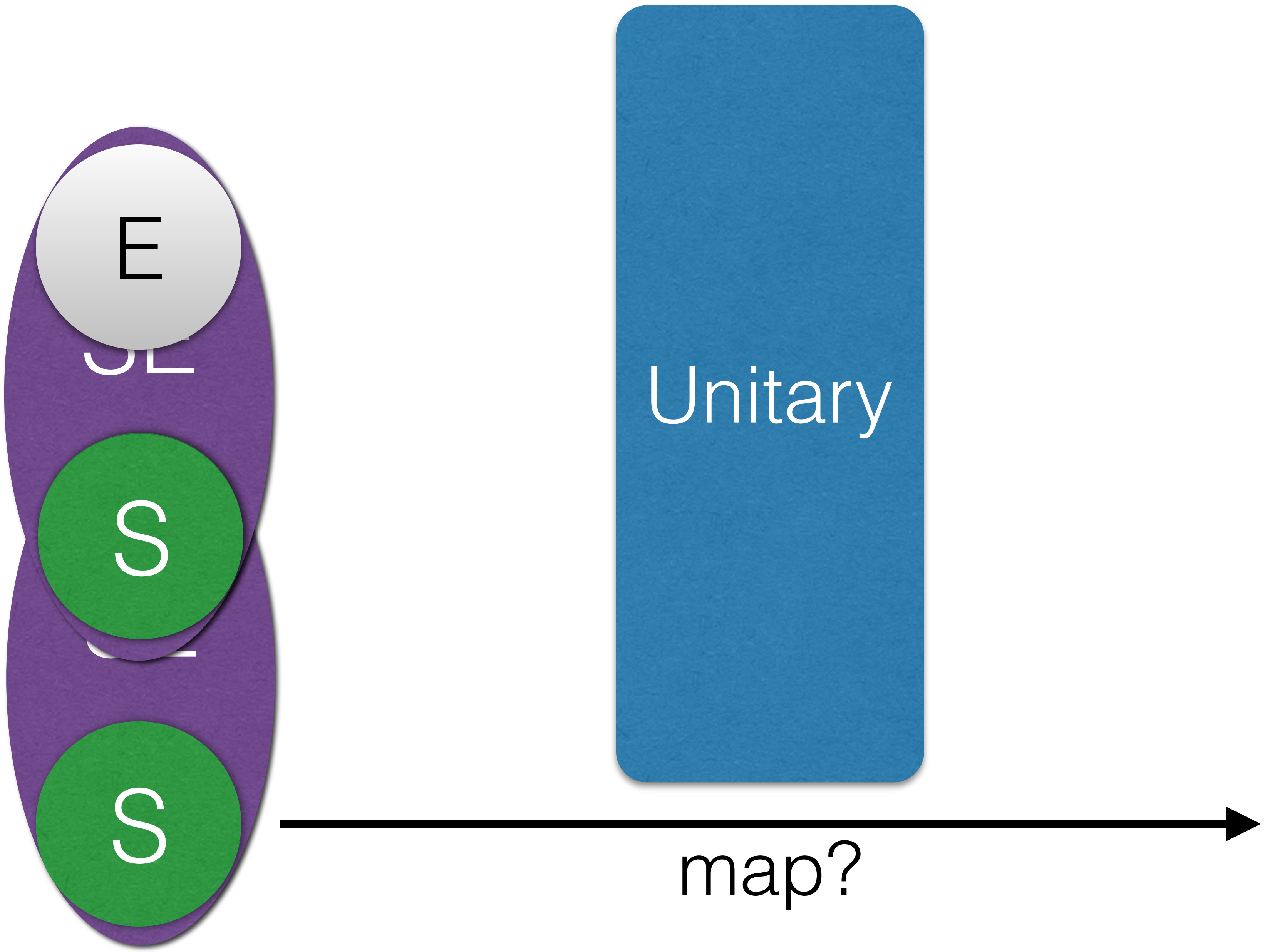
(Received August 15, 1960)

The most general dynamical law for a quantum mechanical system with a finite number of levels is formulated. A fundamental role is played by the so-called "dynamical matrix" whose properties are stated in a sequence of theorems. A necessary and sufficient criterion for distinguishing dynamical matrices corresponding to a Hamiltonian time-dependence is formulated. The non-Hamiltonian case is discussed in detail and the application to paramagnetic relaxation is outlined.

Nearly a decade before Kraus!

Beyond completely positivity?

Initially correlated SE



Pechukas: We must give up **complete positivity** or **linearity**

Alicki: Pechukas theorem is not operational

Pechukas:

$$\rho_i^s \rightarrow \Phi(\rho_i^s) = \rho_i^{se}$$

$$\Phi(x\rho_i^s + y\sigma_i^s) = x\Phi(\rho_i^s) + y\Phi(\sigma_i^s) \qquad \mathrm{Tr}_e\Phi(\rho_i^s) = \rho_i^s \qquad \Phi(\rho_i^s) \geq 0$$

$$\Phi(\rho_i^s) = \rho_i^s \otimes \rho^e$$

Alicki:

Unfortunately, it is impossible to specify such a domain of positivity for a general case, and moreover there exists no physical motivation in terms of operational prescription which could lead to the assignment [Eq. (2)]. The point of this Comment is to propose a mathematically consistent scheme with a clear operational meaning for the reduced dynamics. Let us introduce a completely positive map T ("operation") $\rho_S \mapsto T\rho_S = \sum_n V_n \rho_S V_n^*$, $\sum_n V_n^* V_n = I$ (a sum can be replaced by an integral) which represents the influence of a certain instrument preparing the state, and define the associated assignment map as

$$\Phi\rho_S = \sum_n V_n \rho_S V_n^* \otimes \mathrm{tr}_S(V_n^* V_n \rho_{SR}^{\mathrm{eq}})/\mathrm{tr}(V_n^* V_n \rho_{SR}^{\mathrm{eq}}) . \qquad (3)$$

In conclusion, one should stress that beyond the weak coupling regime there exists no unique definition of the quantum reduced dynamics. It is due to the fact that any physical process of preparation of the initial state of S disturbs the state of R as well. Choosing mathematical models of the preparation process (operation T), one can define consistently various assignment maps and hence various reduced dynamics. All of them satisfy the fundamental positivity condition and can be expressed in terms of completely positive maps.

$$\rho^{se} = \rho_i \otimes \rho^{\mathrm{env}} + \chi^{se} \geq 0$$

Shaji, Jordan, Sudarshan looked at entangled states

Carteret, Terno, Życzkowski looked at separable states

Rodríguez-Rosario, Modi, Kuah, Shaji, Sudarshan looked at classical states

Subsequently paper by Shabani and Lidar incorrectly claimed N&S condition

Quantum computing and quantum process tomography

Realization of quantum process tomography in NMR

Andrew M. Childs, Isaac L. Chuang, and Debbie W. Leung
Phys. Rev. A **64**, 012314 – Published 13 June 2001

Quantum Process Tomography of a Controlled-NOT Gate

J. L. O'Brien, G. J. Pryde, A. Gilchrist, D. F. V. James, N. K. Langford, T. C. Ralph, and A. G. White
Phys. Rev. Lett. **93**, 080502 – Published 20 August 2004

Quantum process tomography on vibrational states of atoms in an optical lattice

S. H. Myrskog, J. K. Fox, M. W. Mitchell, and A. M. Steinberg
Phys. Rev. A **72**, 013615 – Published 25 July 2005

Quantum process tomography and Linblad estimation of a solid-state qubit

M Howard[1], J Twamley[2,4], C Wittmann[3], T Gaebel[3], F Jelezko[3] and J Wrachtrup[3]
Published 6 March 2006 • IOP Publishing and Deutsche Physikalische Gesellschaft
New Journal of Physics, Volume 8, March 2006

why are the maps NCP?

Giving up CP?

Holevo bound — Masillo, Scolarici, Solombrino, J Math Phys 52, 012101 (2011)

Data processing inequality — Buscemi, PRL 113, 140502 (2014)

Entropy production — Argentieri, Benatti, Floreanini,… EPL 107, 50007 (2014)

Operational analysis

Quantum process tomography

$\{\rho_1, \rho_2, \rho_3, \ldots\}$ $\{\rho_1', \rho_2', \rho_3', \ldots\}$

$$\text{map} = f(\text{input states, output states})$$

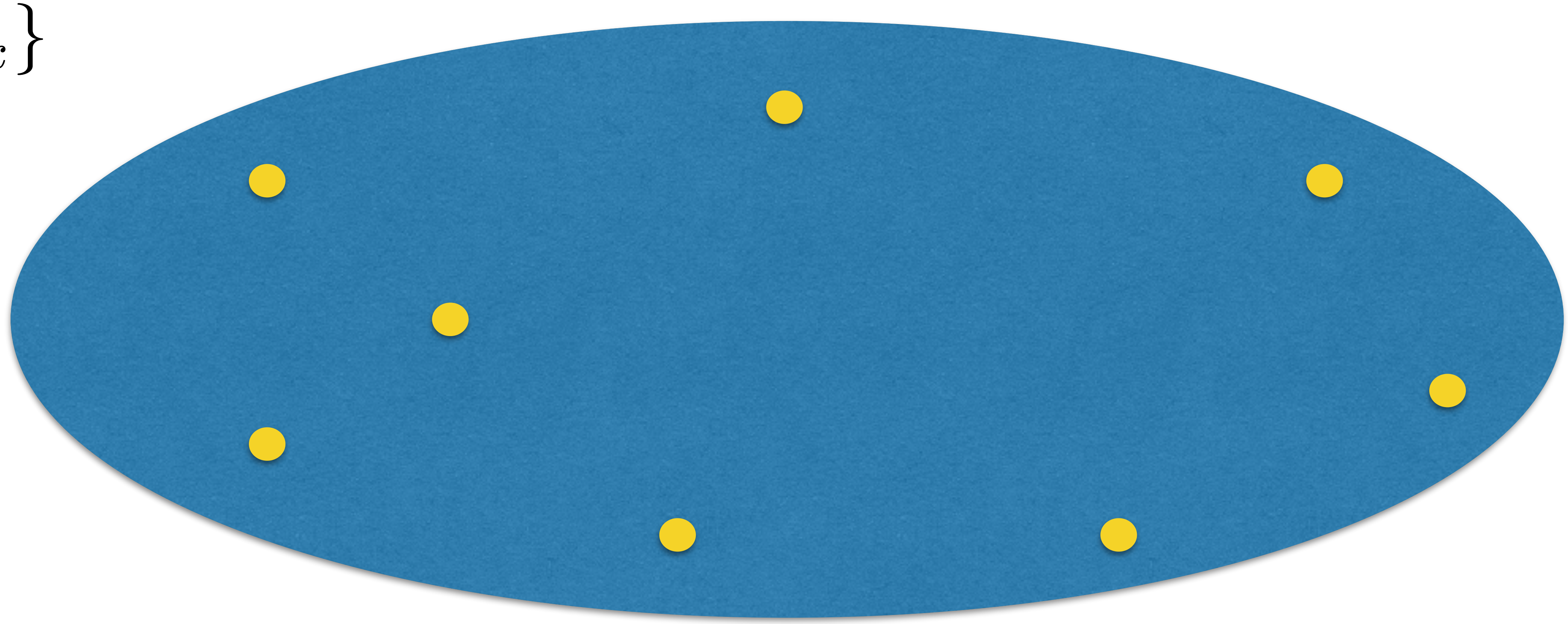
Quantum process tomography

$\{\rho_1, \rho_2, \rho_3, \ldots\}$ $\{\rho'_1, \rho'_2, \rho'_3, \ldots\}$

$$\mathrm{map} = f(\text{input states, output states})$$

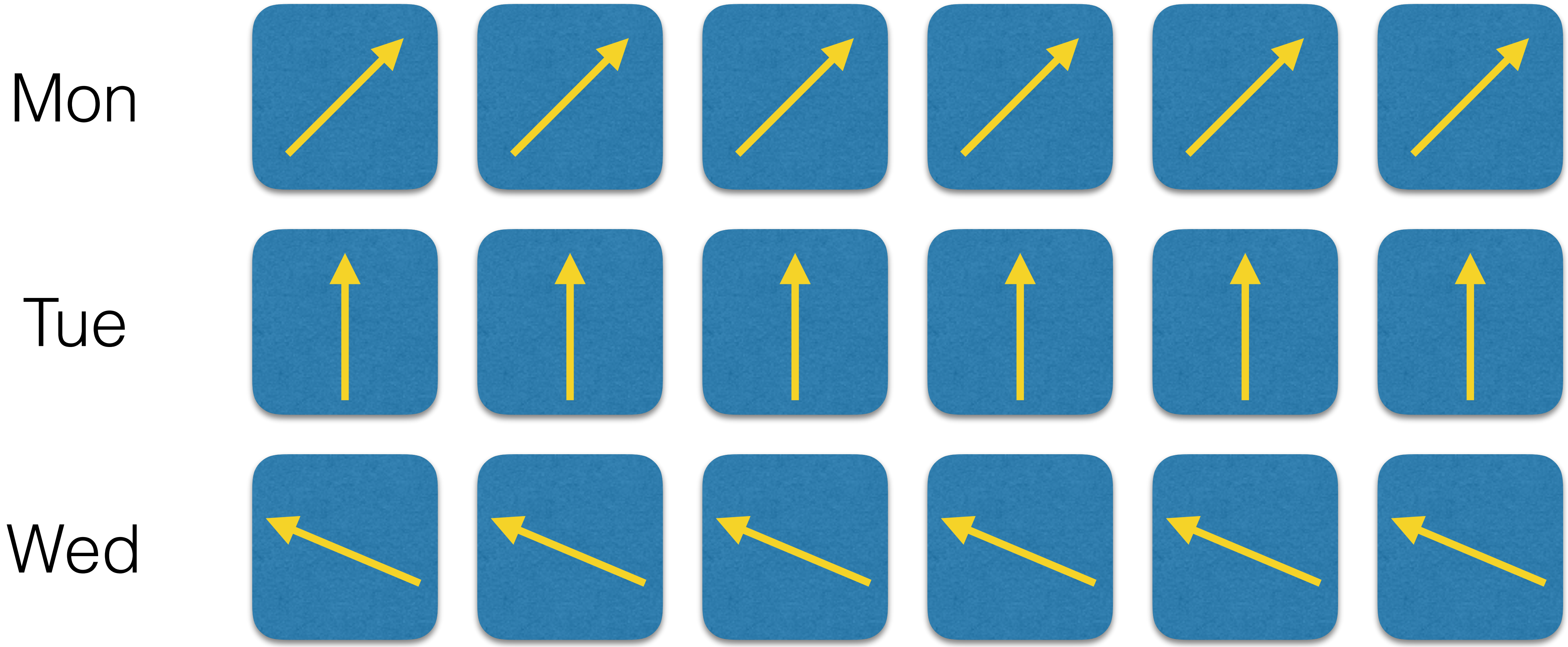
What can we say about initial state?

$\{\rho_k\}$



Alicki: Pechukas theorem is not operational

tomography requires a lot experiment experiments



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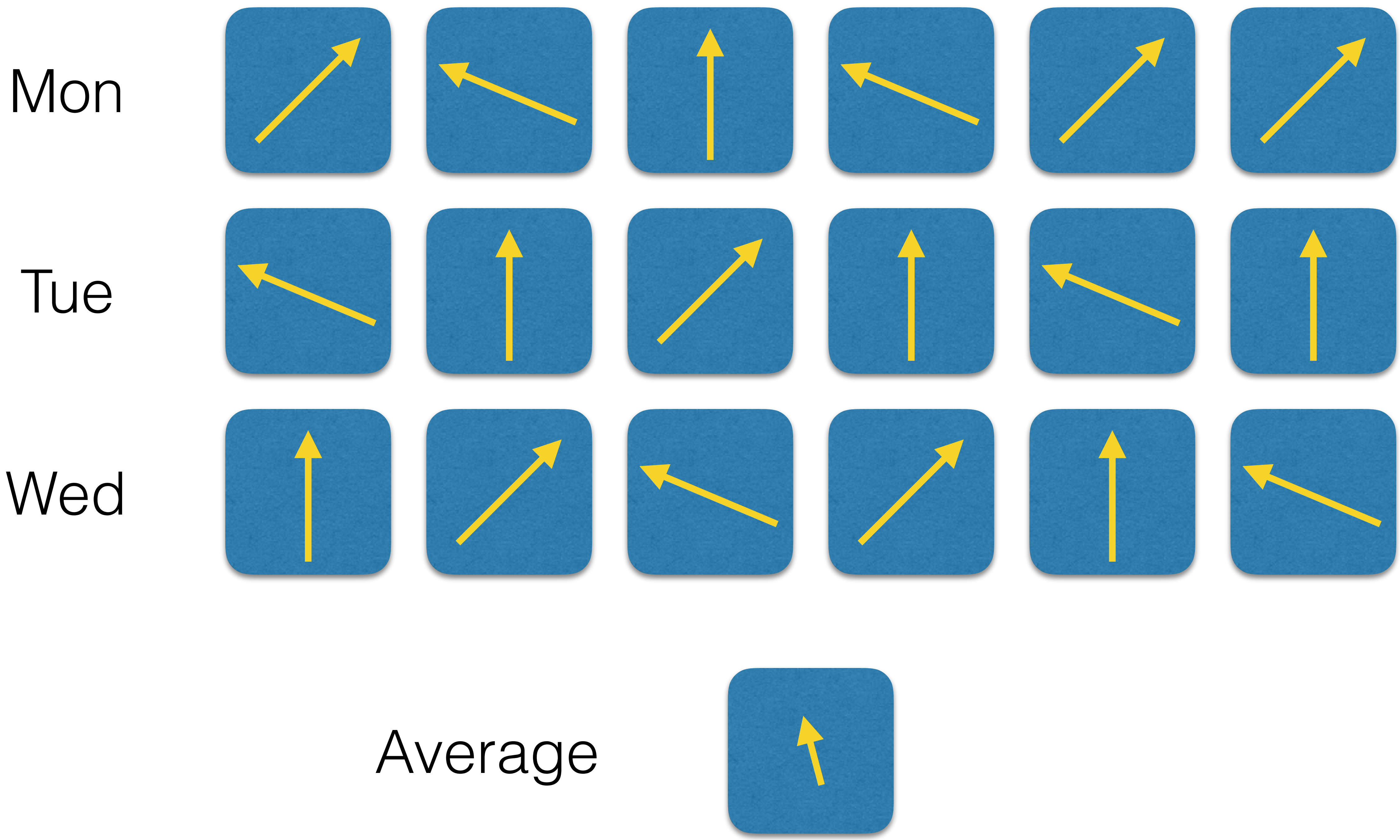
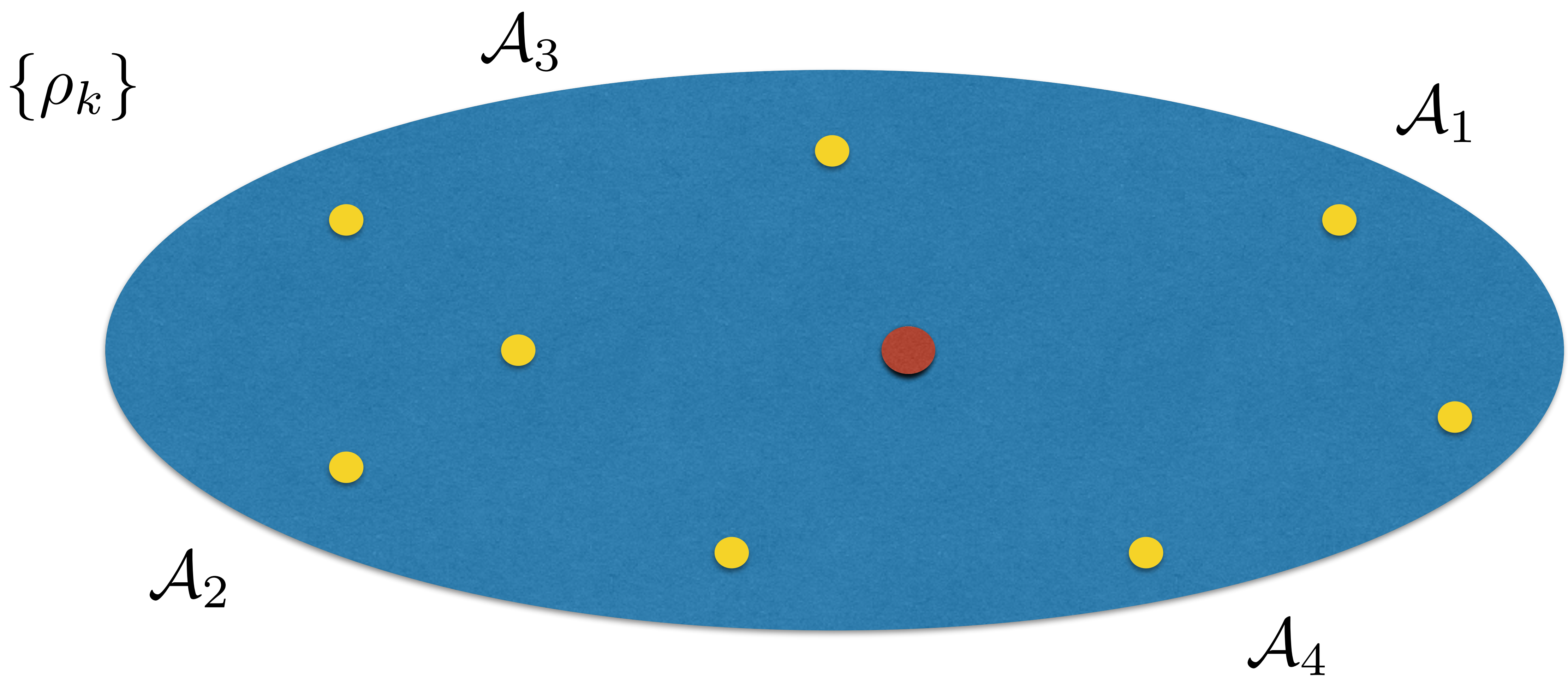
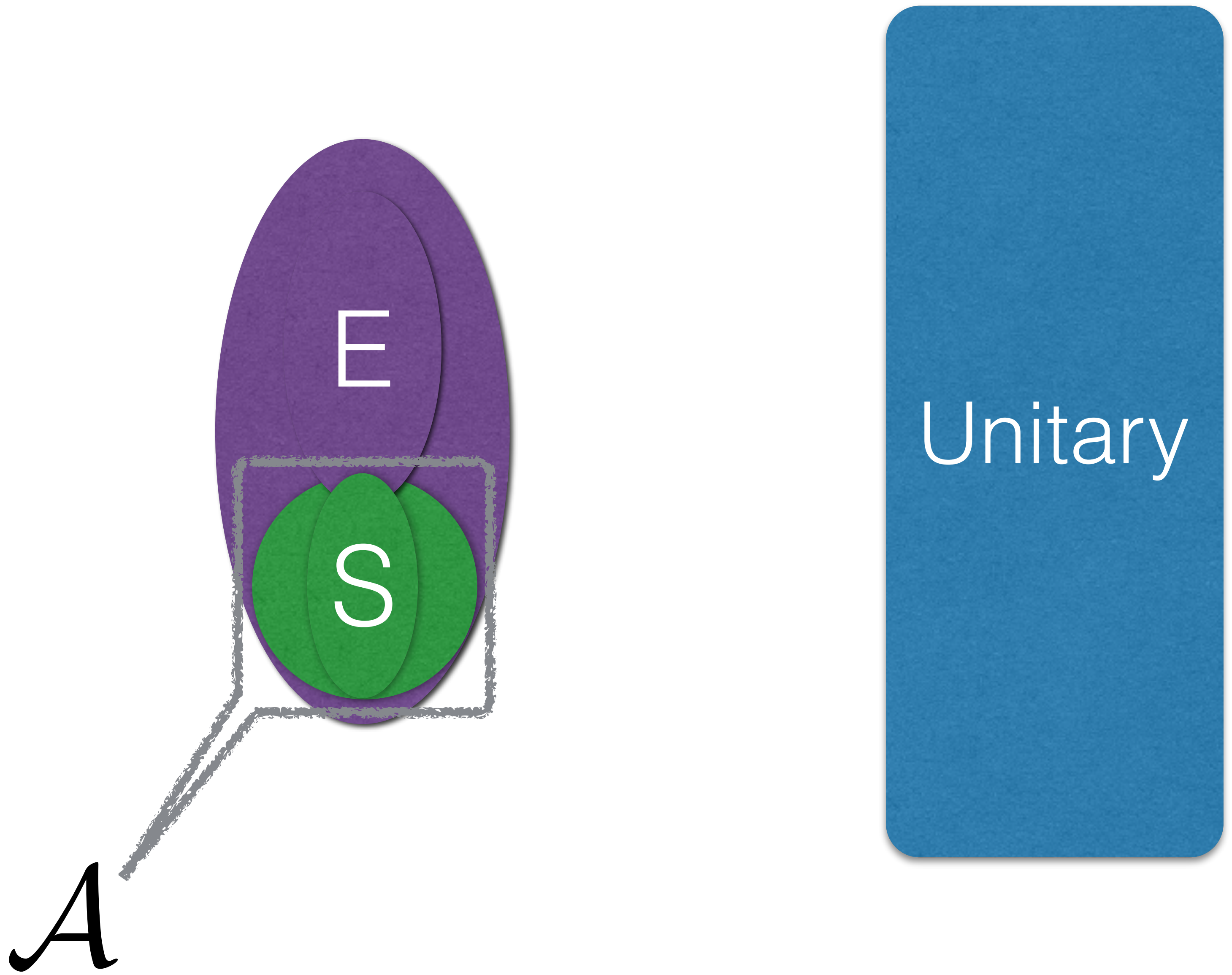


What can we say about initial state?



$$\rho_{\text{avg}} = \sum_k p_k \rho_k$$

Initially correlated SE

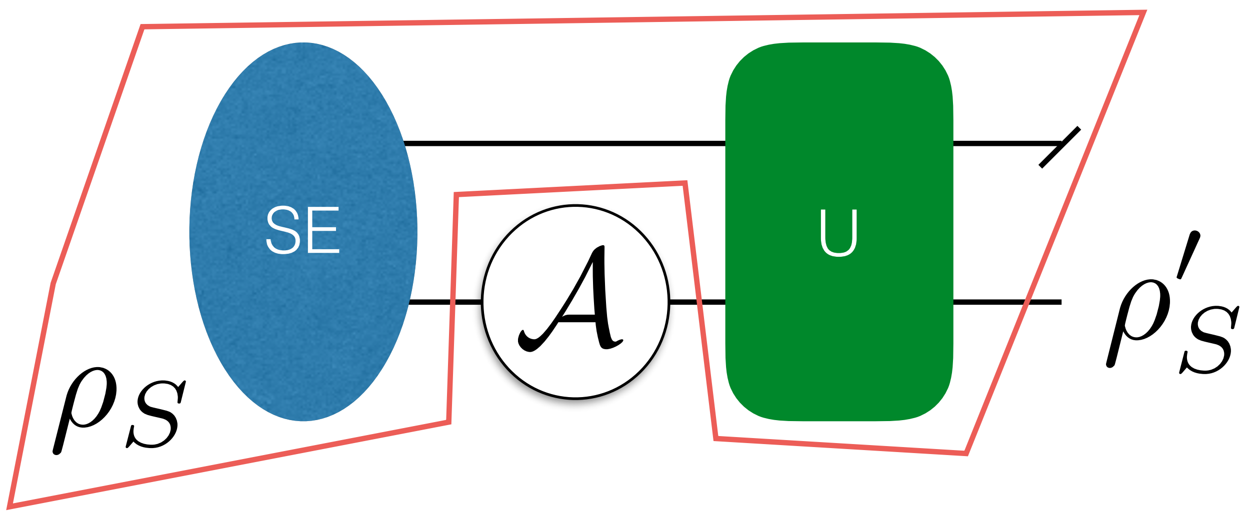


Grad student presses buttons

We can give up states as inputs

Superchannel

Completely positive and linear



$$\rho_f = \mathrm{Tr}_{\mathrm{env}}[U \mathcal{A}^s \otimes \mathcal{I}^e(\rho^{se}) U^\dagger] \qquad \mathcal{M}[\mathcal{A}] = \rho_f$$

Modi Sci. Rep. 2, 581 (2012)

Ringbauer, Wood, Modi, Gilchrist, White, Fedrizzi, PRL 114, 090402 (2015)

Using CP

Holevo bound

Masillo, Scolarici, Solombrino, J Math Phys 52, 012101 (2011)

Data processing inequality

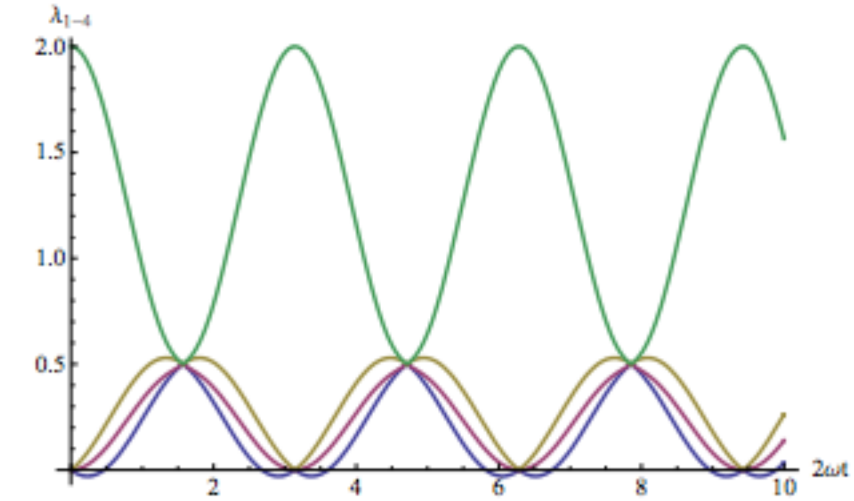
Buscemi, PRL 113, 140502 (2014)

Entropy production

Argentieri, Benatti, Floreanini,… EPL 107, 50007 (2014)

Vinjanampathy & Modi *PRA* 92, 052310 (2015)
Vinjanampathy & Modi, *Int. J. Quantum Inf.* 14, 1640033 (2016)

Be careful with preparations



Errors in preparations

$$V^{(1,-)}\,|1\rangle \to \frac{1}{\sqrt{2}}\left(\sqrt{1-\epsilon}\,|1\rangle - \sqrt{1+\epsilon}\,|0\rangle\right),$$

Variations in preparation

$$P^{\mathbb{I}} = \frac{1}{2}\mathbb{I},\;\; P^{(1+)},\;\; P^{(2+)},\;\; P^{(3+)}.$$

Ambiguity in preparation

$$\Theta\left(\rho^{\mathcal{S}\mathcal{E}}\right) = \frac{1}{2}\{\mathbb{I}\otimes\mathbb{I} + p\sigma_3 + c_{23}\sigma_2\otimes\sigma_3\}.$$

$$\begin{aligned}\Theta\left(\rho^{\mathcal{S}\mathcal{E}}\right) &= \left(pP^{(3,+)} + (1-p)P^{(3,-)}\right)\otimes\mathbb{I}\\ &= \frac{1}{2}\{\mathbb{I} + p\sigma_3\}\otimes\mathbb{I},\end{aligned}$$

Role of preparation in quantum process tomography

Kavan Modi and E. C. G. Sudarshan
Phys. Rev. A **81**, 052119 – Published 27 May 2010

Kavan Modi, *Open Syst. Inf. Dyn.*, **18**, 253 (2011).
https://doi.org/10.1142/S1230161211000170

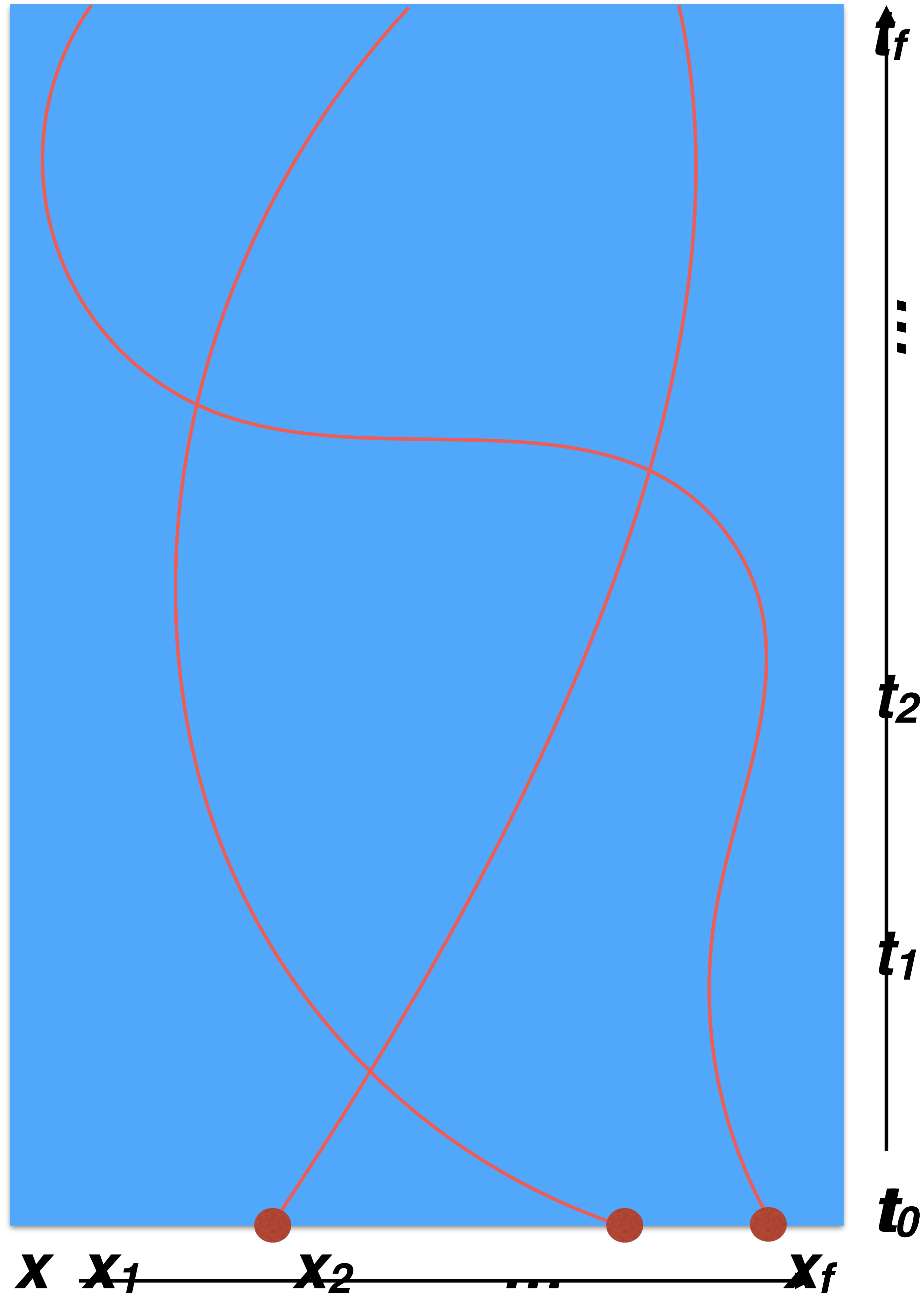
Preparation of States in Open Quantum Mechanics

Non-markovianity

Classical

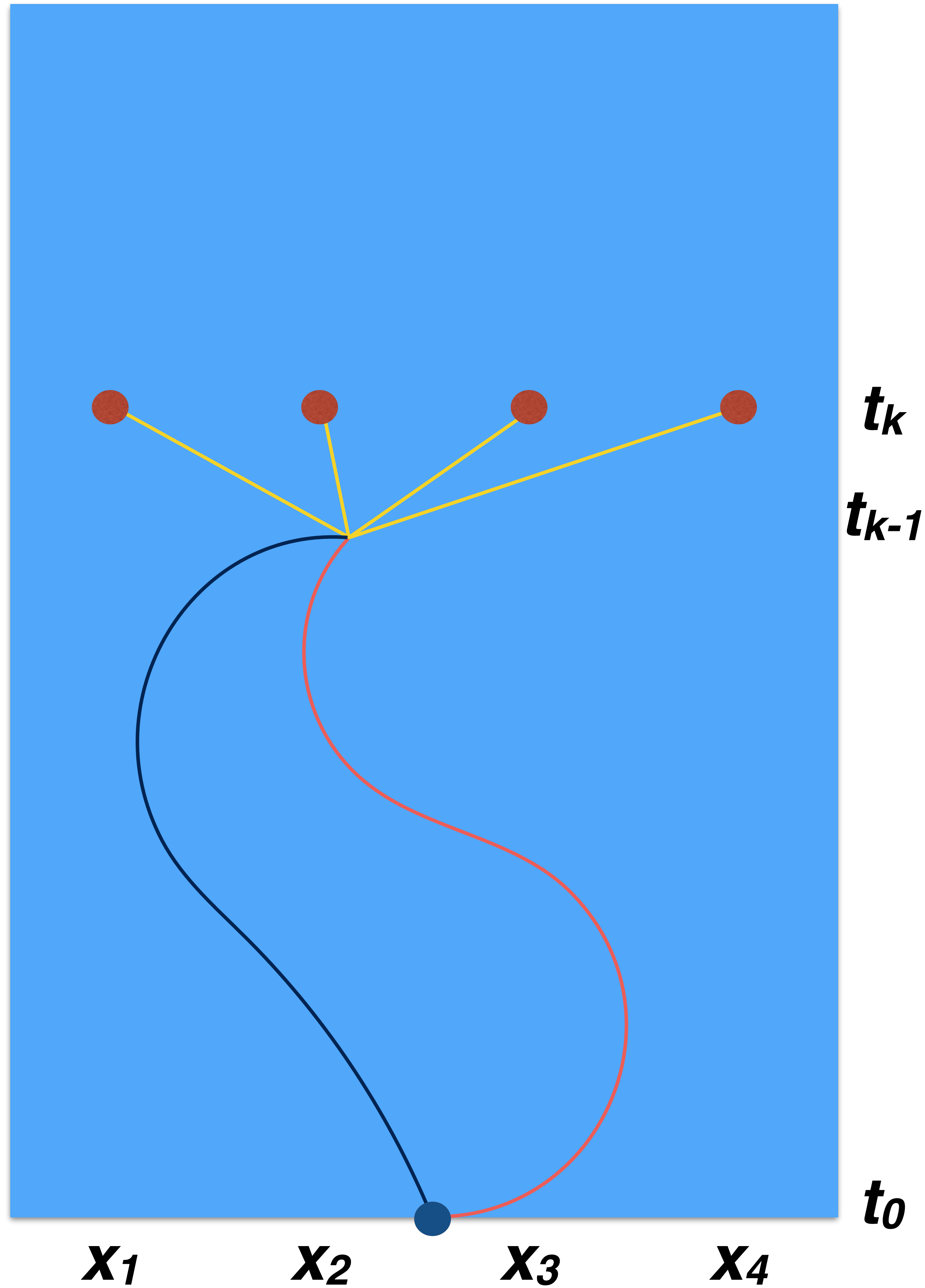
Classical Stochastic process

$$P(x_f\, t_f\, ;\, \ldots\, ;\, x_2\, t_2\, ;\, x_1\, t_1)$$



Classical Stochastic process

$$P(x_k\, t_k) = \int (x_{k-1}\, t_{k-1};\ trajectory)$$



$$P(x_k\, t_k \mid x_{k-1}\, t_{k-1};\ blue)$$

$$\neq$$

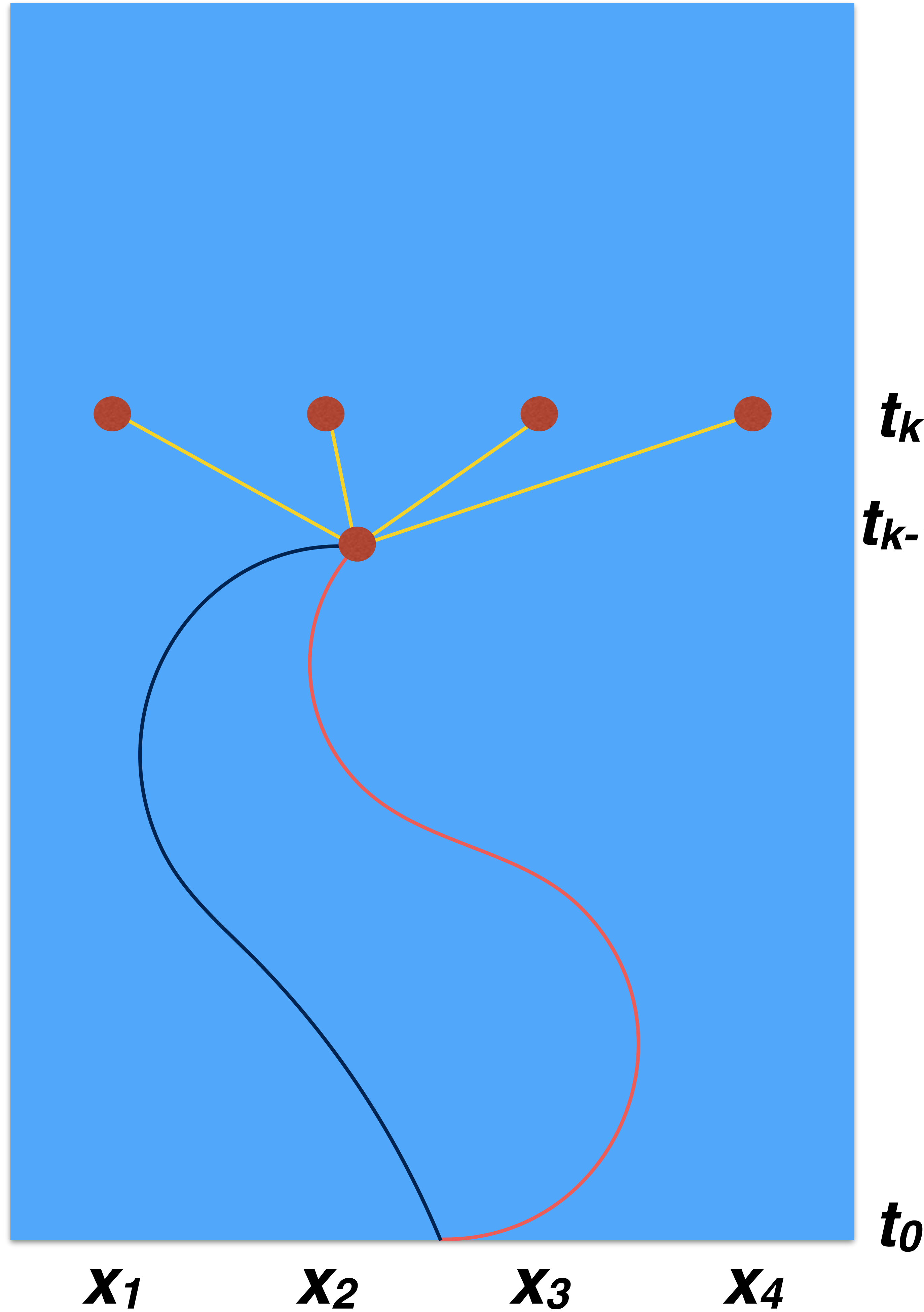
$$P(x_k\, t_k \mid x_{k-1}\, t_{k-1};\ red)$$

Classical Stochastic process

Classical Markov process

$$P(x_k\, t_k) = f(x_{k-1}\, t_{k-1})$$

this function is the well-known stochastic map



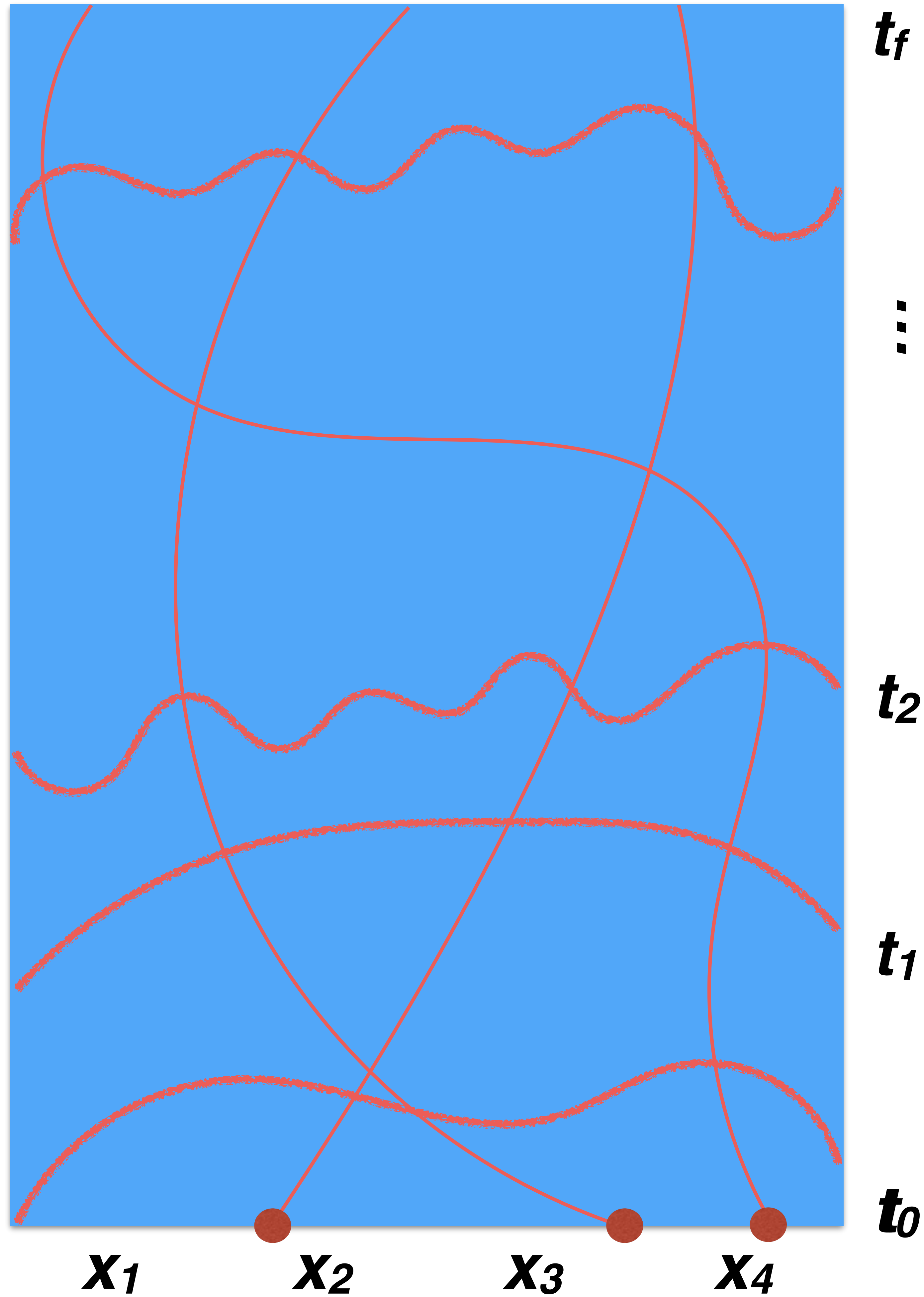
Quantum problem

Classical Stochastic process

$$P \to \rho$$

$$P_{ABC} \to \rho_{ABC}$$

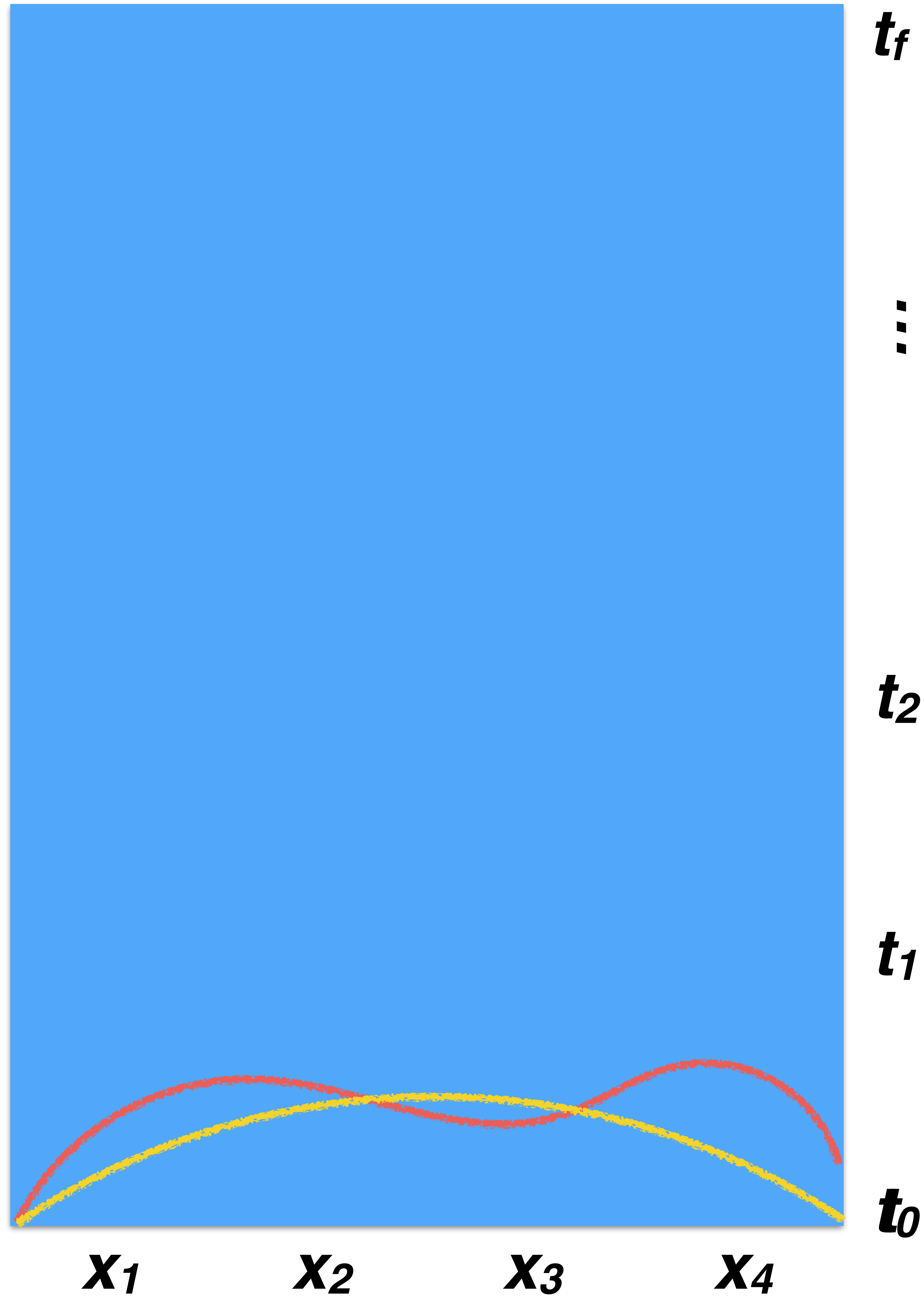
$$P_{x_3 t_3; x_2 t_2; x_1 t_1} \overset{?}{\to} \rho_{x_3 t_3; x_2 t_2; x_1 t_1}$$



Quantum Stochastic process

Add control operations

We're not astronomers!



The framework

Challenge

p c r o o n c t e r s o s l

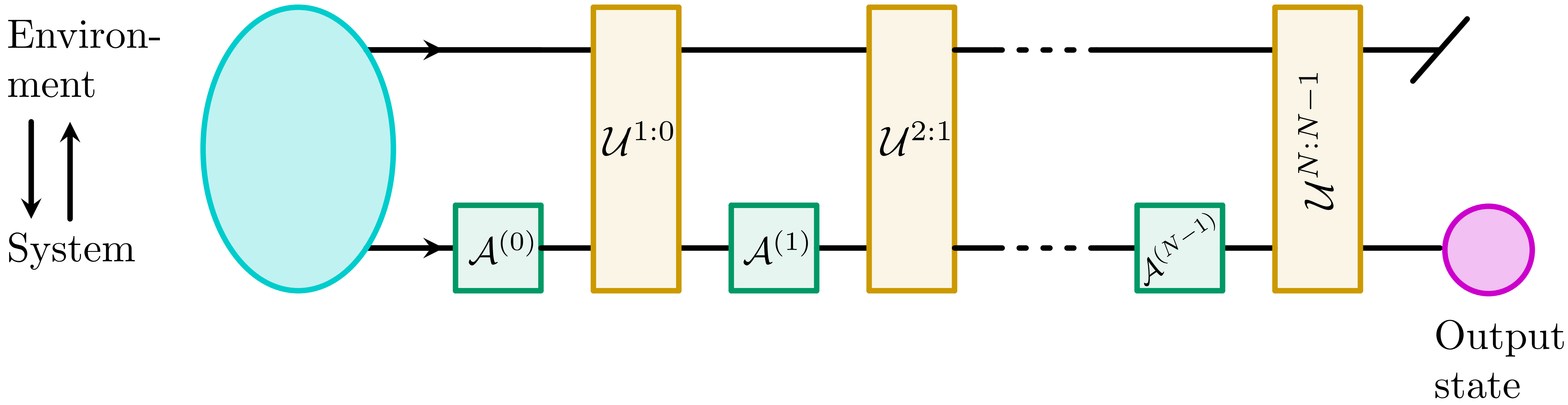
process [control] ⟶ quantum states (probabilities)

Modify definition

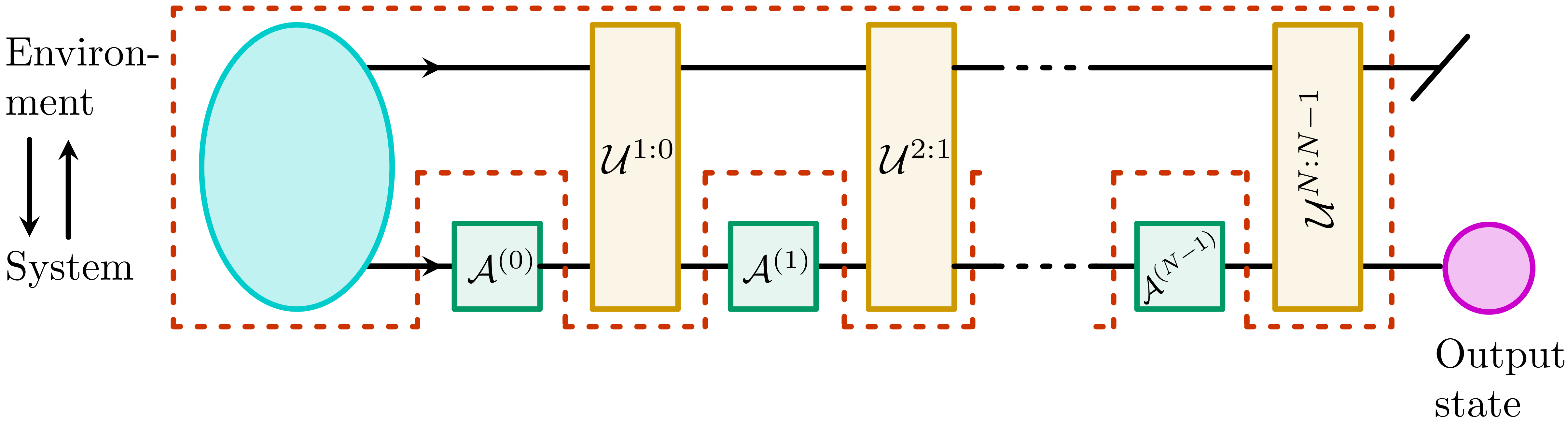
Classical stochastic process is a mapping from observed past events to future states (probability distribution)

Quantum stochastic process is a mapping from past control operations to future states (density matrix)

Open quantum mechanics



Open quantum mechanics



$$\mathcal{T}^{N:0}\left(\mathcal{A}^{(0)}, \mathcal{A}^{(1)}, \ldots, \mathcal{A}^{(N-1)}\right) =$$

$\mathcal{T}$ is called the Process Tensor

A mapping from control operations to states

How good is this framework?

Representation theorem

Open quantum evolution → $\mathcal{T}$

$$\rho_k^{SE} = \mathcal{U}_{k:k-1}\,\mathcal{A}_{k-1}\,\mathcal{U}_{k-1:k-2}\ldots\mathcal{A}_1\,\mathcal{U}_{1:0}\,\mathcal{A}_0\,[\rho_0^{SE}]$$

We can cast the dynamics of the system as

$\mathcal{A}$ act only on the system

$\mathcal{U}$ act on both the system and environment

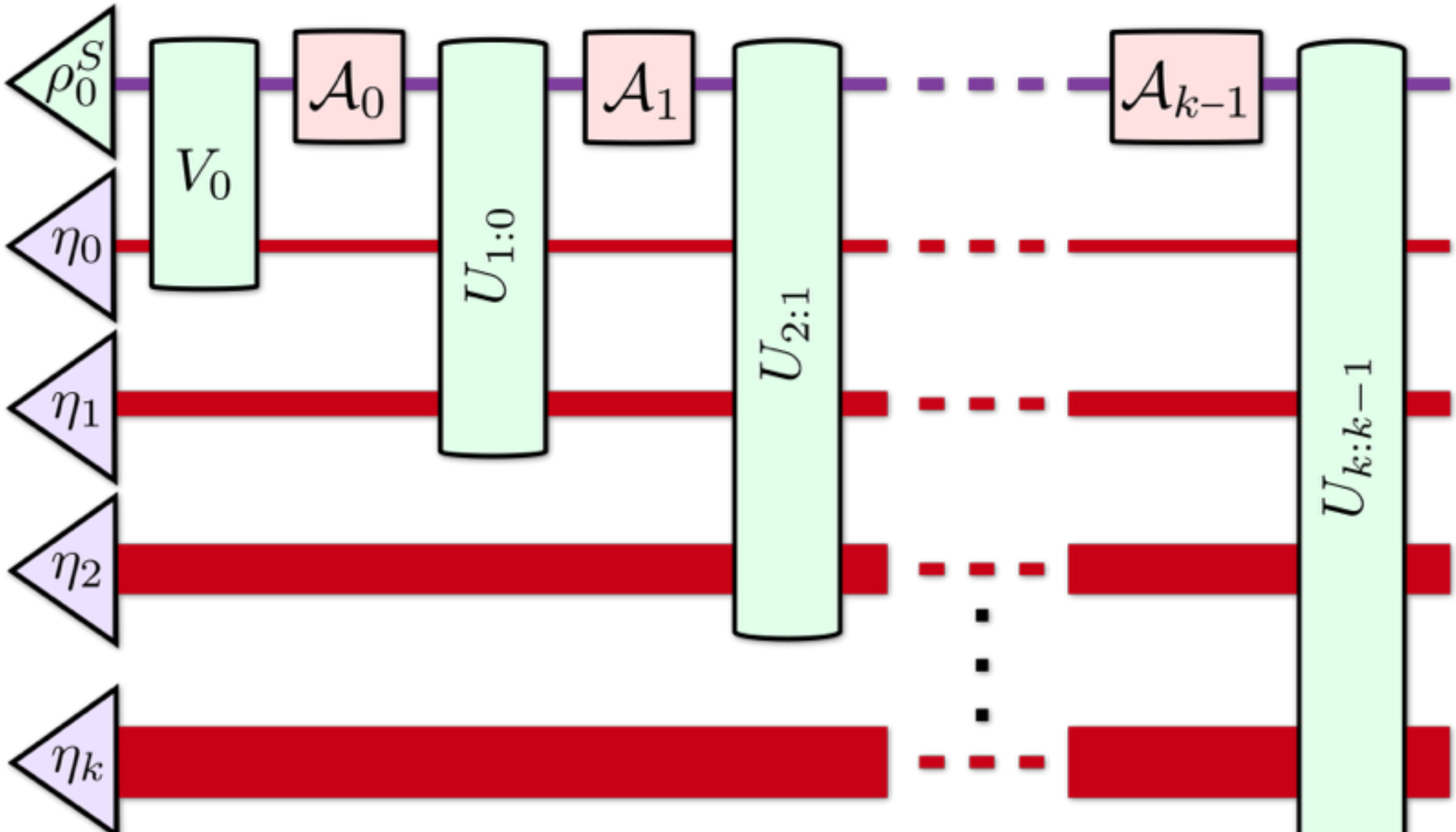
$$\rho_k = \mathcal{T}_{0:k}[\mathbf{A}_{0:k}] = \sum_l (T_{k:0})_l\,\mathbf{A}_{0:k}\,(T_{0:k})_l^\dagger$$

In terms of control operations $\mathbf{A}$

$$\mathbf{A}_{k-1:0} = [\mathcal{A}_{k-1};\mathcal{A}_{k-2};\ldots;\mathcal{A}_1;\mathcal{A}_0]$$

Linear and Completely positive

Open quantum evolution $\mathcal{T}$



How good is this framework?

It's universal

F. Pollock, C. Rodríguez-Rosario, T. Frauenheim, M. Paternostro, K. Modi Phys. Rev. A 97, 012127 (2018)
(on arXiv since late 2015)

Advances in Mathematics

Volume 20, Issue 3, June 1976, Pages 329–366

Nonrelativistic quantum mechanics as a noncommutative Markof process

Luigi Accardi[1, 2]

Comm. Math. Phys.
Volume 65, Number 3 (1979), 281-294.

Non-Markovian quantum stochastic processes and their entropy

Göran Lindblad

But we can make it better!

Quantum Channels with Memory
Dennis Kretschmann, Reinhard F. Werner

Quantum Circuits Architecture
Giulio Chiribella, Giacomo Mauro D'Ariano, Paolo Perinotti

Multiple-time states and multiple-time measurements in quantum mechanics
Y. Aharonov, S. Popescu, J. Tollaksen, L. Vaidman

The Operator Tensor Formulation of Quantum Theory
Lucien Hardy

Complete framework for efficient characterisation of non-Markovian processes
Felix A. Pollock, César Rodríguez-Rosario, Thomas Frauenheim, Mauro Paternostro, Kavan Modi

Causal Boxes: Quantum Information-Processing Systems Closed under Composition
Christopher Portmann, Christian Matt, Ueli Maurer, Renato Renner, Björn Tackmann

Quantum causal modelling
Fabio Costa, Sally Shrapnel

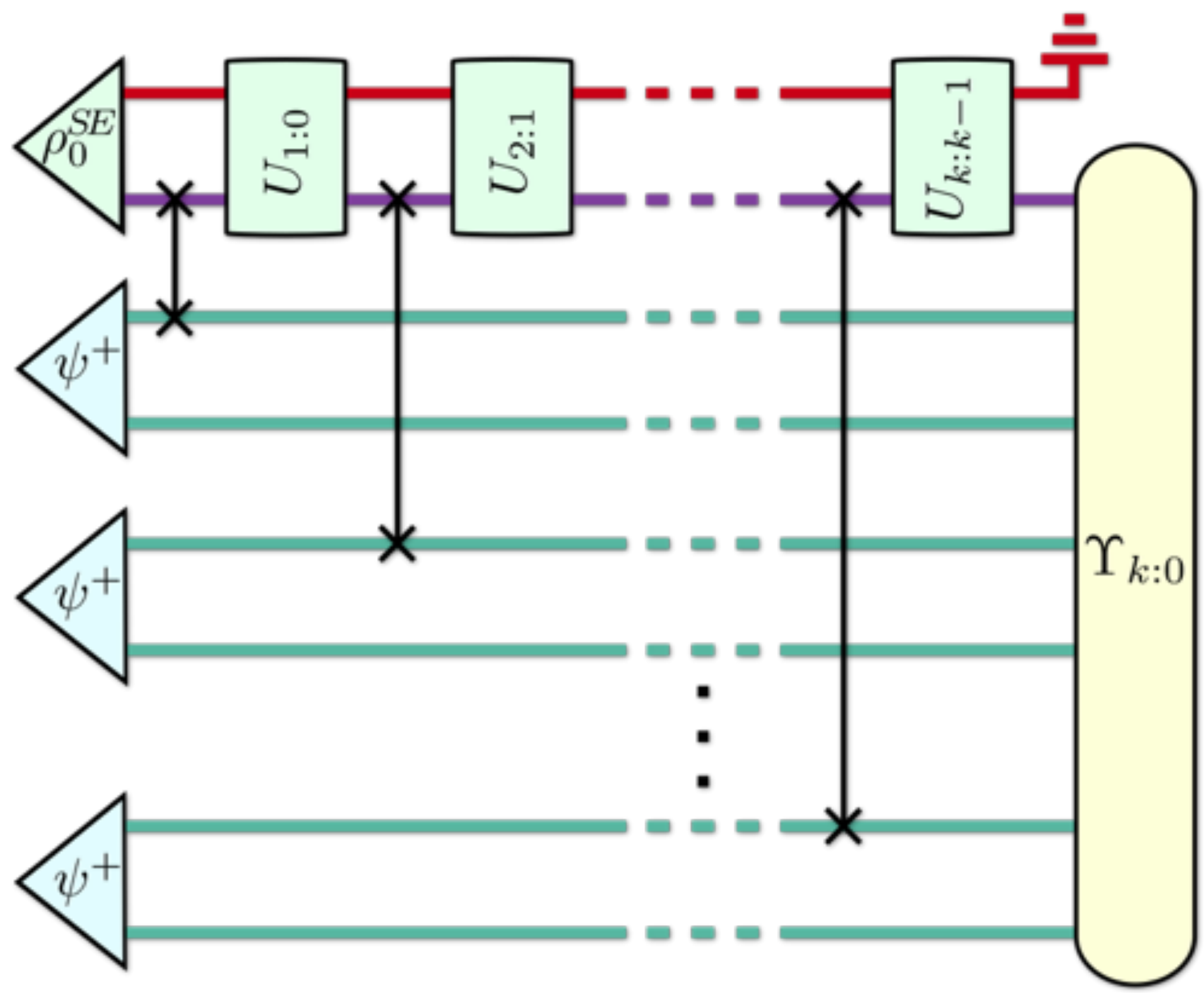
Quantum common causes and quantum causal models
John-Mark A. Allen, Jonathan Barrett, Dominic C. Horsman, Ciaran M. Lee, Robert W. Spekkens

Superdensity Operators for Spacetime Quantum Mechanics
Jordan Cotler, Chao-Ming Jian, Xiao-Liang Qi, Frank Wilczek

time ↓

What can we do with this framework?

Encode into state



$\Upsilon_{k:0}$ is a matrix product density operator.

Formally define quantum stochastic processes

Kolmogorov extension theorem

$$P(x_9, t_9; x_7, t_7; x_6, t_6; x_3, t_3; x_2, t_2) \subset P(x_9, t_9; x_8, t_8; x_7, t_7; x_6, t_6; x_5, t_5; x_4, t_4; x_3, t_3; x_2, t_2; x_1, t_1)$$

Proves the existence of an underlying continuous stochastic process.

Important for proving Brownian motion.

destroy quantum interferences. The fact that the joint probability distributions (24) violate in general the Kolmogorov condition (15) is even true for a closed quantum system. Thus, the quantum joint probability distributions given by Eq. (24) do not represent a classical hierarchy of joint probabilities satisfying the Kolmogorov consistency conditions. More generally, one can consider other joint probability distributions corresponding to different quantum operations describing nonprojective, generalized measurements.

For an open system coupled to some environment measurements performed on the open system not only influence quantum interferences but also system-environment correlations. For example, if the system-environment state prior to a projective measurement at time t_i is given by $\rho_{SE}(t_i)$, the state after the measurement conditioned on the outcome x is given by

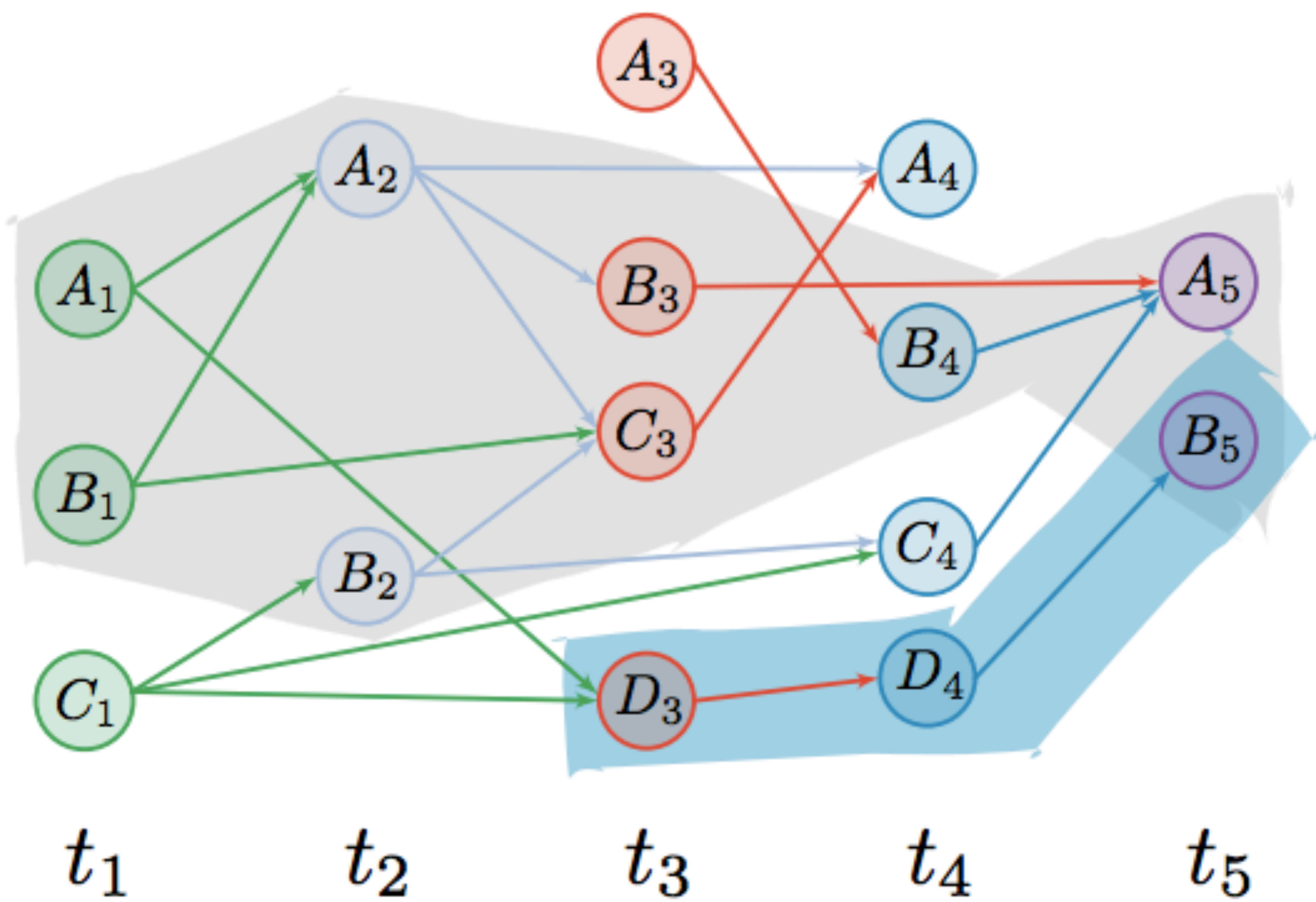
$$\rho'_{SE}(t_i) = \frac{\mathcal{M}_x \rho_{SE}(t_i)}{\mathrm{tr}\mathcal{M}_x \rho_{SE}(t_i)} = |\varphi_x\rangle\langle\varphi_x| \otimes \rho_E^x(t_i), \qquad (25)$$

where ρ_E^x is an environmental state which may depend on the measurement result x. Hence, projective measurements completely destroy system-environment correlations, leading to an uncorrelated tensor product state of the total system, and, therefore, strongly influence the subsequent dynamics.

We conclude that an *intrinsic* characterization and quantification of memory effects in the dynamics of open quantum systems, which is independent of any prescribed measurement scheme influencing the time evolution, has to be based solely on the properties of the dynamics of the open system's density matrix $\rho_S(t)$.

H.-P. Breuer *et al.*, Rev. Mod. Phys. 88, 021002 (2016).

Kolmogorov extension theorem for general (quantum) stochastic processes.

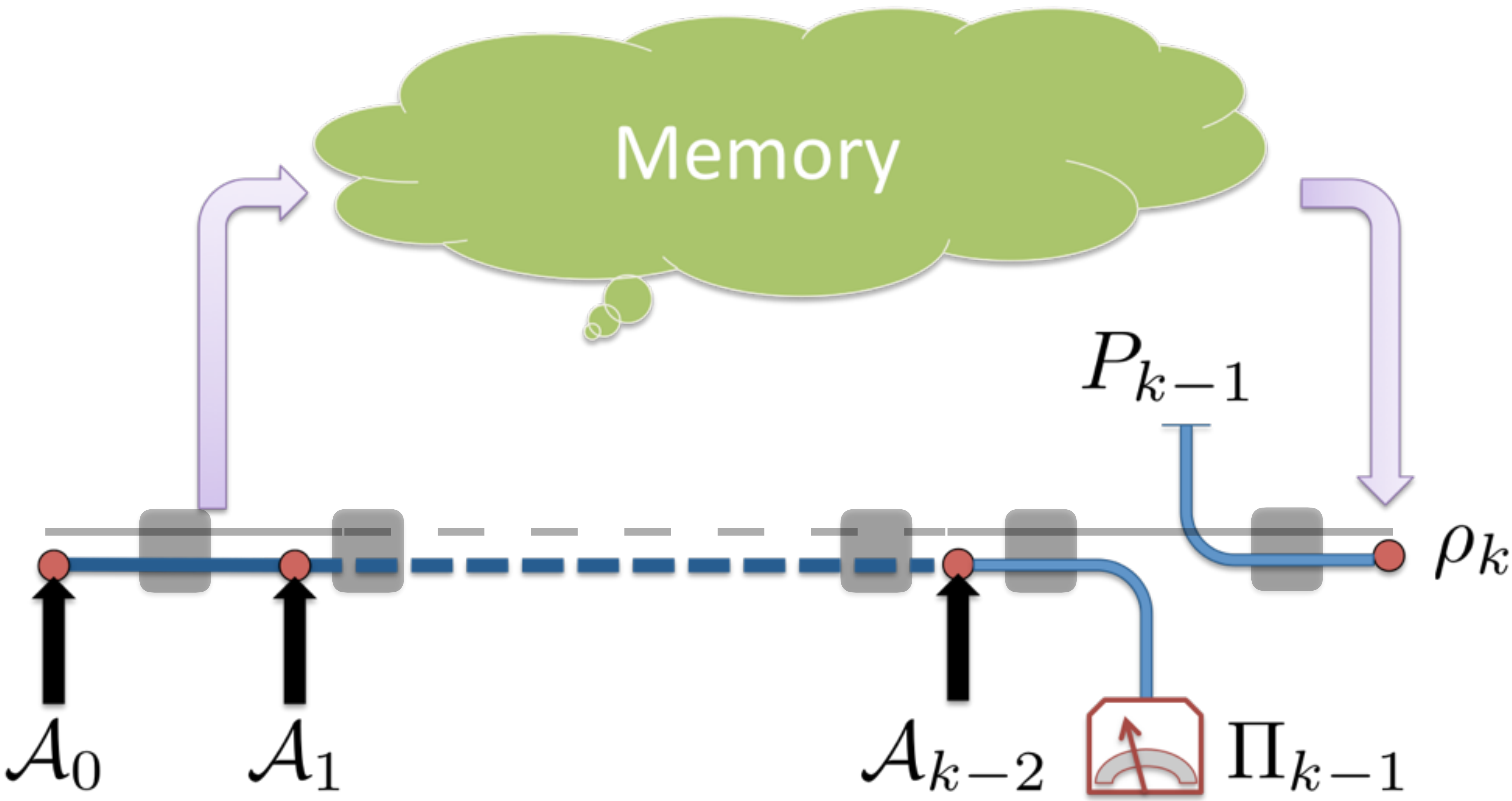


S. Milz, F. Sakuldee, F. Pollock, K. Modi arXiv:1712.02589 (2017)

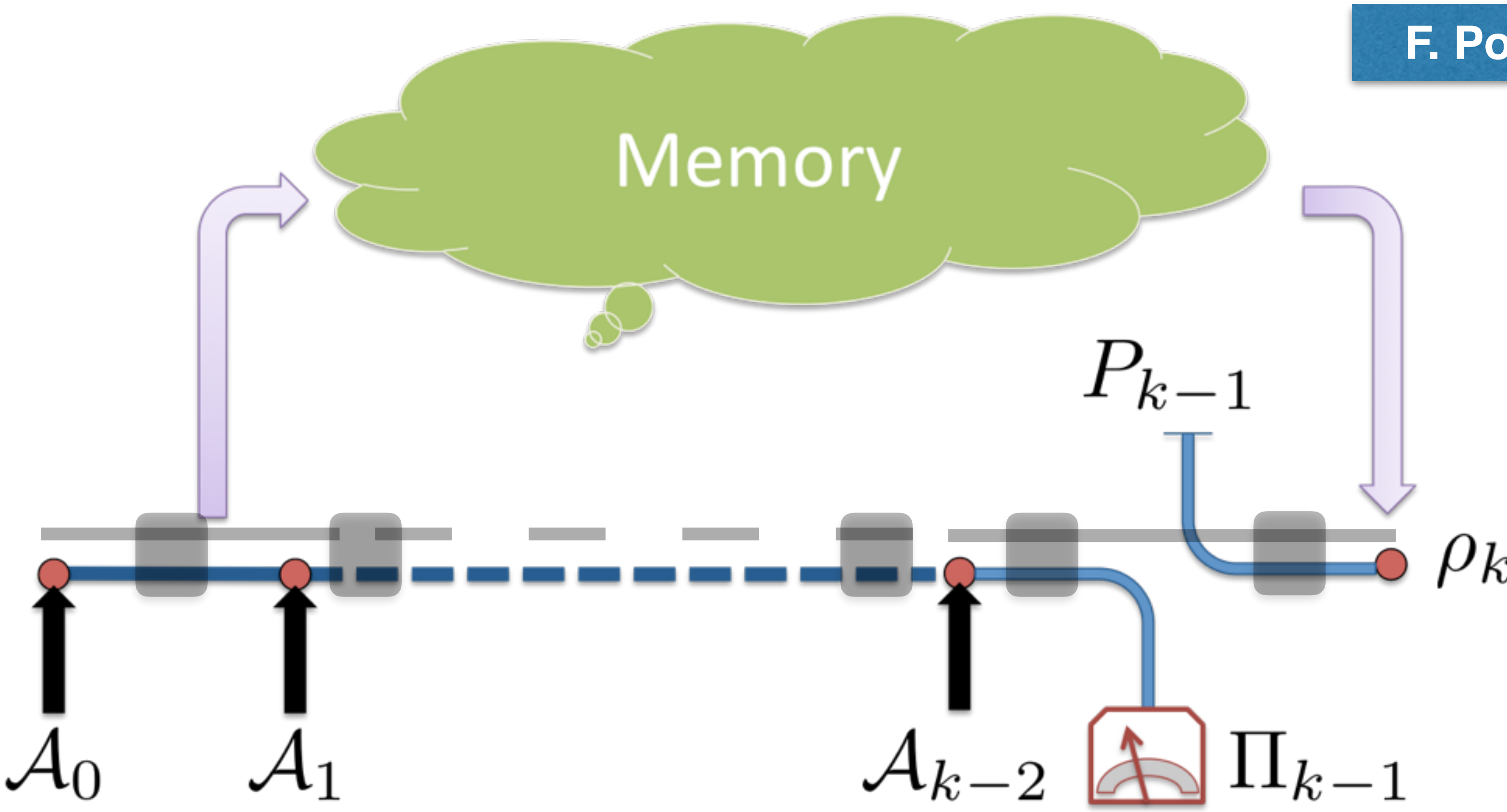
Operational Markov condition and

Shift switch of grad student

Causal break



F. Pollock et al. Phys. Rev. Lett. 120, 040405 (2018)



$$\rho_k(P_{k-1}|\Pi_{k-1};\mathbf{A}_{k-2:0}) = \rho_k(P_{k-1})$$

$$\rho_k(P_{k-1}|\Pi_{k-1};\mathbf{A}_{k-2;0}) \neq \rho_k(P_{k-1}|\Pi'_{k-1};\mathbf{A}'_{k-2;0})$$

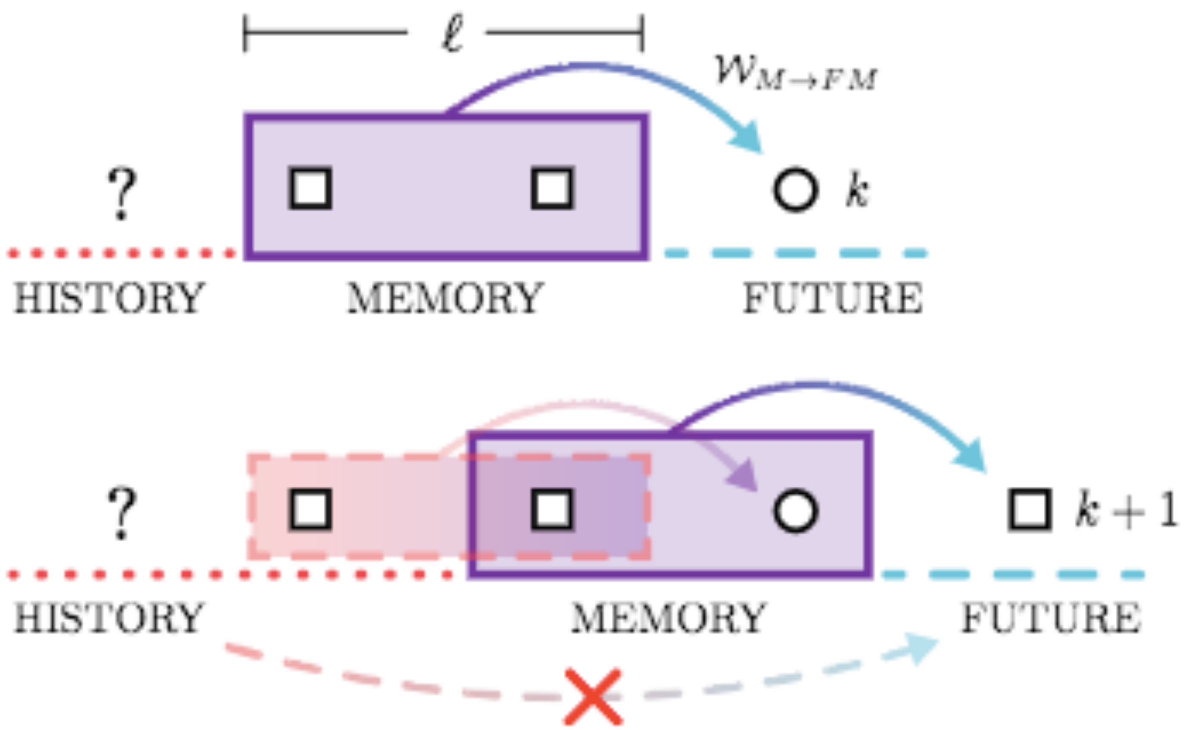
Markovian → Divisible

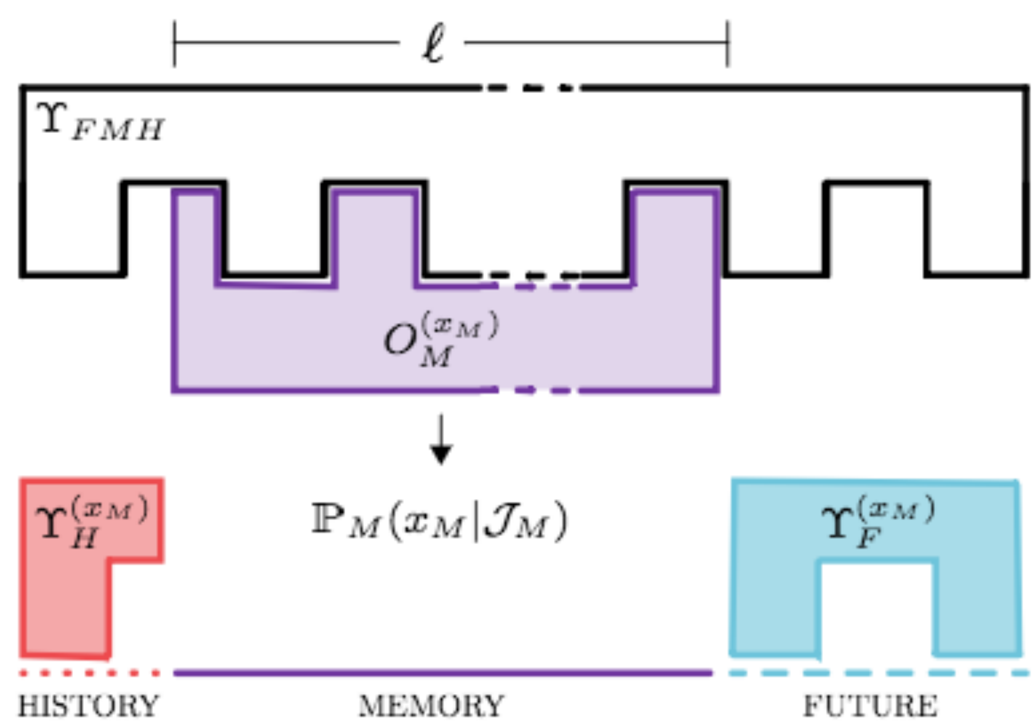
(← ✗)

Semigroup

Markov order

Quantum Markov order





P. Taranto et al. arXiv:1805.11341

Operational measure for non-Markovianity

F. Pollock et al. Phys. Rev. Lett. 120, 040405 (2018)

Measuring non-Markovianity with Relative entropy

$\Upsilon_{7:0}$ | S A1B1 A2B2 A3B3 A4B4 A5B5 A6B6 A7B7

$\Upsilon_{7:0}^{\text{Markov}}$ | S | A1B1 | A2B2 | A3B3 | A4B4 | A5B5 | A6B6 | A7B7

$$\mathcal{N} = R(\Upsilon_{7:0} \| \Upsilon_{7:0}^{\text{Markov}})$$

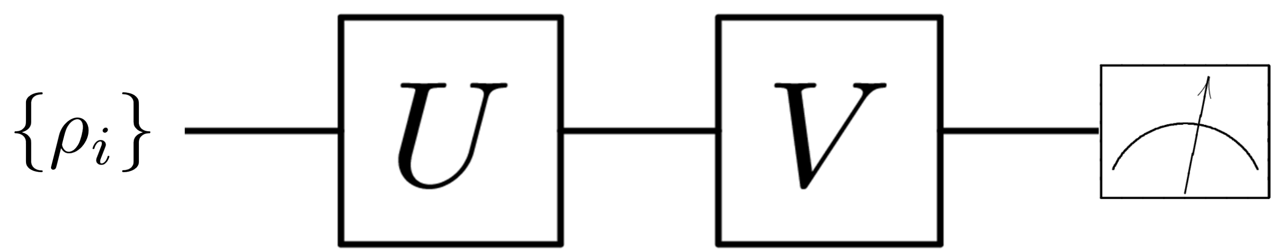
$$\text{Confusion probability} = e^{-n\mathcal{N}}$$

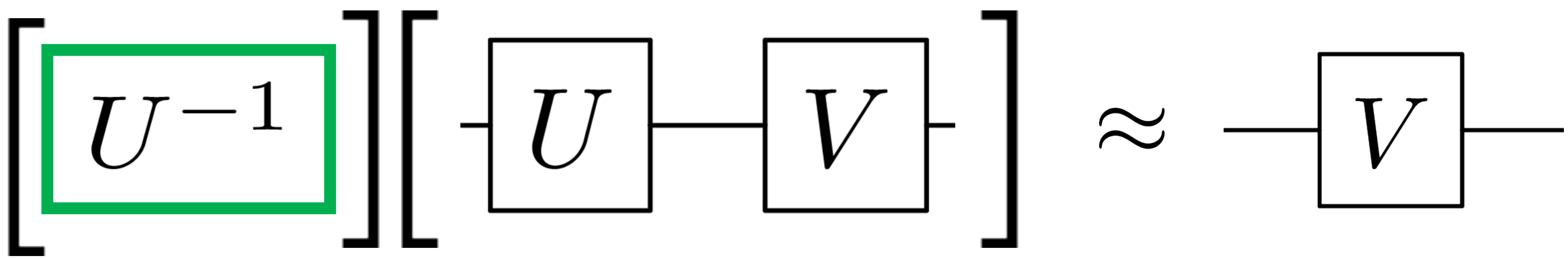
You have a Markovian model to describe a non-Markov process.
How surprised are you when model gives wrong answer

We can change the model and systematically develop better description for the experiment

Non-Markovianity in IBM Experience

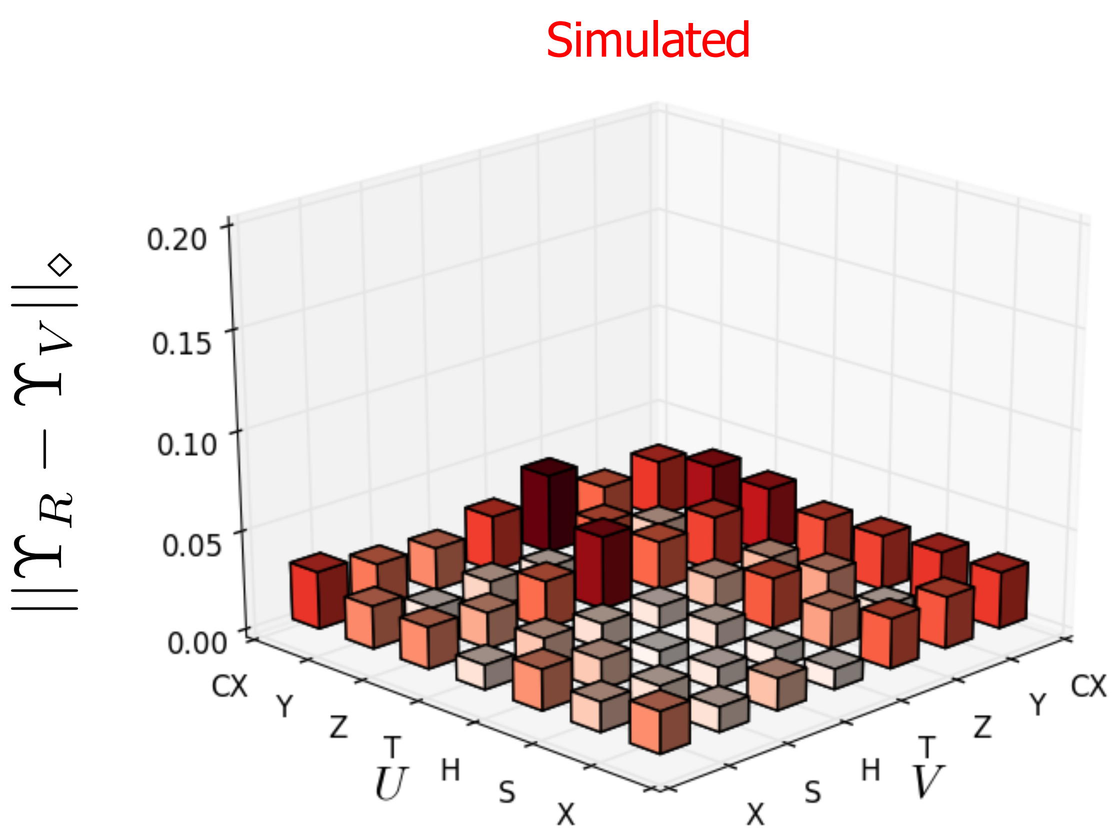
Two Step Processes

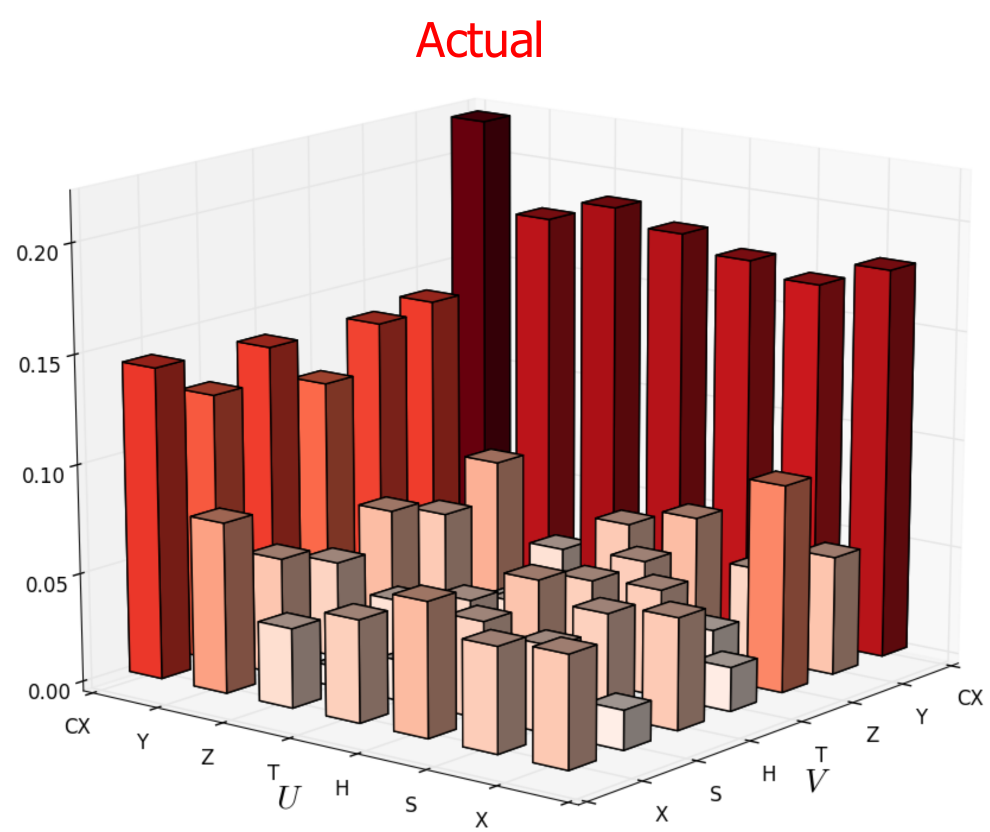


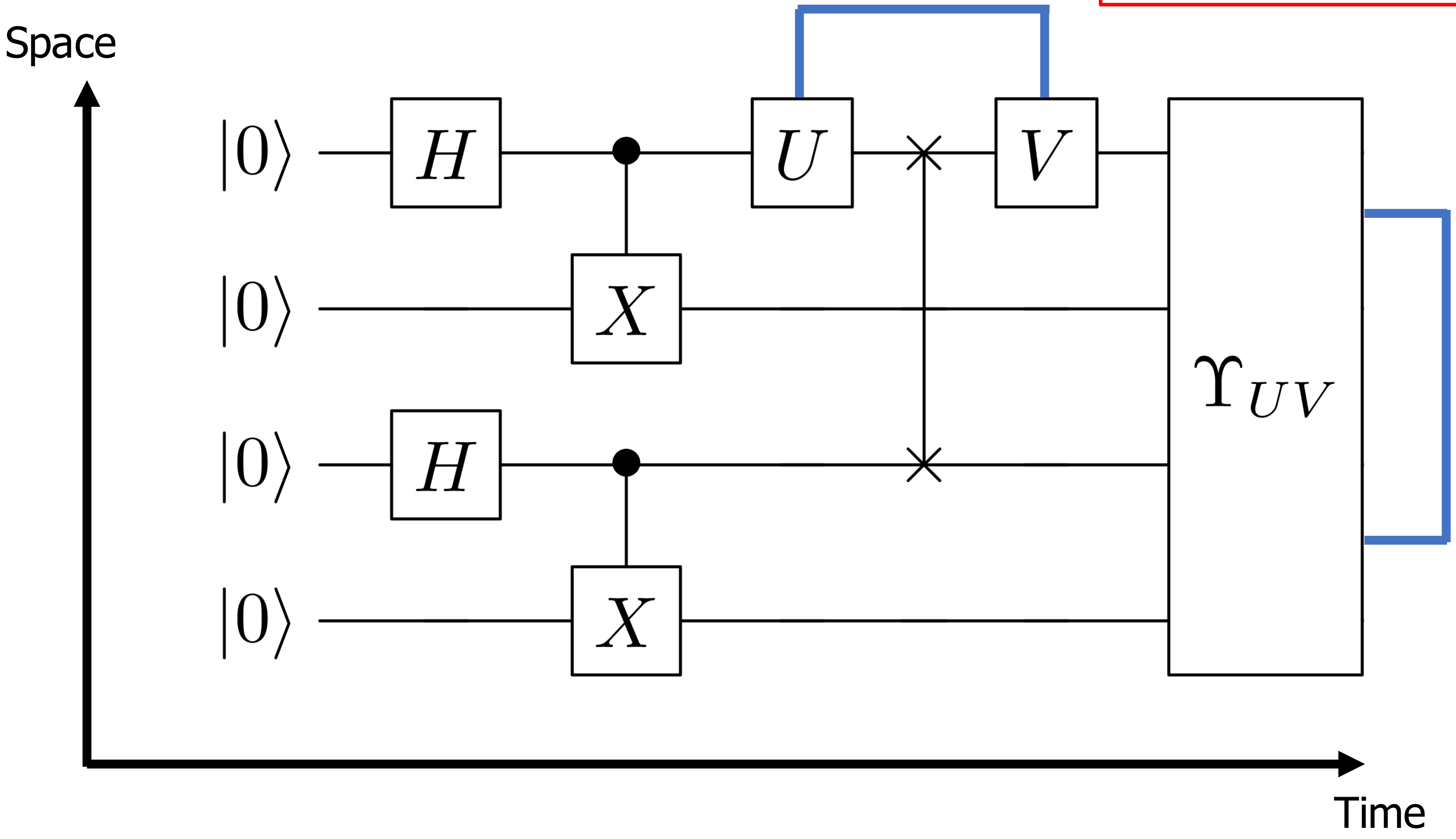


$$\mathbb{L}_V \mathbb{L}_U \mathbb{L}_U^{-1} = \mathbb{L}_R \approx \mathbb{L}_{V_{ideal}}$$

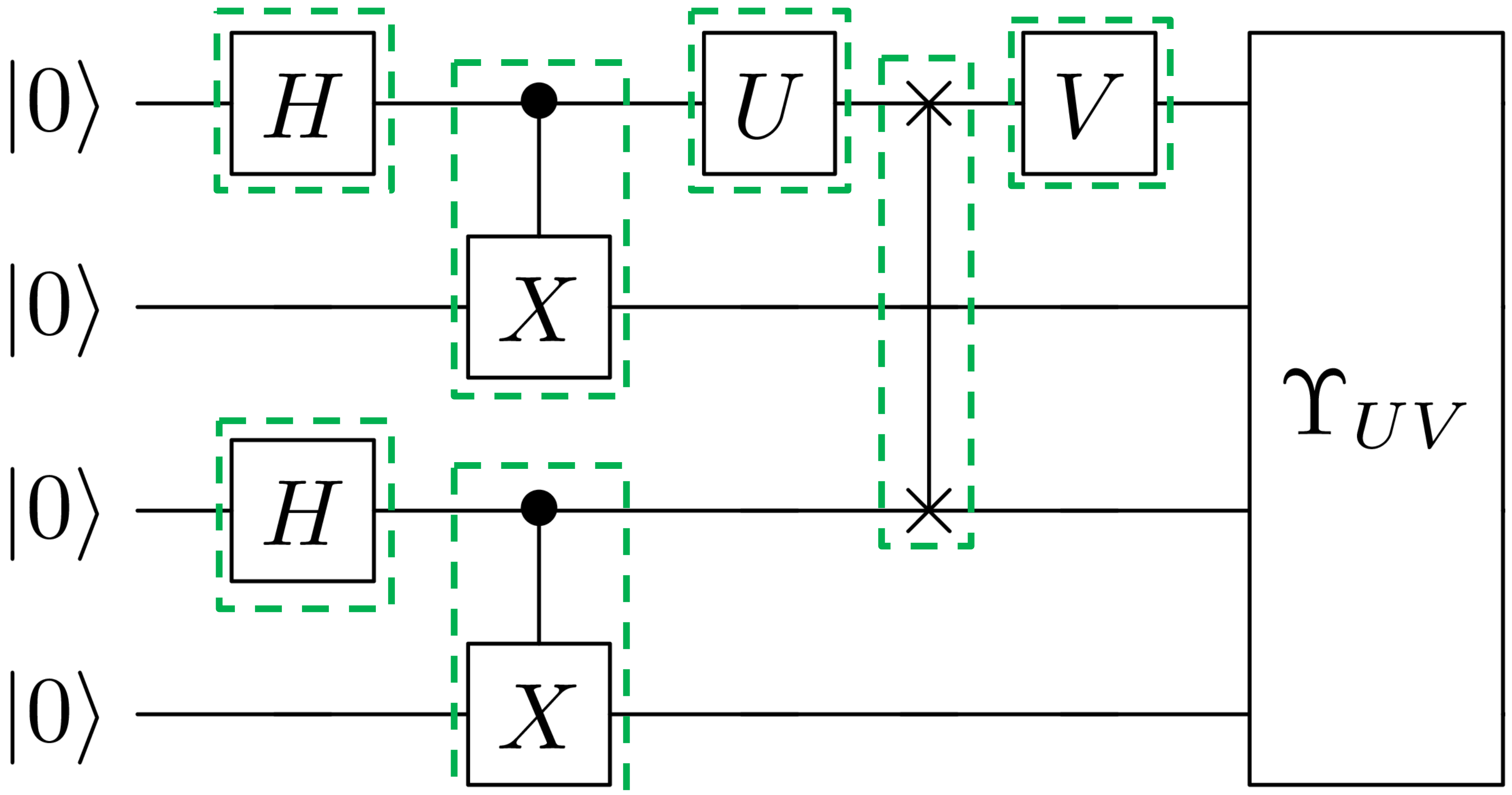
Conditional Errors






Space
|0⟩
H
U
V
|0⟩
X
ϒ_UV
|0⟩
H
|0⟩
X
Time

The Solution



$$= \Upsilon_{UV}^{Mon}$$

Simulator Vs Simulator

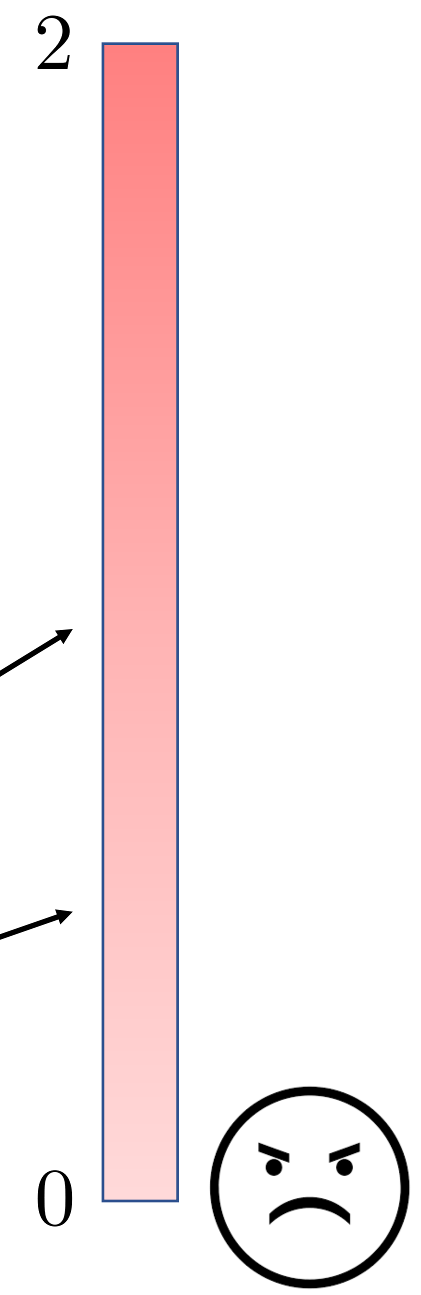
In order to compute the degree of memory due solely to correlated operations we compute the quantum relative entropy of the Choi states

$$S(\rho||\sigma) = \mathrm{Tr}[\rho(\log(\rho) - \log(\sigma))]$$

IBM: $S(\Upsilon_{UV}||\Upsilon_{UV}^{IBM}) = 1.02 \pm 0.04$

Monash: $S(\Upsilon_{UV}||\Upsilon_{UV}^{Mon}) = 0.68 \pm 0.04$

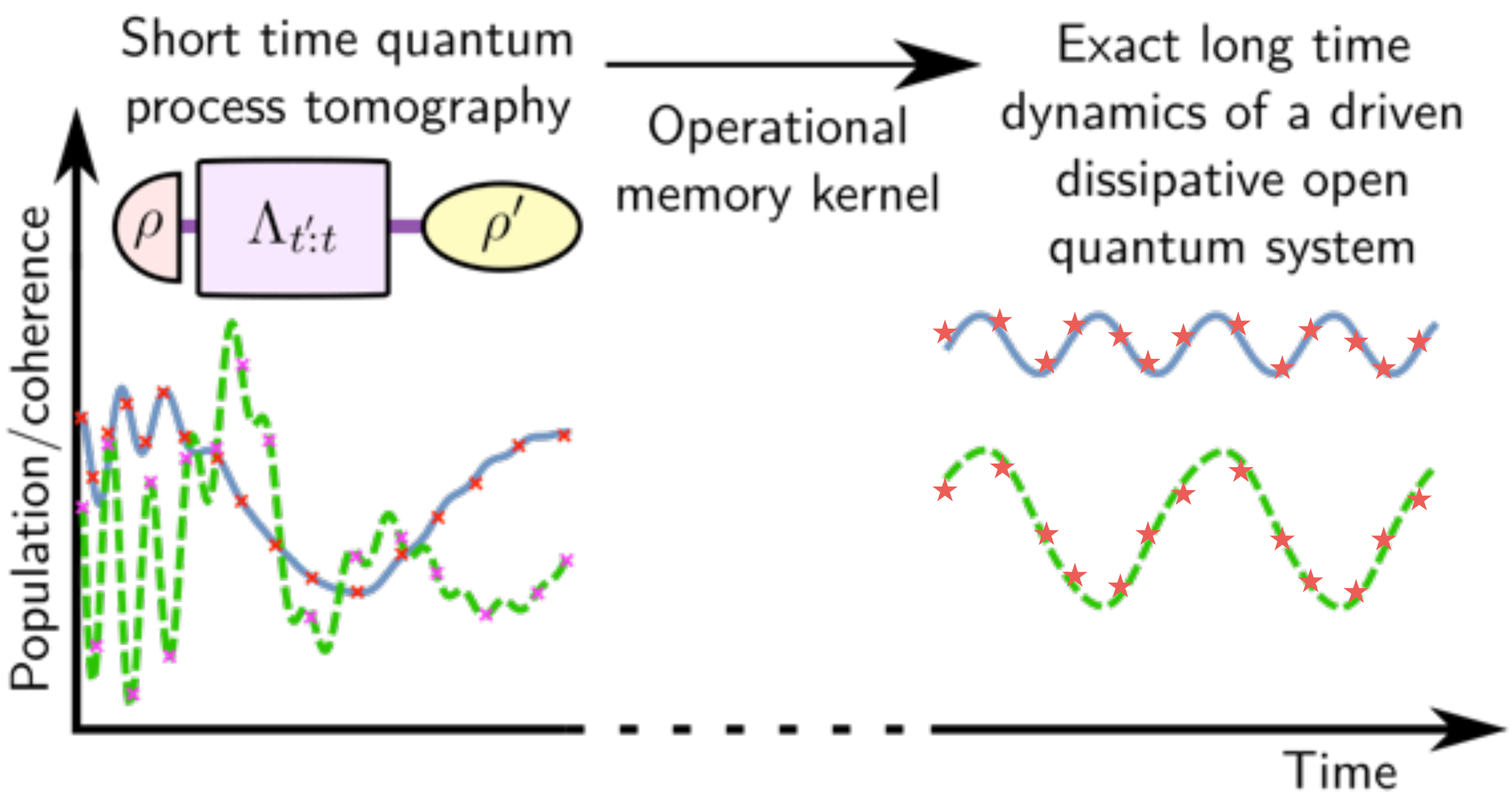
$$\Pr_{conf} = \exp(-nS(\rho||\sigma))$$



In preparation.

Efficient long time dynamics

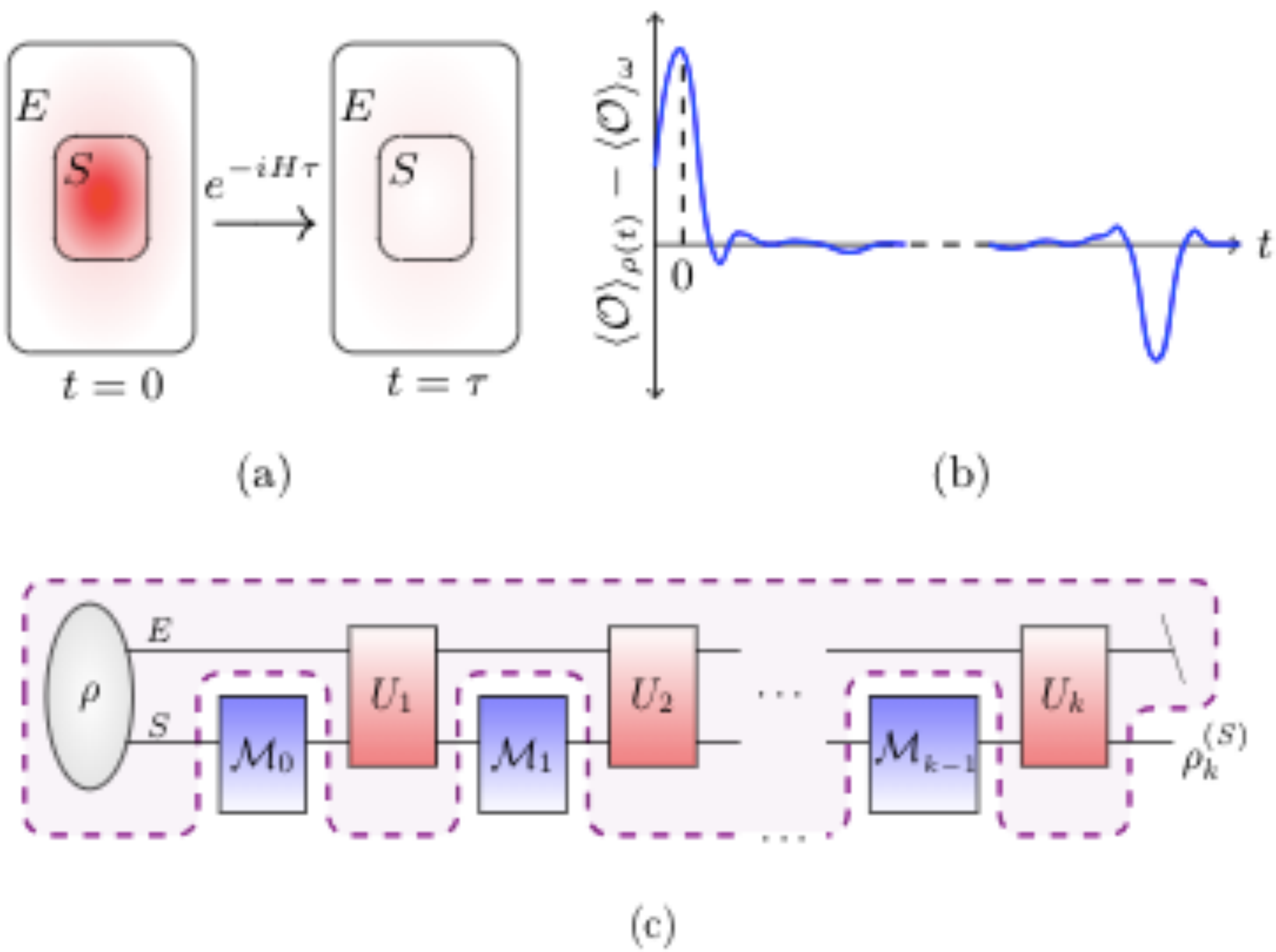
Reconstructed Nakajima-Zwanzing master equations

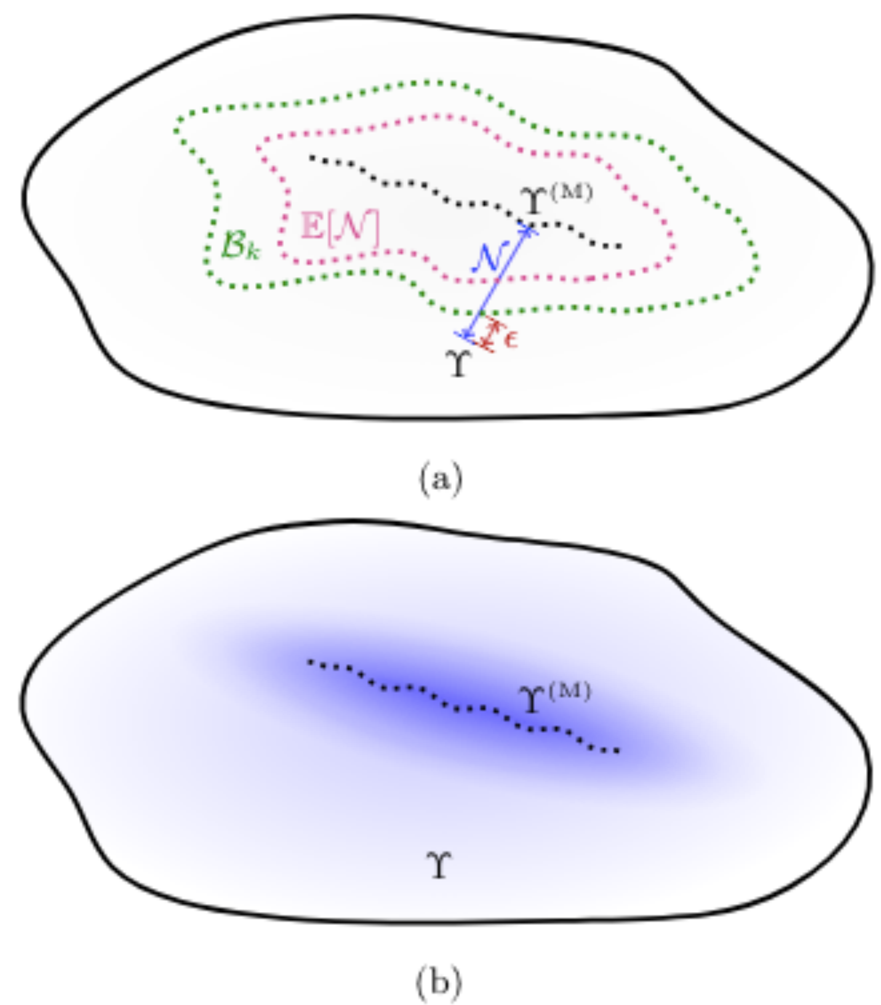


Pollock, Modi *arXiv*:1704.06204 (2017)

Typicality

Almost Markovian processes from closed dynamics

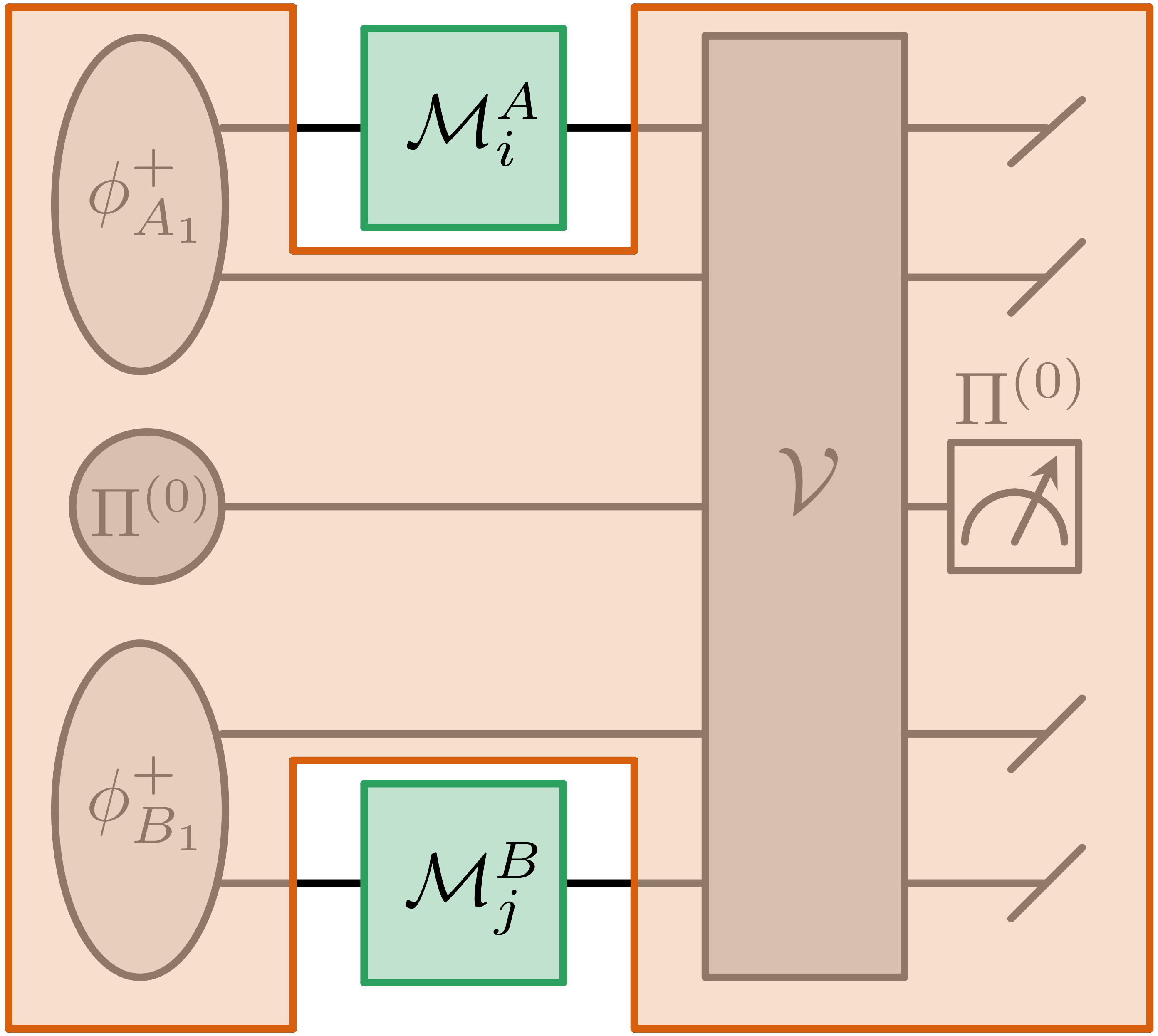




P. Figueroa-Romero, K. Modi, F. Pollock arXiv:1802.10344 (2018)

Relation to process matrix

Entanglement, non-Markovianity, and causal non-separability



S. Milz, F. Pollock, T. Le, G. Chiribella, K. Modi New J. Phys. 20, 033033 (2018)

Conclusions

We have a universal descriptor for arbitrary quantum processes and whole lot of applications…

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MonQIS

http://monqis.physics.monash.edu

S. Milz, F. Pollock, K. Modi Open Sys. Info. Dyn. 24, 1740016 (2017)

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