EBERHARD KARLS UNIVERSITÄT TÜBINGEN
FACULTY OF SCIENCE
Institute of Theoretical Physics

Quantum-chaotic sensors

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KITP Santa Barbara, April 2019

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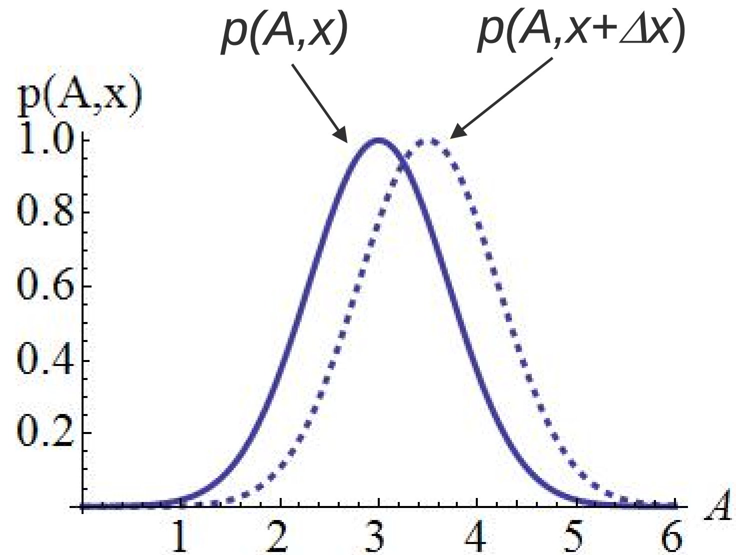
Parameter estimation



- Known distribution *p(A,x)*
- Sample random variable *A* => *M* results A_i
- Estimate parameter *x* given the A_i with *estimator function*

$$x_{est}(A_1, \ldots, A_M)$$

Unbiased estimator: $\langle x_{\text{est}} \rangle = x$

Uncertainty of estimation of x:

$$\delta x \equiv \sigma(x_{est}(A_1, \ldots, A_M))$$

portraits of statisticians, www.york.ac.uk

Smallest uncertainty, optimized over all unbiased estimators:

$$\delta x \geq \delta x_{\min} = \frac{1}{\sqrt{M I_{\text{Fisher}}}}$$

Cramér-Rao bound

$$I_{\text{Fisher}} = \int dA\, p(A,x) \left(\frac{\partial \ln p(A,x)}{\partial x} \right)^2$$

H. Cramér '46; C.R. Rao '45

Parameter estimation

- Known distribution *p(A,x)*
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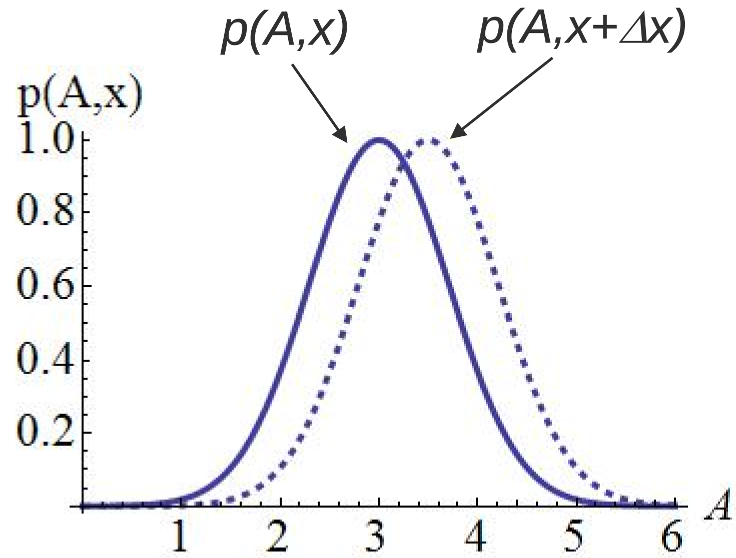

Unbiased estimator: $\langle x_{\text{est}} \rangle = x$

Uncertainty of estimation of x:

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Can be saturated for $M \longrightarrow \infty$ with max likelihood estimator

Smallest uncertainty, optimized over all unbiased estimators: $$\delta x \geq \delta x_{\min} = \frac{1}{\sqrt{M I_{\text{Fisher}}}}$$

$$I_{\text{Fisher}} = \int dA\, p(A,x) \left(\frac{\partial \ln p(A,x)}{\partial x} \right)^2$$

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Quantum Parameter Estimation

QM: state ρ_x. Choice of measurement! Create *p(A,x)*.

$$\delta x \geq \delta x_{\mathrm{min}} = \frac{1}{\sqrt{M\, I_{\mathrm{QFisher}}}}$$

$$I_{\mathrm{QFisher}} = \mathrm{tr}(\rho_x L_{\rho_x}^2)$$

$$\frac{\partial \rho_x}{\partial_x} = \frac{1}{2}(\rho_x L_{\rho_x} + L_{\rho_x}\rho_x)$$

symmetric logarithmic derivative

Quantum Cramér-Rao bound

Helstrom ’67,’68,’76;
Holevo ’73,’74

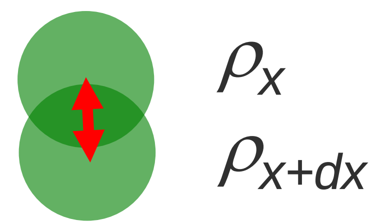
Quantum Parameter Estimation (q-pet)

Physical meaning of Quantum Cramér-Rao bound:

$$I_{\mathrm{IQFisher}} = 4 ds^2_{\mathrm{Bures}}(\rho_x, \rho_{x+dx})/dx^2$$

Braunstein & Caves, PRL '94

- (Quantum) information-theoretical interpretation:
  - distinguishability of quantum states: Bures distance
  - function of quantum state ρ_x



- Ultimate achievable lower bound
  - for all possible data-analysis schemes (unbiased estimator)
  - for all possible measurements (POVMs)
- Relevant, once all technical noise problems are solved

Quantum-chaotic sensors

General motivation:

- Probes for measurements always (?) taken as integrable so far
    - e.g. harmonic oscillator (mode of light field)
    - precessing spin (magnetometer)

- What happens for non-integrable (chaotic) dynamics?

- Possible to render integrable dynamics chaotic by making hamiltonian time-dependent (e.g. kicked top)

L. Fiderer and D. Braun, Nat. Com. (2018) 9:1351

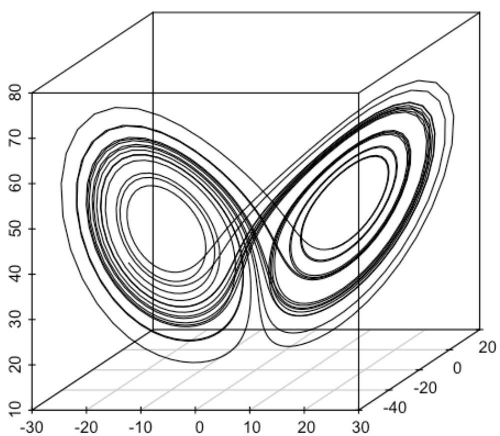
Classical chaos

- Extreme sensitivity to initial conditions:
  - Close-by trajectories diverge exponentially
  - Quantified by Lyapunov exponent:
  - λ is function of starting point

$$\lambda = \lim_{t\to\infty} \lim_{d_0\to 0} \left[\frac{1}{t} \ln\left(\frac{d(t)}{d_0}\right)\right]$$

- Very few systems have proven full chaos (mixing, e.g. Sinai billard)
- Most physical systems show mixed phase space

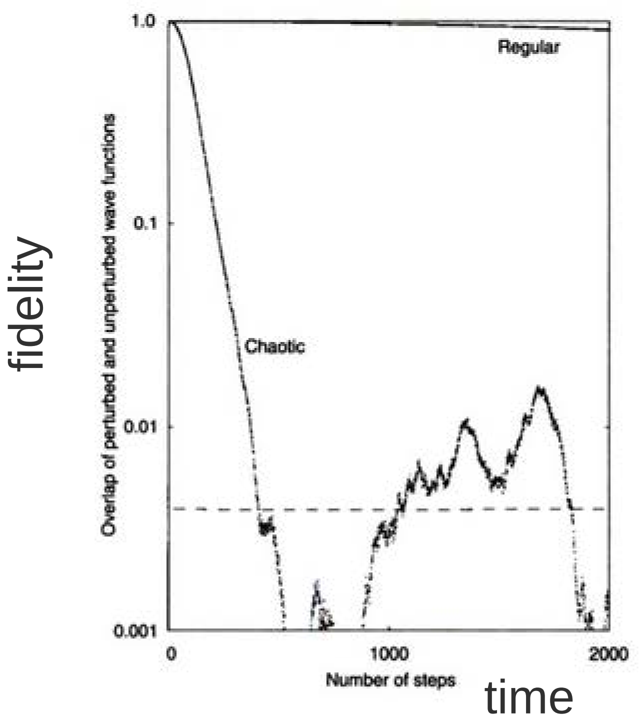
e.g. Henon attractor



Quantum chaos

- Exponential divergence of distance between quantum states impossible for unitary time evolution (scalar product conserved)
- But: possible exponential sensitivity to parameters of system

=> useful for metrology?



A. Peres 1995

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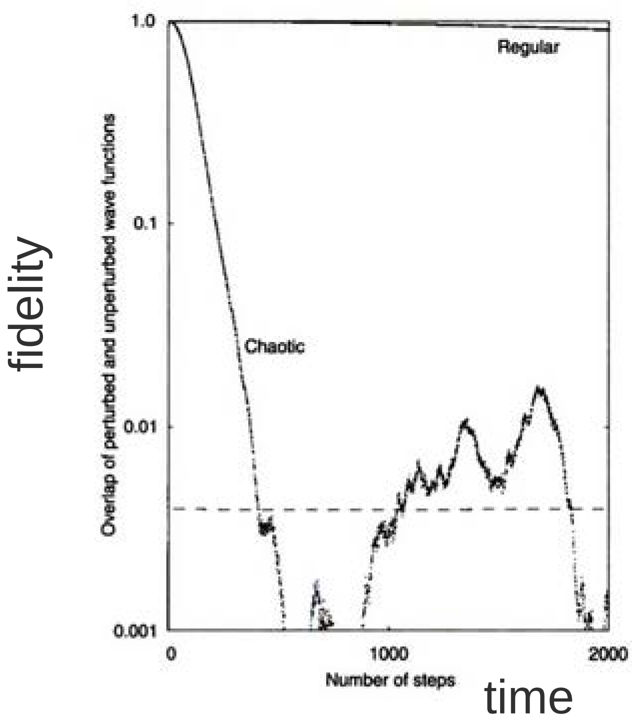

Loschmidt echo:

$$F_{\delta\alpha}(t) = |\langle\psi|U_{\alpha+\delta\alpha}(-t)U_{\alpha}(t)|\psi\rangle|^2$$

A. Peres 1995

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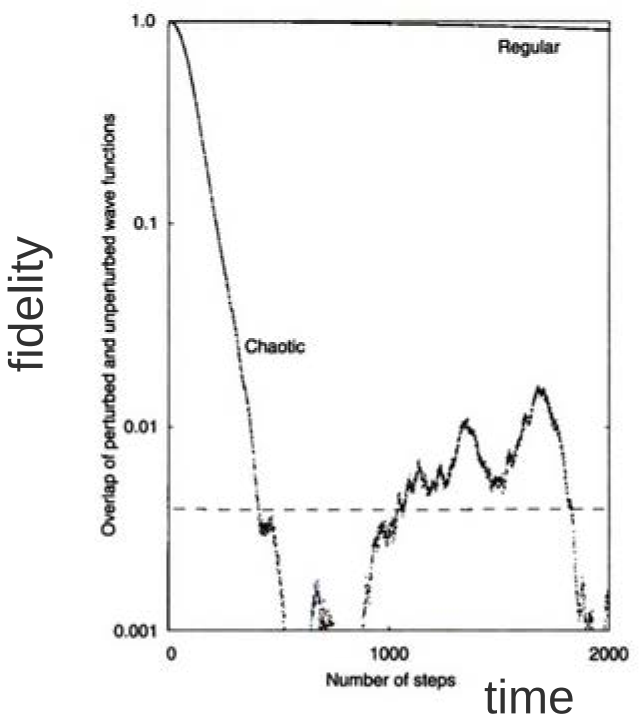

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Related to pure state QFI:

$$I_\alpha(t) = \lim_{\delta\alpha\to 0} 4\frac{1-F_{\delta\alpha}(t)}{\delta\alpha^2}$$

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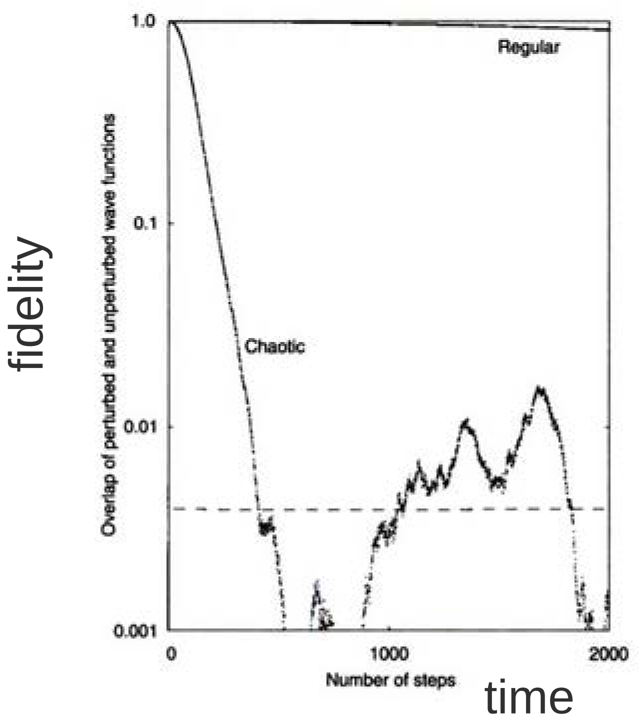

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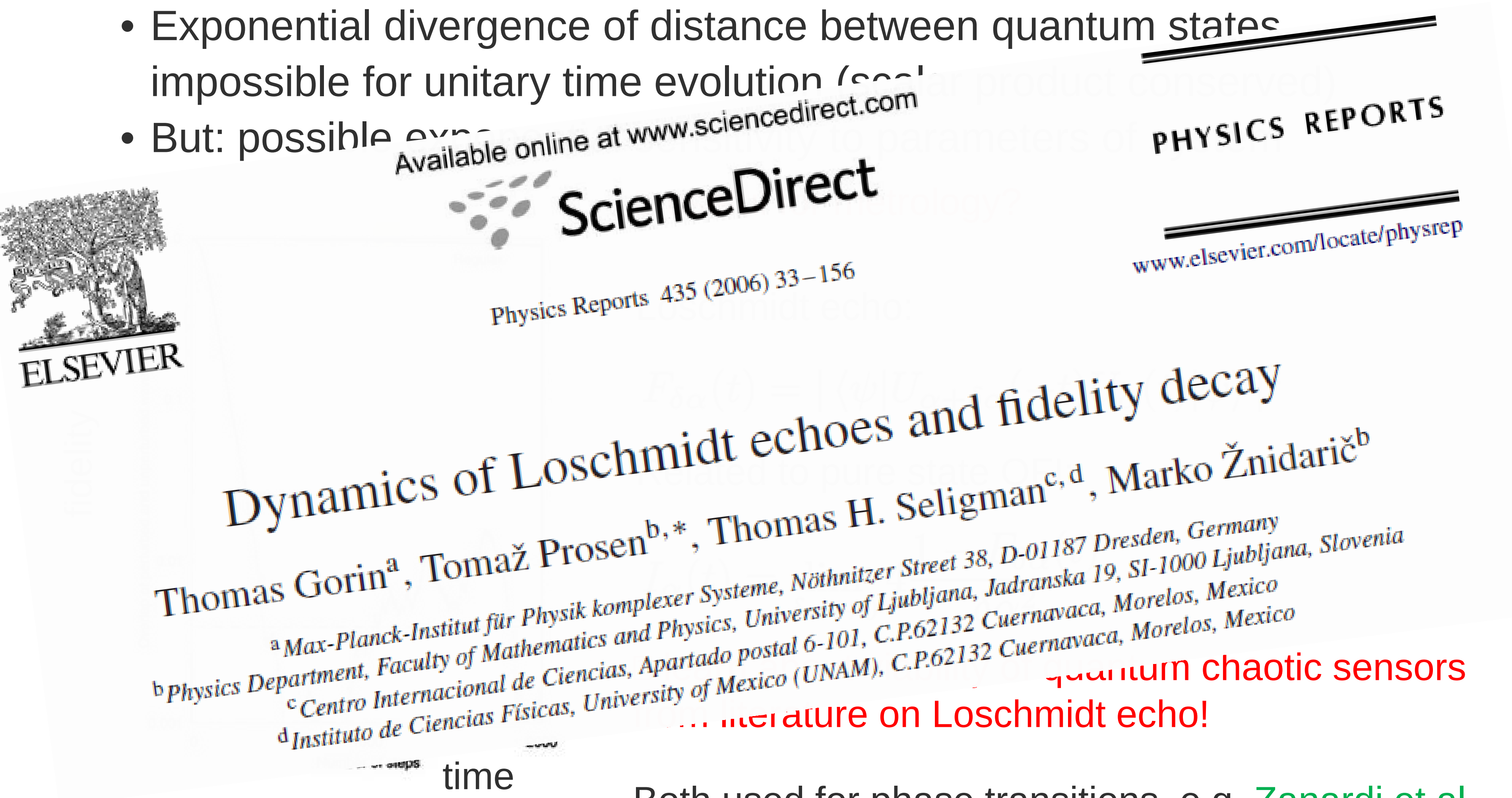
Related to pure state QFI:

$$I_\alpha(t) = \lim_{\delta\alpha\to 0} 4\frac{1-F_{\delta\alpha}(t)}{\delta\alpha^2}$$

=> learn about viability of quantum chaotic sensors from literature on Loschmidt echo!

Quantum chaos

- Exponential divergence of distance between quantum states impossible for unitary time evolution (scalar product conserved)
- But: possible exp


Available online at www.sciencedirect.com
ScienceDirect
PHYSICS REPORTS
www.elsevier.com/locate/physrep
ELSEVIER
Physics Reports 435 (2006) 33–156

Dynamics of Loschmidt echoes and fidelity decay

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quantum chaotic sensors
literature on Loschmidt echo!

time

A. Peres 1995

Both used for phase transitions, e.g. Zanardi et al. PRA 2009!

Model system: Kicked top

- (Pseudo-)angular momentum of size j, J=j+1/2

$$\mathcal{H}_{\mathrm{KT}} = \alpha J_z + \frac{k}{(2j+1)\hbar} J_y^2 \sum_{n=-\infty}^{\infty} \tau\delta(t-n\tau) \qquad (\hbar = \tau = 1)$$

- F. Haake, M. Kus, R. Scharf, Z. Phys. Cond. Matt. (1987)
- F. Haake, „Quantum Signatures of Chaos“ (Springer, 1992, 3rd ed. 2010, 4th ed. 2018)

Kicked top

- (Pseudo-)angular momentum of size j, J=j+1/2

$$\mathcal{H}_{\mathrm{KT}} = \alpha J_z + \frac{k}{(2j+1)\hbar} J_y^2 \sum_{n=-\infty}^{\infty} \tau\delta(t-n\tau) \qquad (\hbar=\tau=1)$$

$$U_\alpha(k) = T\exp\left(-i\int_t^{t+1} dt' H_{\mathrm{KT}}(t')\right) = e^{-ik\frac{J_y^2}{2J}} e^{-i\alpha J_z}$$

non-linearity

- F. Haake, M. Kus, R. Scharf, Z. Phys. Cond. Matt. (1987)
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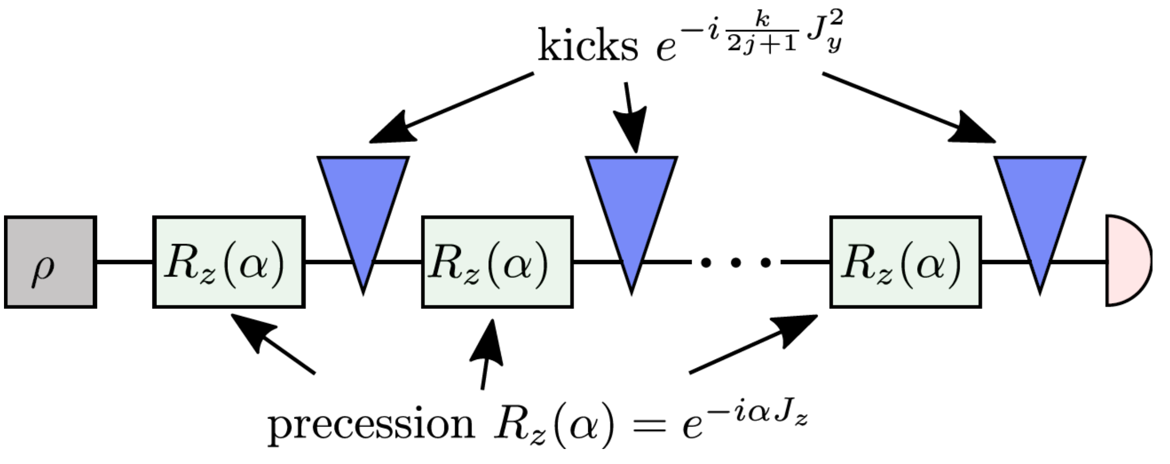
Kicked top

- (Pseudo-)angular momentum of size j, J=j+1/2

$$\mathcal{H}_{\rm KT} = \alpha J_z + \frac{k}{(2j+1)\hbar} J_y^2 \sum_{n=-\infty}^{\infty} \tau\delta(t-n\tau) \qquad (\hbar=\tau=1)$$

$$U_\alpha(k) = T\exp\left(-i\int_t^{t+1} dt' H_{\rm KT}(t')\right) = e^{-ik\frac{J_y^2}{2J}} e^{-i\alpha J_z}$$

non-linearity



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Two time scales

- Ehrenfest time: Time that an initial minimal quantum uncertainty spreads to entire accessible phase space

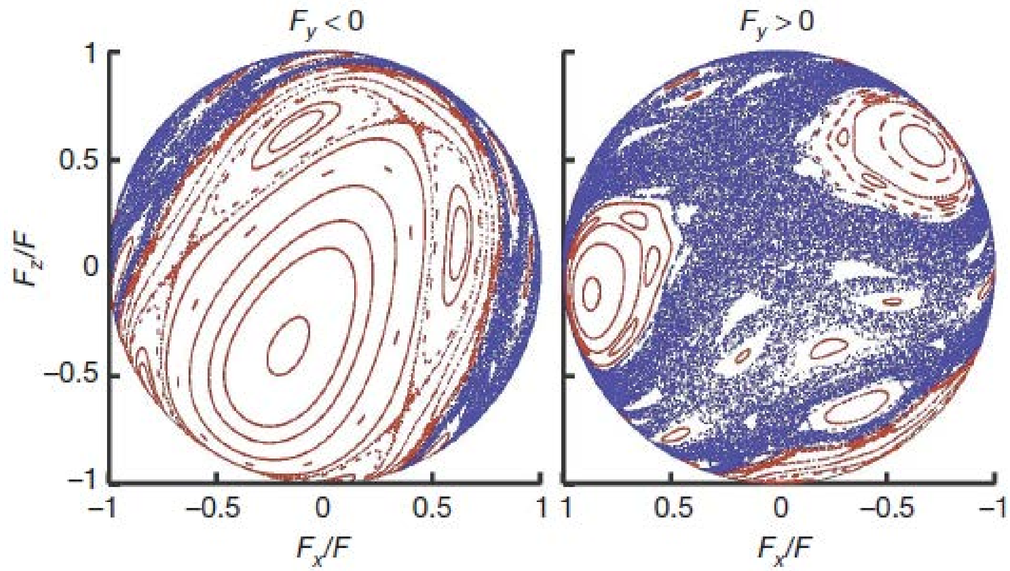
$$t_E = \frac{1}{\lambda} \ln\left(\frac{\Omega_V}{h^d}\right)$$

- Heisenberg time: Inverse typical level spacing

$$t_H = \hbar/\Delta$$

Phase space structure

Poincaré sections: Phase space $(p,q)=(J_z/J,\phi)$



Chaudhury et al.
Nature 2009

α=0.99, k=2

- k=0 integrable, k≈ 2.5 mixed phase space, k ≈ 3.5 almost fully chaotic
- Initial SU(2) coherent state (most classical state possible), area 1/J ~ $\hbar$ in phase space for quantum dynamics

Benchmarks without kicking

- QFI for top without kicking, initial SU(2) coherent state

$$I_\alpha(t) = 2t^2 j \sin^2\theta$$

=> Standard Quantum Limit

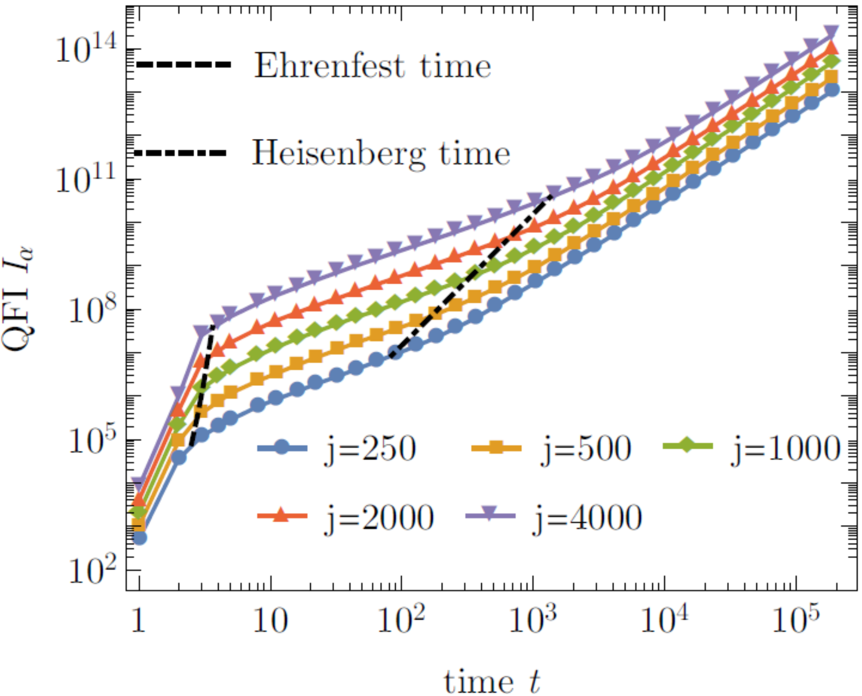
$j = N/2$

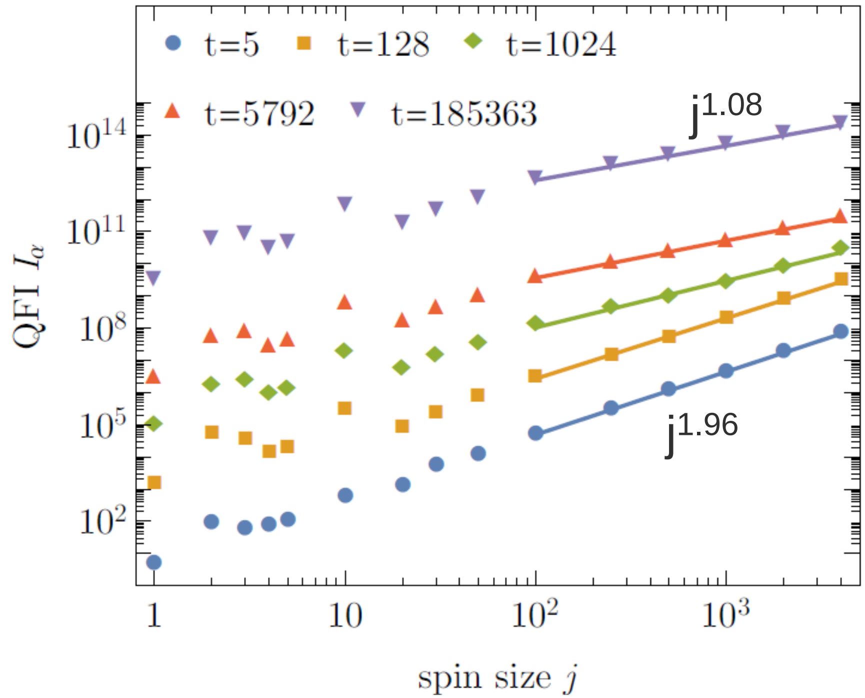
- QFI for top without kicking, initial GHZ state (N spins-1/2)

$$I_\alpha(t) = 4t^2 j^2 \equiv I_{\mathrm{top,GHZ}}$$

=> Heisenberg limit

Results: Kicked top vs integrable top





- Ininitial coherent state at $\theta=\pi/2, \phi=\pi/2$
- Reproduces behavior expected from known Loschmidt echo results:

$$t \simeq t_E\colon\ I_\alpha \propto tj^2 \qquad t_E =\sim \frac{1}{\lambda}\ln(2j+1)$$

$$t \gg t_H\colon\ I(t) = 8s\sigma_{\rm cl}t^2J \qquad t_H =\sim J$$

(s=3, σ_{cl} transport coefficient)

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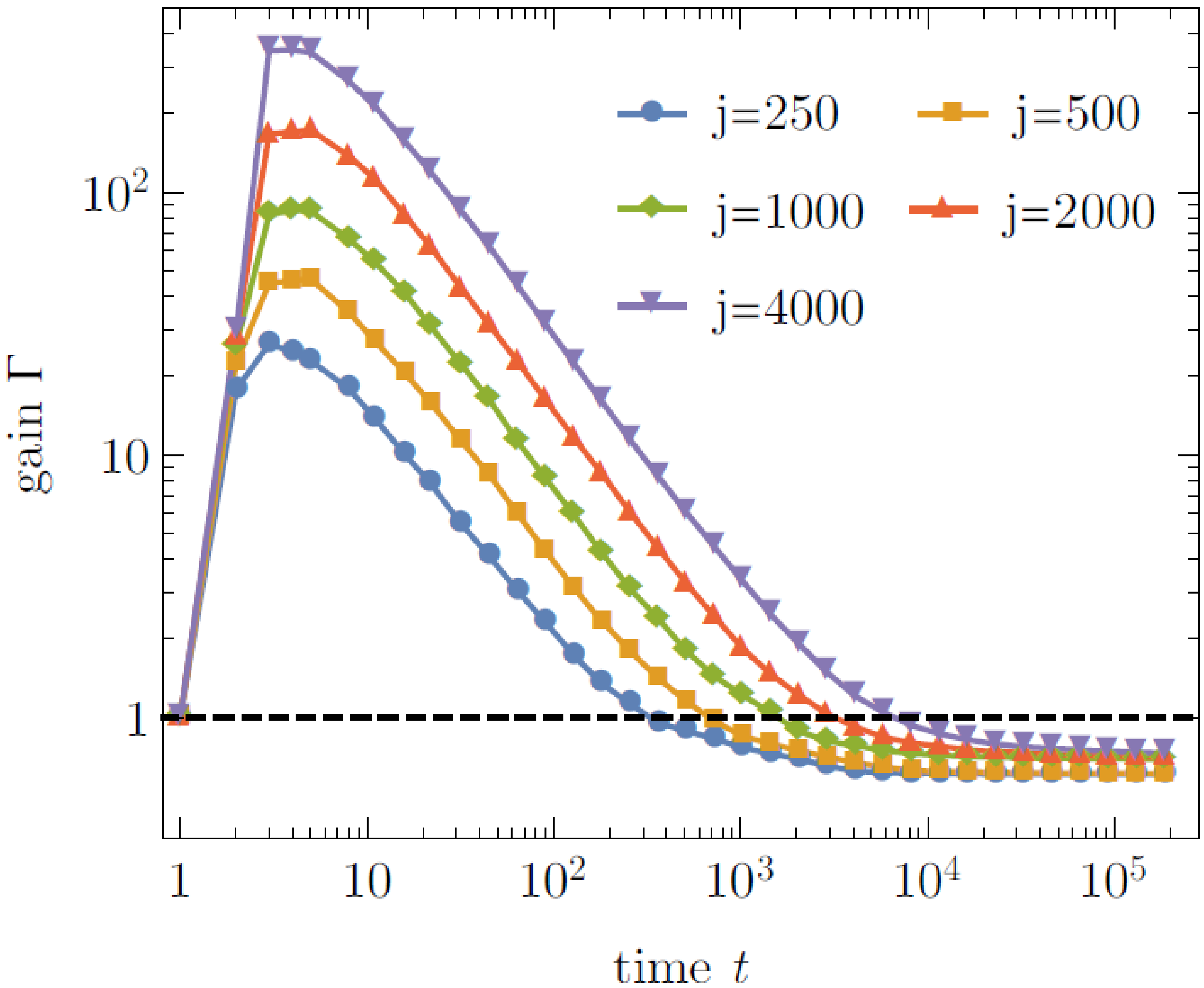
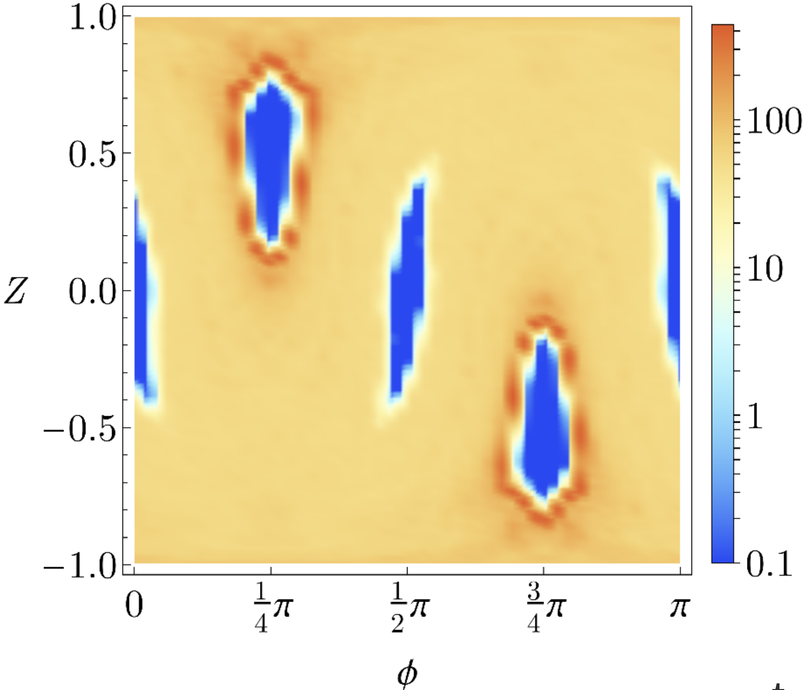
Gain $\Gamma = I_{\alpha,\mathrm{KT}} / I_{\mathrm{top,CS}}$



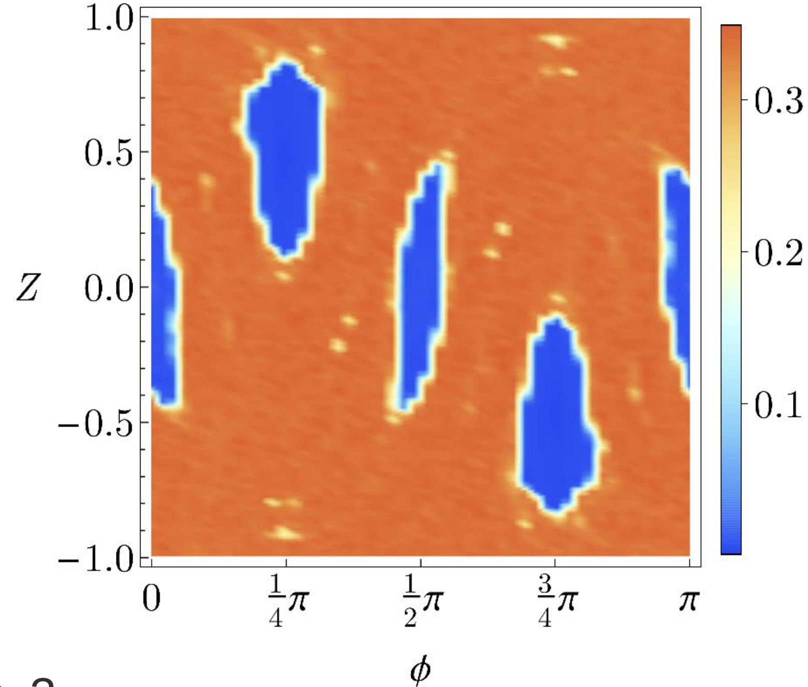
Ininitial coherent state at $\theta=\pi/2, \phi=\pi/2$

Dependence on initial state

Gain in QFI by kicking



Lyapunov exponent



$t=2^{15}$, k=3

- 2D Phase space, coordinates ($Z=J_z/J,\phi$)
- Gain correlated with chaoticity (Lyapunov exponent)
- For large times, largest gain for „edge states“, border to chaotic sea

Dissipative kicked top

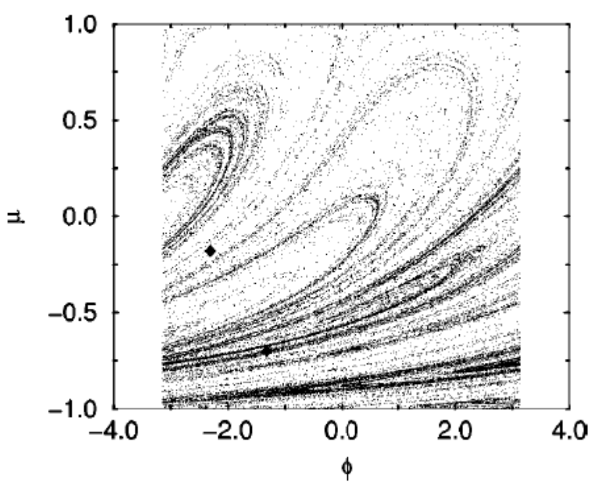
- „Superradiant damping“

$$\frac{d}{dt}\rho(t) = \gamma([J_-,\rho(t)J_+] + [J_-\rho(t),J_+]) \equiv \Lambda\rho(t)$$

- Commutes with precession about z-axis, and negliglible during kicks

$$\rho(t+\tau) = P\rho(t) = U_\alpha(k)\left(\exp(\Lambda\tau)\rho(t)\right)U_\alpha^\dagger(k)$$

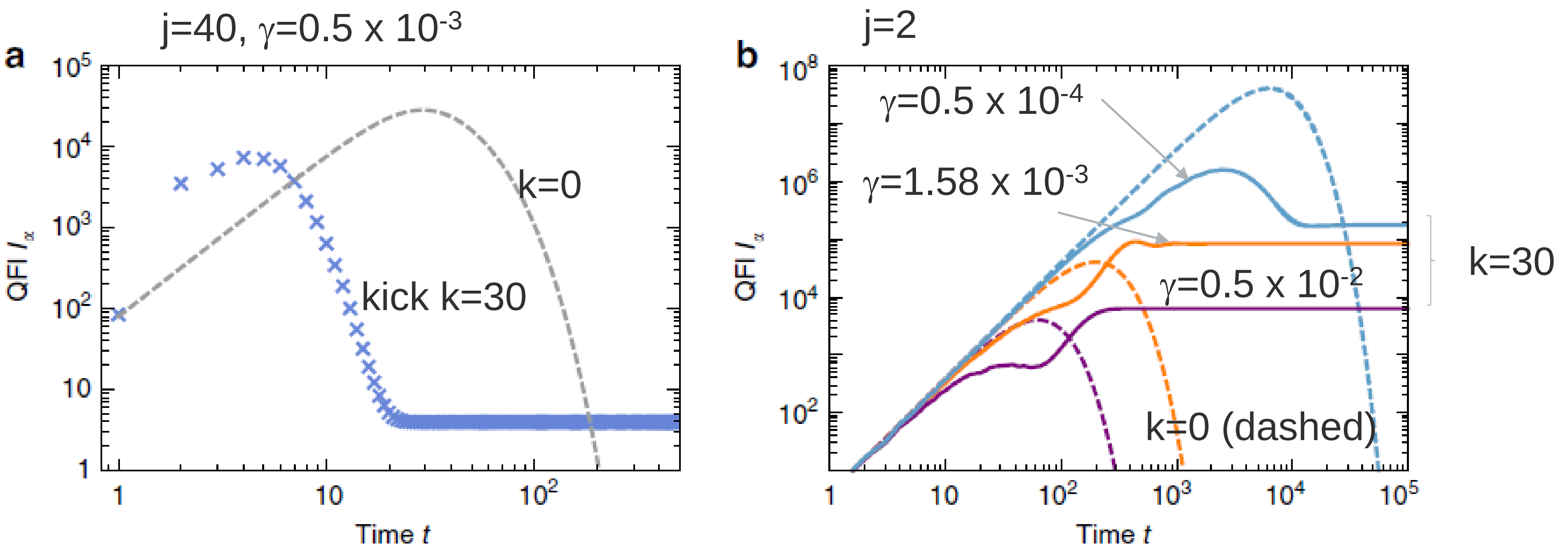
- Classically: Strange attractor in phase space (multifractal)



k=8, α=2, 2J$\gamma\tau$=1

DB Chaos 1999

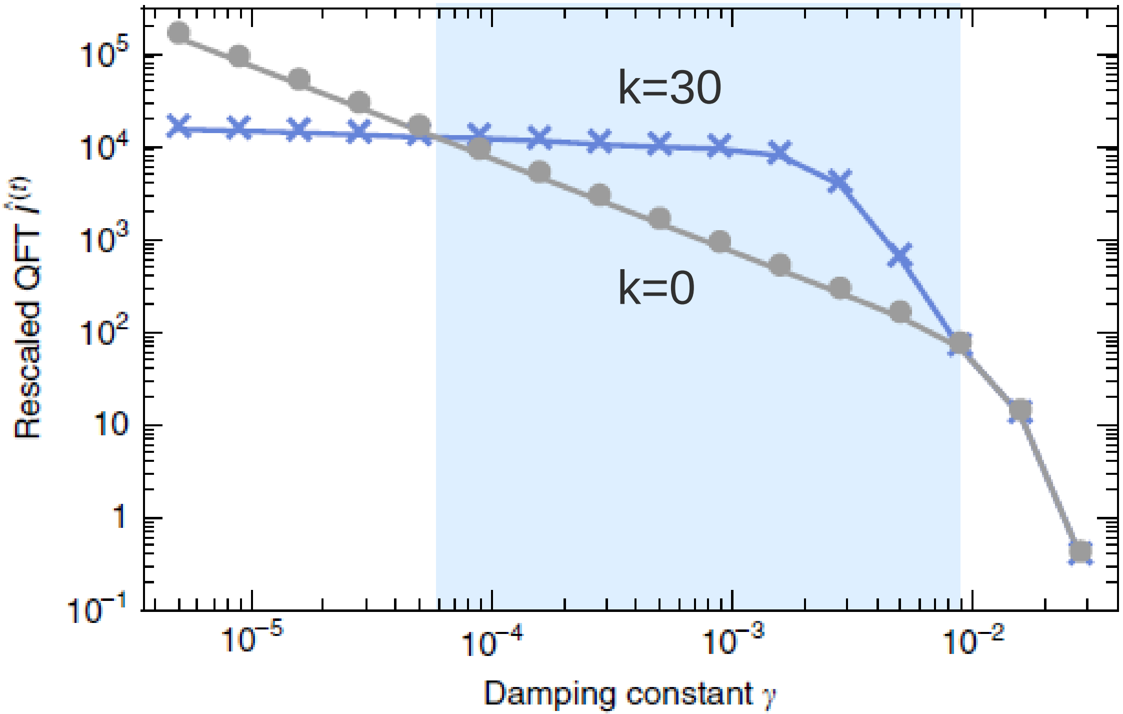
Dissipative kicked top: results



- Comparable maximum (as function of time) of QFI
- Finite plateau value for large time: non-equilibrium steady state contains information about parameter!
- Max value reached much earlier => useful when time counts!
- Relatively large plateau for j=2

Rescaled QFI: sensitivities per Hz$^{1/2}$

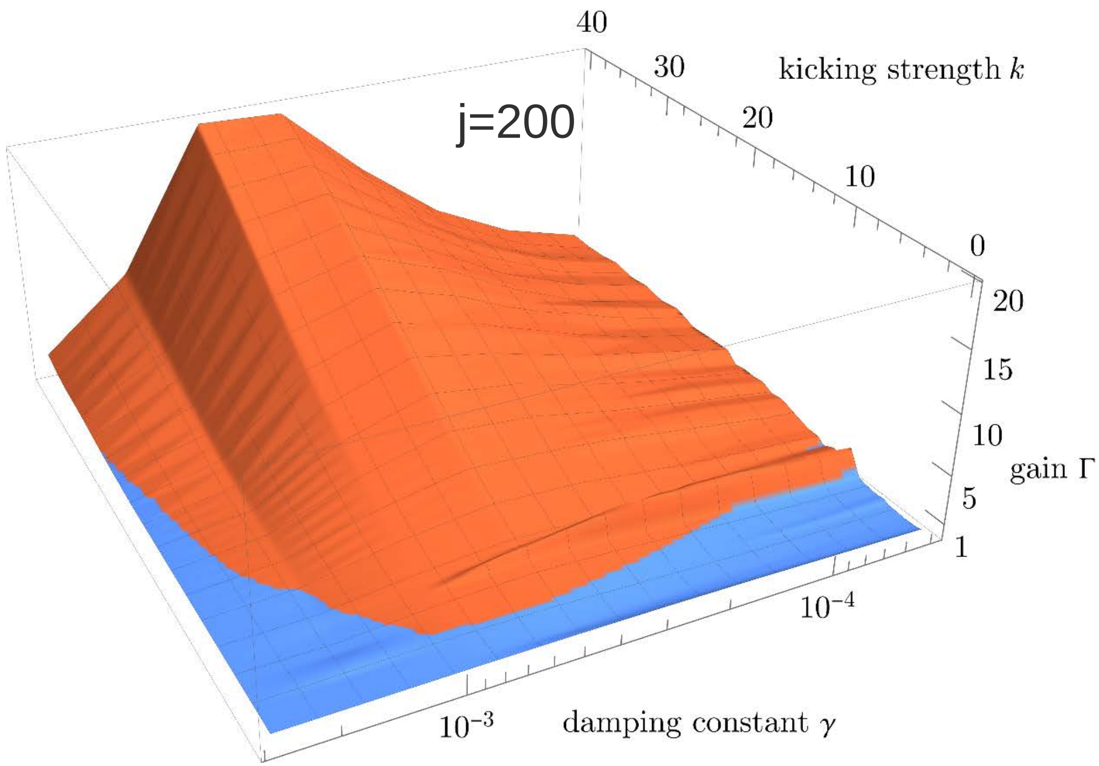
$$I_\alpha^{(t)} \equiv I_\alpha / T_{\rm av}$$



j=100

- Max rescaled QFI of kicked dissipative top outperforms the one without kicks in broad range of dampings
- Optimization in both cases over location of initial coherent state

Dependence of sensitivity gain on non-linearity and dissipation



$$\Gamma = \frac{\hat{I}^{(t)}_{\alpha,\mathrm{DKT}}(\theta{=}\pi/2,\phi{=}\pi/2)}{\hat{I}^{(t)}_{\alpha,\mathrm{DT}}}$$

- Fixed initial state for dissipative kicked top
- QFI for top without kicks optimized over initial state
- Large gain in a broad damping regime through strong kicking

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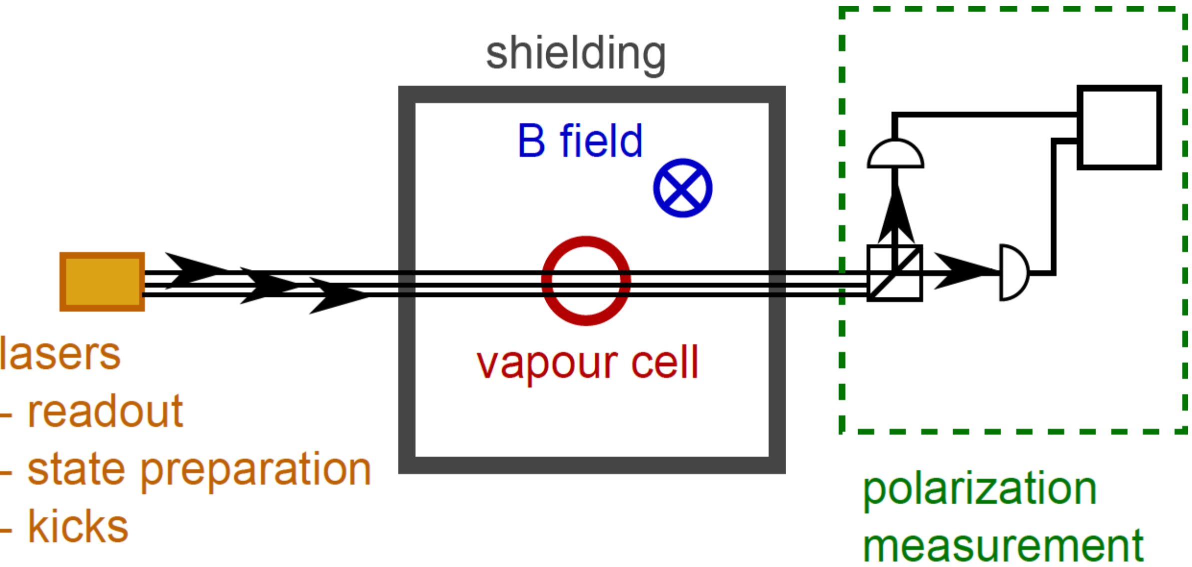
Improving a state-of-the-art SERF magnetometer

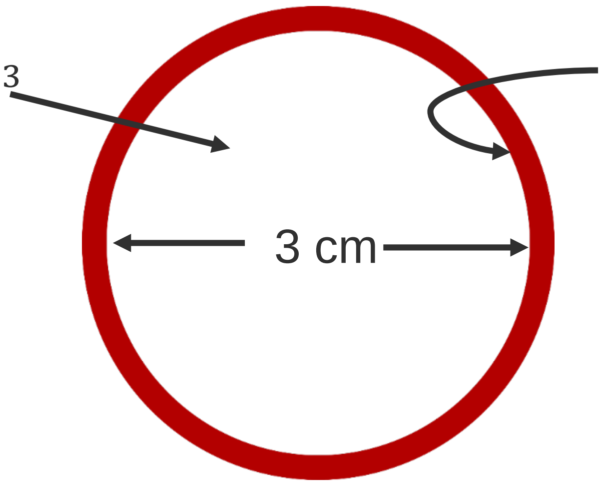
SERF magnetometer



- Cesium vapor magnetometer
- Precession of atomic spins in magnetic field
- „Sping exchange relaxation free“ regime

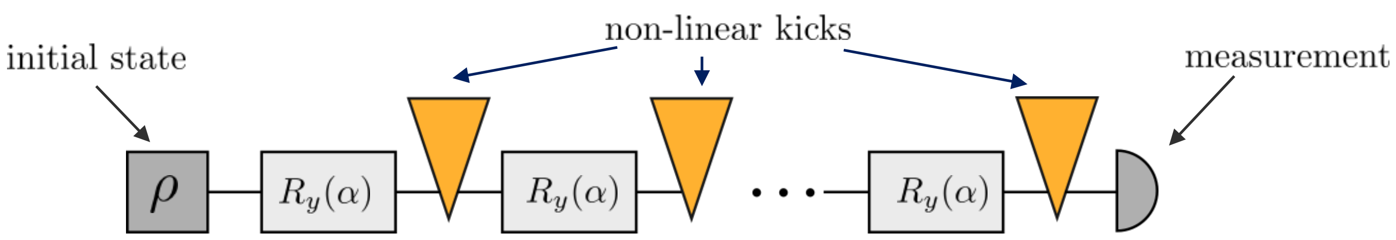
- State preparation: optical pumping to almost fully (0.95) polarized hyperfine state, $j \rightarrow F=3$
- B_z => linear parameter α
- Non-linear kicks: off-resonant laser pulses (AC-Stark shift)
- Read-out: rotation of polarization of probe beam (Faraday effect)

How to render a cesium-vapor magnetometer chaotic?



vapor cell

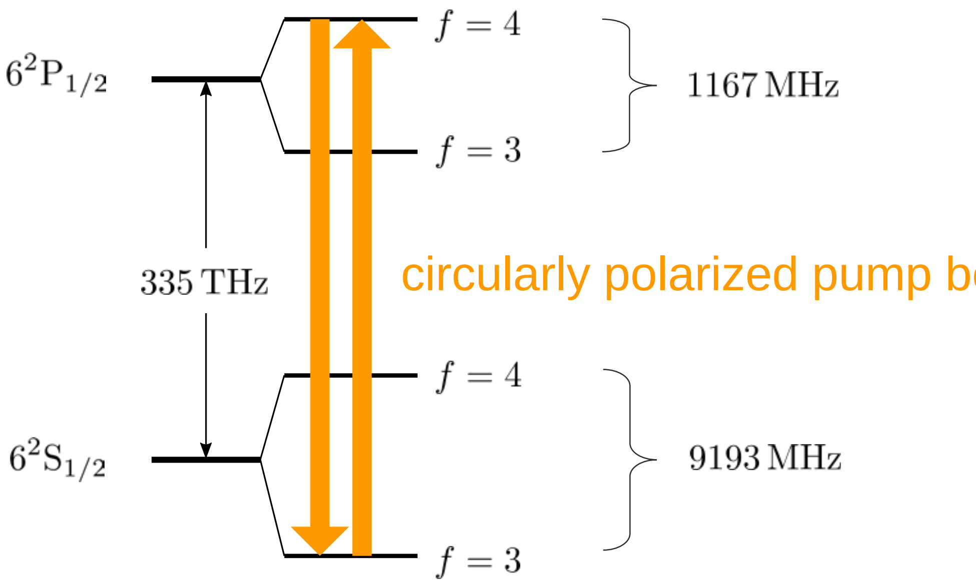
How to render a cesium-vapor magnetometer chaotic?



How to render a cesium-vapor magnetometer chaotic?

$\boxed{\rho}$

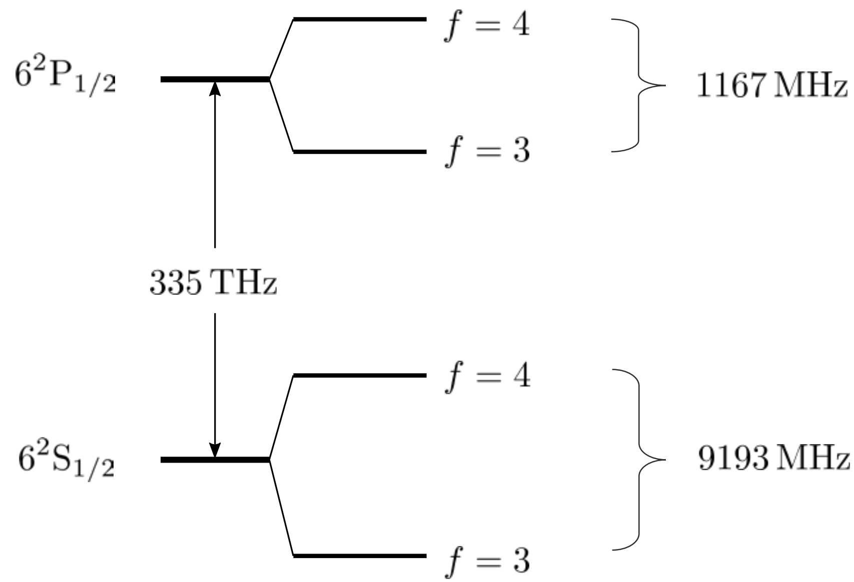
z-polarized state:
spin-temperature distribution



How to render a cesium-vapor magnetometer chaotic?

$\boxed{\rho}$ z-polarized state: spin-temperature distribution

$\boxed{R_y(\alpha)}$ precession in the magnetic field B



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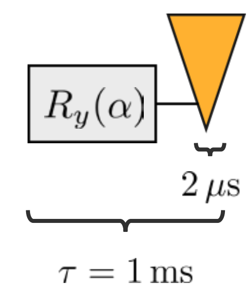
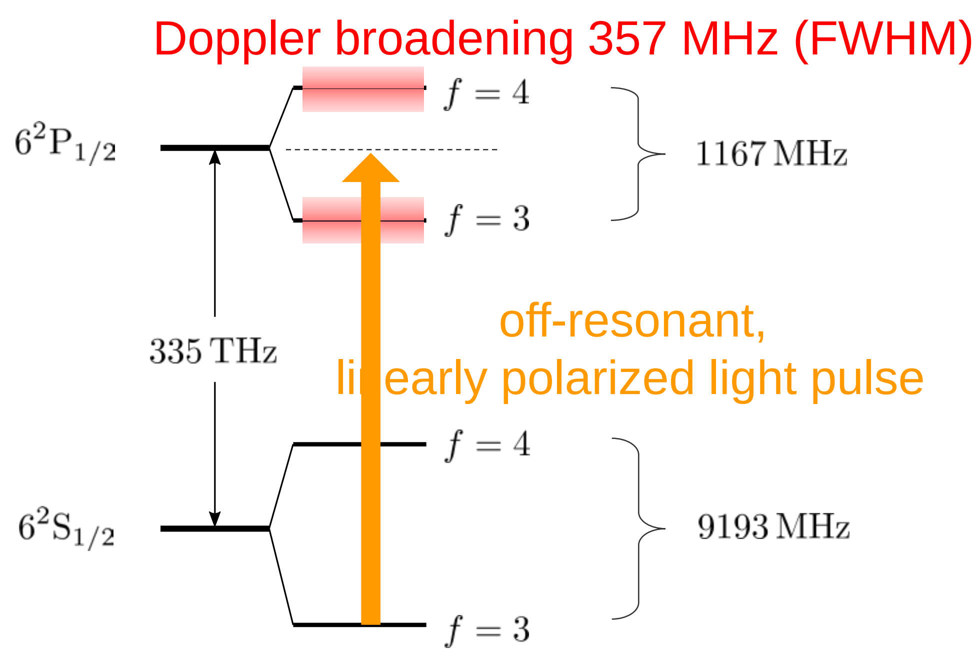
▽ non-linear kicks via the ac Stark effect (rank-2 light shift)



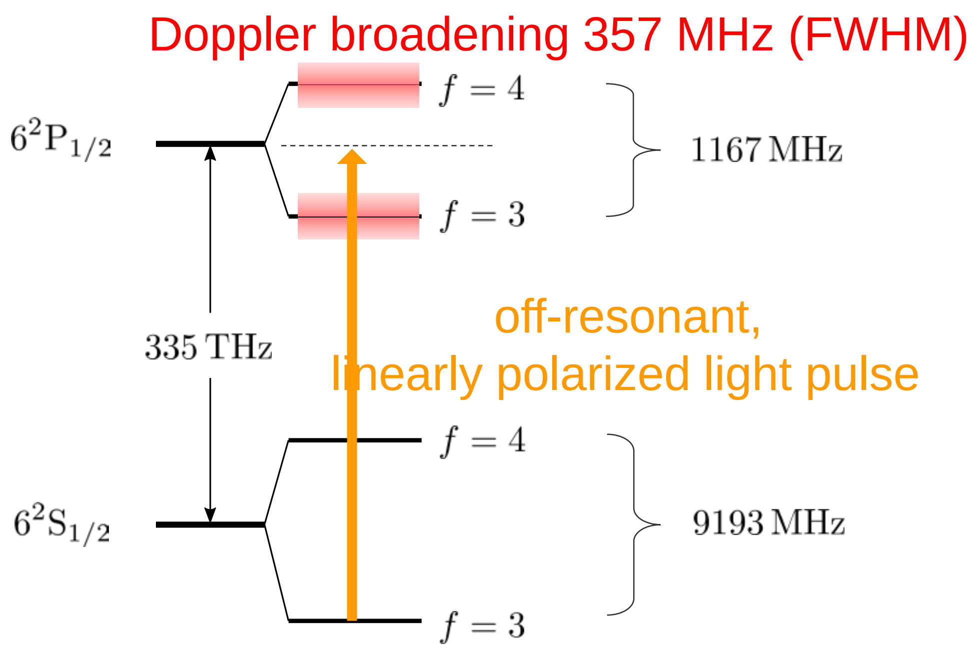
intensity $I_{\text{kick}} = 0.1\,\frac{\text{mW}}{\text{cm}^2}$

$\Rightarrow$ kicking strength $k = 6 \times 10^{-4}$



How to render a cesium-vapor magnetometer chaotic?

- ρ — z-polarized state: spin-temperature distribution
- $R_y(\alpha)$ — precession in the magnetic field B
- non-linear kicks via the ac Stark effect (rank-2 light shift)
- Faraday rotation of off-resonant light detected with polarization measurement



Detailed numerical model

spin exchange

purely nuclear part of ρ

spin distruction

hyperfine coupling
K=nuclear spin

$$\frac{d\rho}{dt} = R_{\mathrm{se}}\left[\varphi(1+4\langle\mathbf{S}\rangle\cdot\mathbf{S})-\rho\right] + R_{\mathrm{sd}}\left[\varphi-\rho\right] + a_{\mathrm{hf}}\frac{[\mathbf{K}\cdot\mathbf{S},\rho]}{i\hbar} + \frac{H_{\mathrm{A}}^{\mathrm{eff}}\rho-\rho H_{\mathrm{A}}^{\mathrm{eff}\dagger}}{i\hbar}$$
$$+\gamma_{\mathrm{nat}}\sum_{q=-1}^{1}\left(\sum_{f,f_1}W_q^{ff_1}\rho_{f_1f_1}\left(W_q^{ff_1}\right)^\dagger + \sum_{f_1\neq f_2}W_q^{f_2f_2}\rho_{f_2f_1}\left(W_q^{f_1f_1}\right)^\dagger\right)$$

$$H_{\mathrm{A},f}^{\mathrm{eff}} = \hbar\Omega_{\mathrm{Lar}}F_y + \sum_{f'}\frac{\hbar\Omega^2C_{j'f'f}^{(2)}}{4(\Delta_{ff'}+i\gamma_{\mathrm{nat}}/2)}\left|\boldsymbol{\epsilon}_{\mathrm{L}}\cdot\mathbf{F}\right|^2$$

precession and kicks

$$\Omega = \gamma_{\mathrm{nat}}\sqrt{I_{\mathrm{kick}}/(2I_{\mathrm{sat}})}$$

detuning from D1 line

Rabi-frequency of the D1 line

$$W_q^{f_bf_a} = \sum_{f'=3}^{4}\frac{\Omega/2}{\Delta_{f_af'}+i\gamma_{\mathrm{nat}}/2}\left(\mathbf{e}_q^*\cdot\mathbf{D}_{f_bf'}\right)\left(\boldsymbol{\epsilon}_{\mathrm{L}}\cdot\mathbf{D}_{f_af'}^\dagger\right)$$

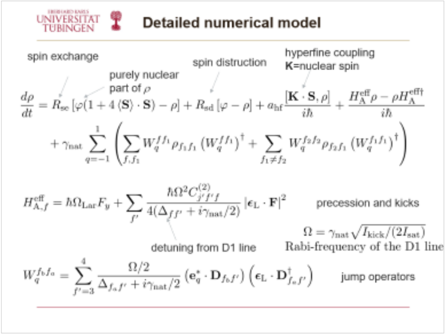
jump operators

Detailed numerical modeling

- Experimentally confirmed modelling:
  - SERF magnetometer ($\Omega_{\text{lamor}} \gg \gamma_{SE}$) Budker et al. PRA 2008; PRL 2010
  - non-linear pulses Chaudhury et al. Nature 2009

- Master equation includes all relevant decoherence mechanisms:
  - spin-exchange collisions
  - spin-destruction collisions
  - jump processes induced by kicks
  - Doppler broadening (average detuning over Maxwell-Boltzmann distribution of thermal cloud of atoms)

- F=3, so no large gain to be expected, but still…



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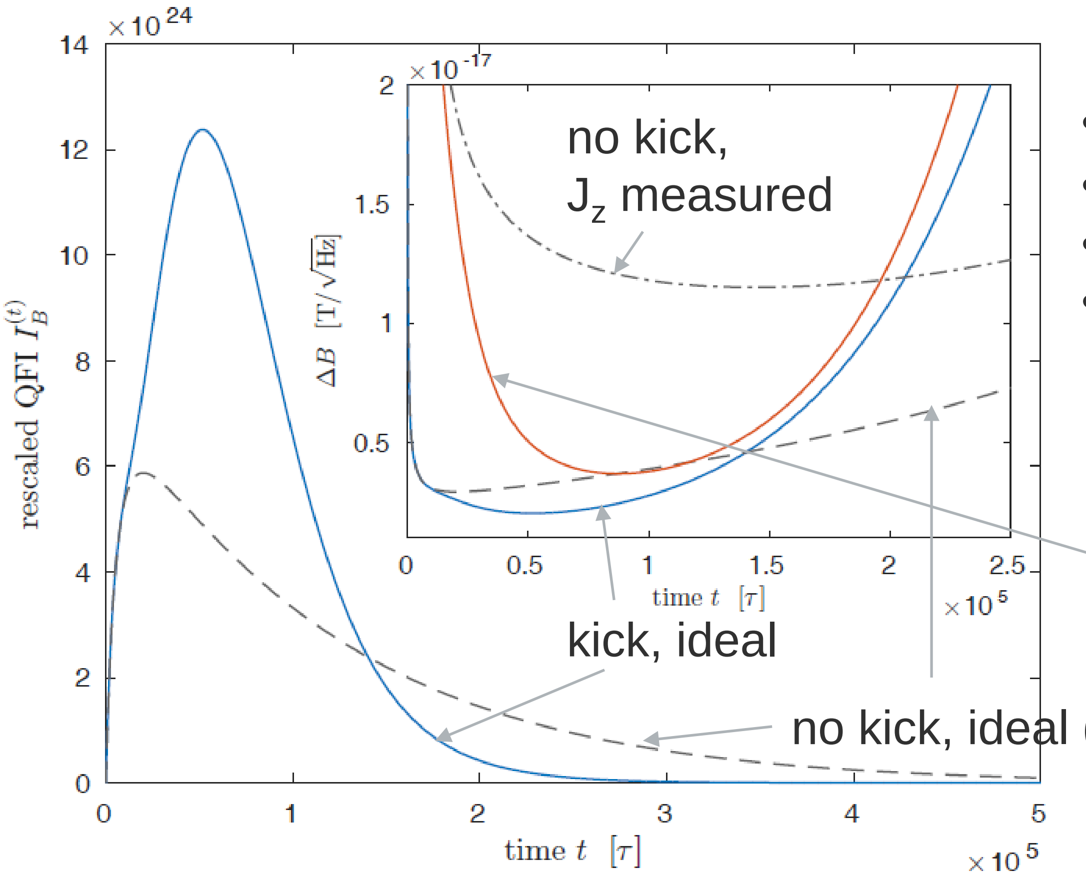
Results for SERF magnetometer



- B_y=4 x10^{-14} T
- n=2 x 10^{10} atoms/cm^3
- period τ=1ms
- kick strength k=6.5 x 10^{-4} from 2μs pulse at end of period, x-direction, Intensity 0.1 mW/cm^2

- 1.5 times smaller Δ B based on rescaled QFI
- 3.1 times smaller Δ B from Faraday effect read-out

Beyond quantum chaos ?

- Large new freedom
  - Kicking times and total number of kicks
  - Kicking strengths
  - Kicking directions

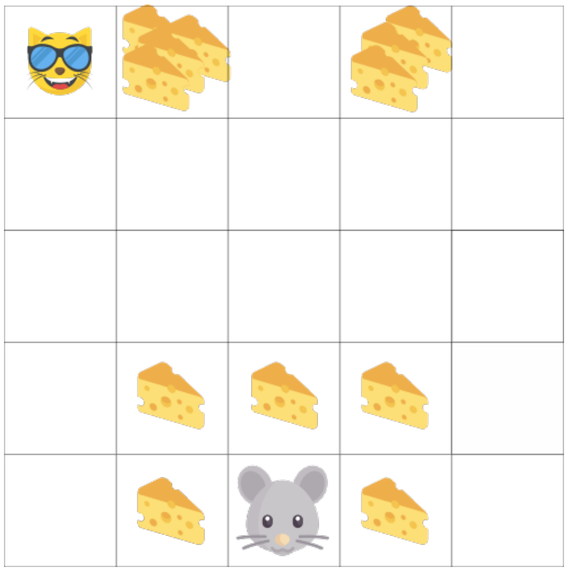
- Gain beyond quantum-chaotic kicked-top?
  - High-dimensional optimization problem!
  - Calls for Machine Learning!

- Classical ML for q-metrology
  - Classical information, quantum actions
  - Reward (final QFI) at end of long sequence of pulses
  - Reinforcement Learning

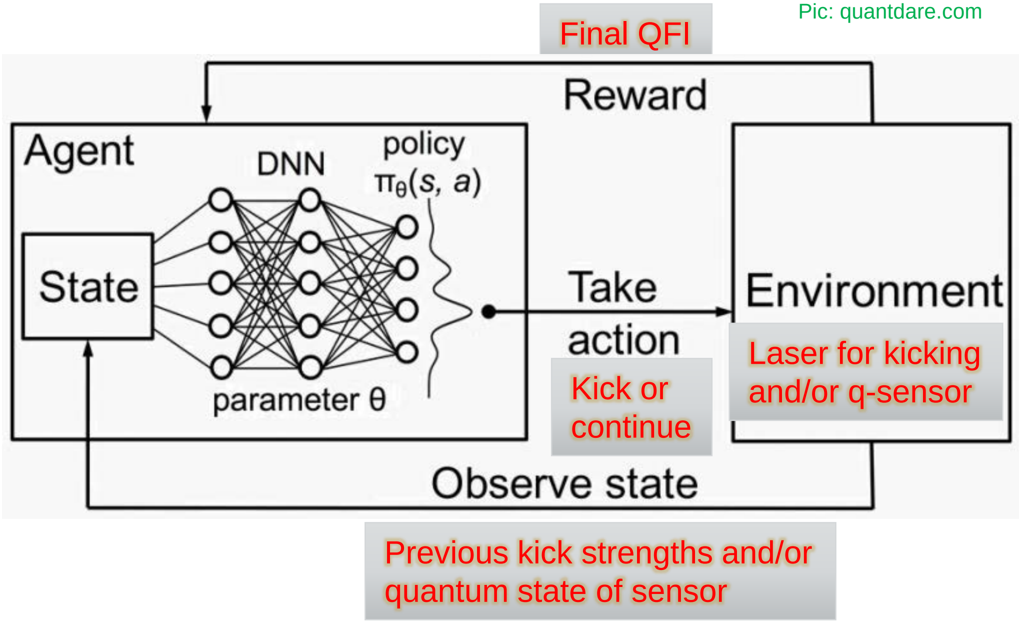
Reinforcement learning

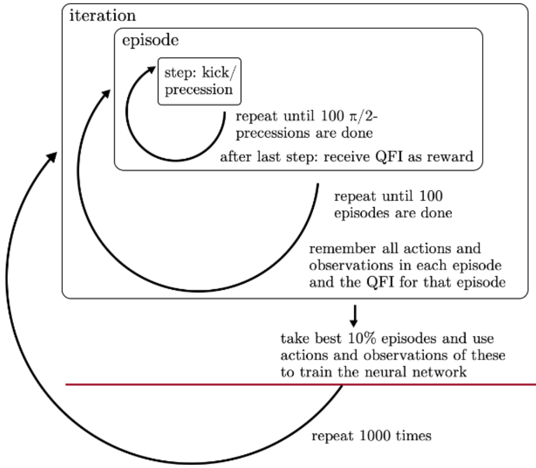
Agent performs (probabilistic) actions based on observed state of environment, and memory and rewards from previous actions

Pic: medium.freecodecamp.org



Pic: quantdare.com

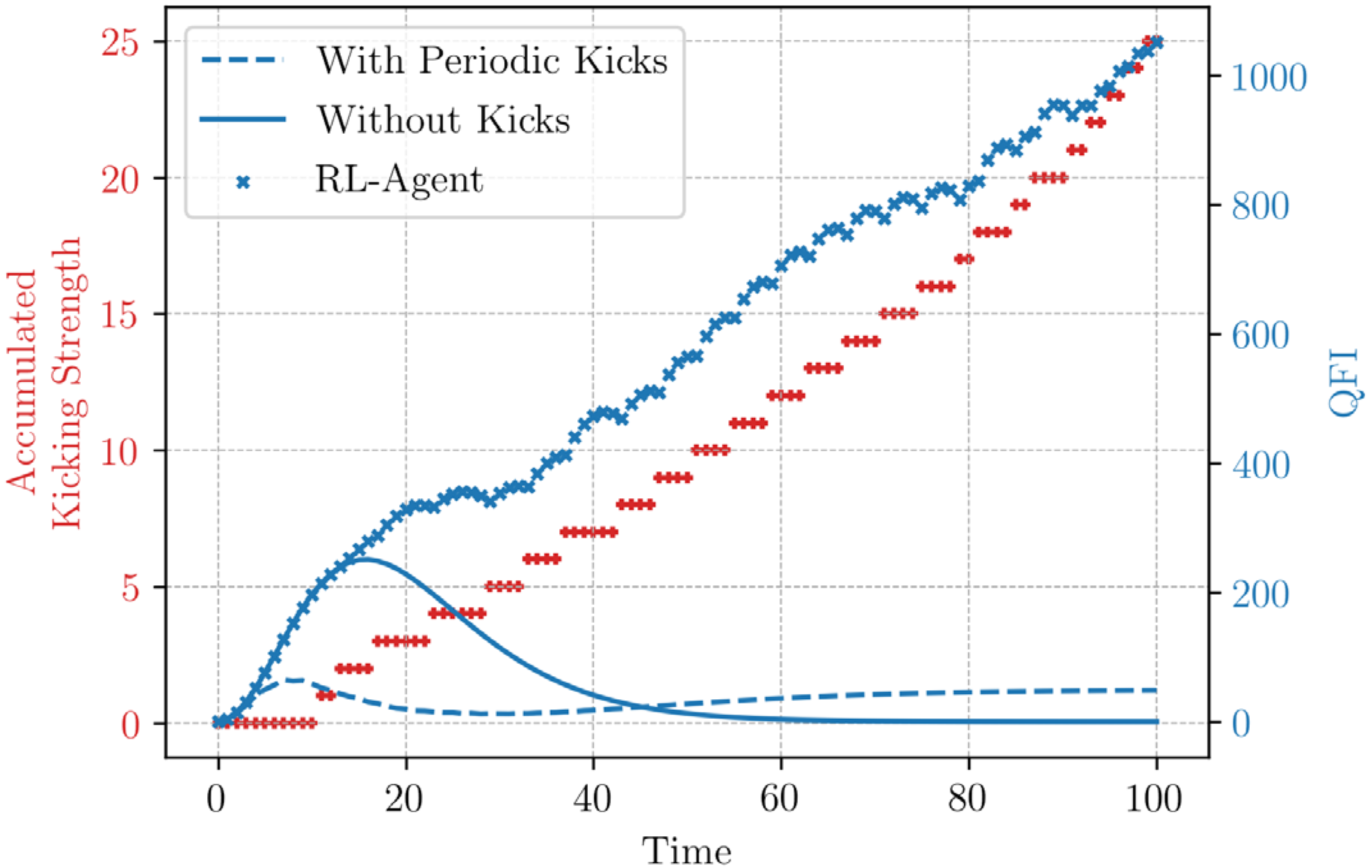




- Final policy typically probabilistic
- Take best sequence from sampling the best policy many times

Results reinforcement learning

Jonas Schuff, bachelor thesis



Superradiant damping, γ=0.02
j=2

2 orders of magnitude increase of stationary-state QFI over dissipative kicked top!

Connection to Antonella de Pasquale's talk

PHYSICAL REVIEW A **96**, 022322 (2017)

Amending entanglement-breaking channels via intermediate unitary operations

Á. Cuevas,[1] A. De Pasquale,[2] A. Mari,[2] A. Orieux,[1,3,4] S. Duranti,[1,5] M. Massaro,[1,6] A. Di Carli,[1,7] E. Roccia,[1,8] J. Ferraz,[1,9] F. Sciarrino,[1] P. Mataloni,[1] and V. Giovannetti[2]

$$\Phi^n = \underbrace{\Phi \circ \Phi \circ \cdots \circ \Phi}_{n \text{ times}}$$

- Decohering quantum channel repeated n times
- Entanglement breaking after a minimum n

$$\underbrace{\Phi \circ \mathcal{F} \circ \Phi \circ \cdots \circ \mathcal{F} \circ \Phi}_{q \text{ times}}$$

- Intercept channel with tailored unitaries (rotations)
- Non-entanglement breaking for arbitrarily large q

- What are the conditions for fighting decoherence with unitary intercepts?
- Connection to/framework of quantum optimal control?
- What are the ultimate limits of this type of fighting decoherence?

Conclusions

J. Fraisse & DB, PRA ’17; QMQM ’17

- New freedom for quantum metrology: tailor dynamics
  - quantum Fisher-information related to Loschmidt echo
  - profit from knowledge in q-chaos
- Gain in sensitivity in model system „kicked top“
  - directly linked to classical chaos (phase space structure!)
  - large gain for large spin
  - robust under superradiant dissipation/decoherence
- Improvement of existing SERF magnetometers
  - detailed model close to experimental Cs-vapor magnetometer
  - 2-3 fold improvement of sensitivity despite small spin, additional decoherence due to kicks

- Machine learning can improve sensitivity even more (drastically so!)

L. Fiderer and D. Braun, Nat. Com. (2018) 9:1351; patent application pending

DFG Deutsche Forschungsgemeinschaft