The detectability lemma: making sense of the notion of quantum constraint violation.

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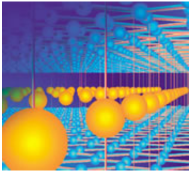
Local Hamiltonians



Fundamental Question:

What is the ground state of a local Hamiltonian?

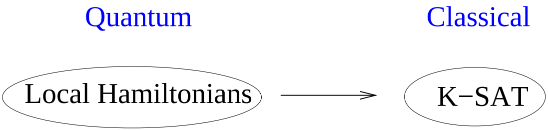
- State space: $B^{\otimes n}$.
- Local Hamiltonians H_i.
- $H = \sum_i H_i$.

Interested in the structure and eigenvalue (energy) associated with the lowest eigenvector

Classical analogue to local Hamiltonians?

Classical analogue:

Restrict to the case where H_i commute. So there exists a basis where all the H_i are diagonal.



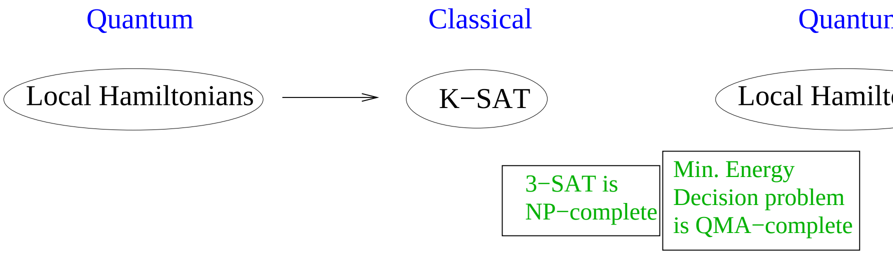
K-SAT:

- Boolean variables: $x_1, x_2, \ldots, x_n$
- Constraints $c_1, c_2, \ldots, c_M$
  (e.g $x_1 \vee \neg x_2 \vee x_3$.)

Questions:

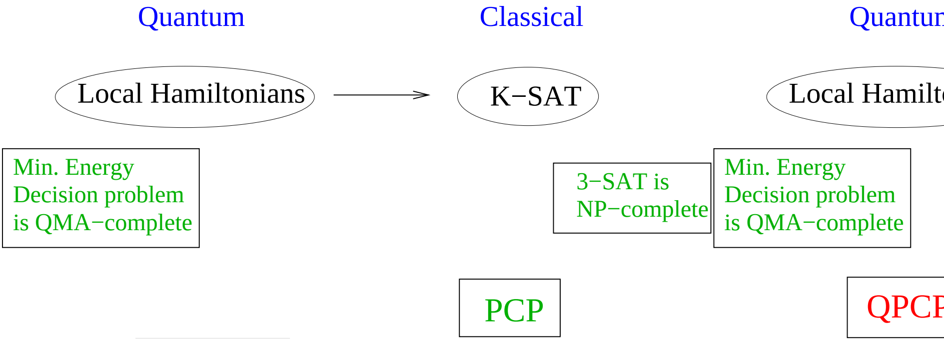
What is the minimum number of constraints that must be violated?

Complete Problems



QMA complete problem (Kitaev):
Is the energy of $H = 0$ or above $\frac{1}{poly(n)}$?

PCP



QPCP Question:
Is deciding whether

Normalized energy: lowest eigenvalue of $\frac{1}{M}H$ is $= 0$ or $\geq c$

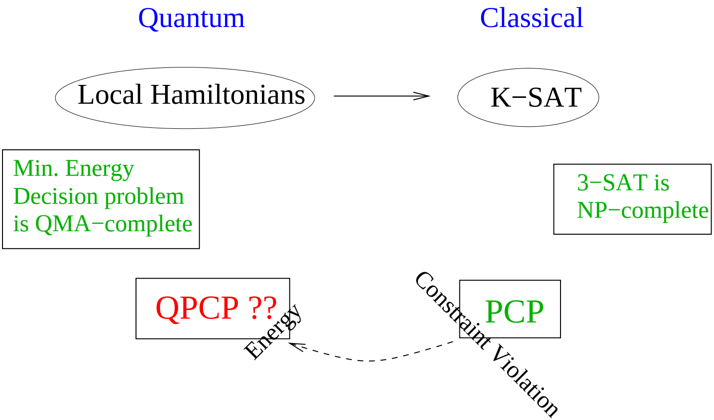
QMA-hard (quantum NP)?

PCP Theorem: Deciding whether

Average # of constraints is 0 or greater than or equal to c is

is NP-hard.

Constraints and Energy



Question:
What is the relationship between violation of constraints and energy?

Constraints and Energy

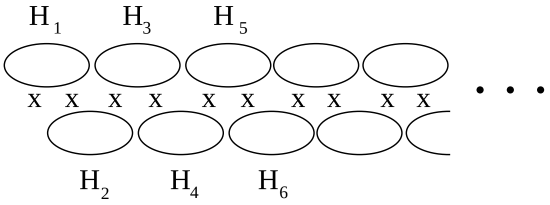
If the local Hamiltonians commute, we are in good shape:

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H = H_1 + H_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- So $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ violates H_1 and satisfies H_2, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ violates both and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ satisfies both.
- Thus any vector can be thought of as a probabilistic mixture of three states, each of which completely satisfy or completely violate each of the constraints H_1 + H_2.
- Then the energy is the expected number of constraints violated if you measured in the diagonal basis.

But when the H_i don't commute, there is no "good" basis and the question of constraint violation of any state doesn't really make sense.

Detectability Lemma



**Extra conditions: H_i projections from a fixed finite set.*

Let $P_i = 1 - H_i$ the projection onto the null space.

$$\pi_{\text{odd}} = P_1 P_3 P_5 \cdots$$
$$\pi_{\text{even}} = P_2 P_4 P_6 \cdots$$

Detectability Lemma [Aharonov, Arad, Landau, Vazirani]

- Sequential measurement has constant probability of detection:

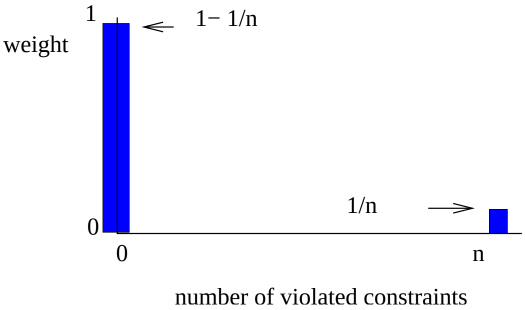
$$||\pi_{\text{even}} \pi_{\text{odd}} \psi||^2 < 1 - c\epsilon$$

- At least one layer has constant probability of detection:

$$||\pi_{\text{odd}} \psi||^2 < 1 - c'\epsilon \quad \text{or} \quad ||\pi_{\text{even}} \psi||^2 < 1 - c'\epsilon$$

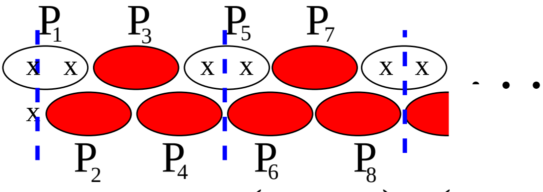
How could this not be true?

Each layer provides a snapshot of the state according to those constraints. What if all those snapshots look like this:



We need a picture that incorporates the non-commuting aspect of things.

Pyramids



$$(X_1 \oplus Y_1) \otimes (X_2 \oplus Y_2) \otimes ($$

The XY Decomposition: a snapshot across layers

- Tensor product of local spaces.
- Each local space = *commuting* $\oplus$ non-commuting

Allows for the simultaneous analysis of all red constraints.

e.g. $(X_1 \otimes X_2) \oplus (X_1 \otimes Y_2) \oplus (Y_1 \otimes X_2) \oplus (Y_1 \otimes Y_2)$

diagonalizes the actions of $P_2, P_3, P_4, P_6, P_7, P_8$.

Exponential Decay: the important structural feature

- e.g. in Y_1, $||P_2 P_3 P_4|| \leq \theta < 1$.
- Norm of $(P_3 P_7 \ldots)(P_2 P_4 P \ldots)$ has exponential decay in # of Y components.

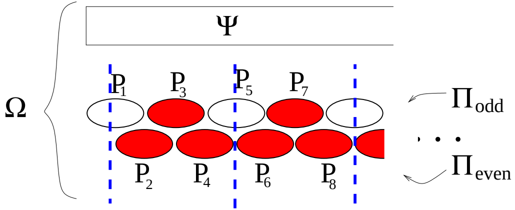
The XY decomposition along with exponential decay allows for the proper analysis.

Back to the Detectability Lemma

Detectability Lemma [Aharonov, Arad, Landau, Vazirani]

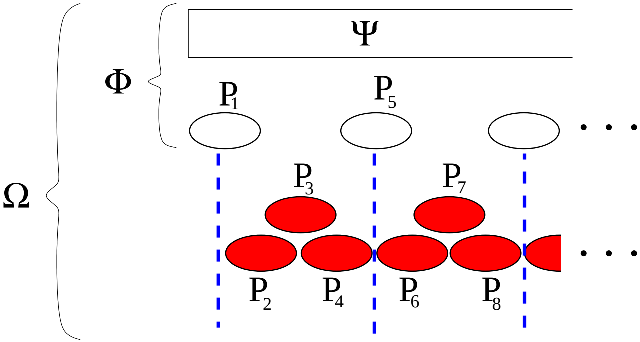
- Sequential measurement has constant probability of detection:

$$||\pi_{\text{even}}\pi_{\text{odd}}\psi||^2 < 1 - c\epsilon$$



Set $\Omega = \pi_{\text{even}}\pi_{\text{odd}}\psi$, so our goal is to show

Detectability Lemma Proof

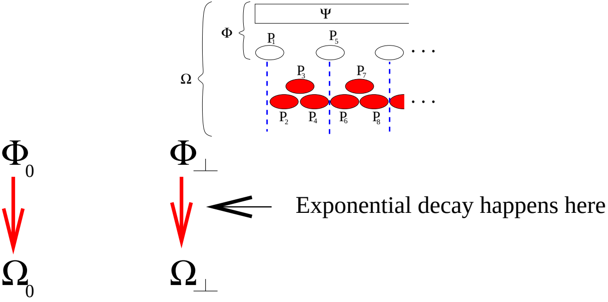


Write

$$\phi = \phi_0 + \phi_\perp, \quad \Omega = \Omega_0 + \Omega_\perp$$

where ϕ_0, Ω_0 are the projections onto the all X's subspace: $X_1 \oplus X_2 \oplus \ldots$.

- Moving from ϕ to ψ is diagonal relative to this decomposition.



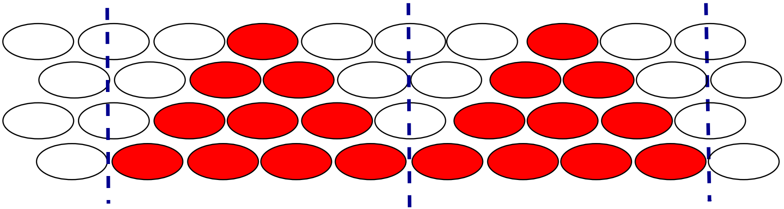
Exponential decay Lemma says:

- going from ϕ to Ω shrinks norm by at least a constant times $||\phi_\perp||$,
- portion of energy of $\Omega_\perp$ from $H_3 + H_7 + \ldots$ can only be as big as $c||\phi_\perp||^2$ because it has incurred exponential decay; but this energy is at least $c'\epsilon$ so:

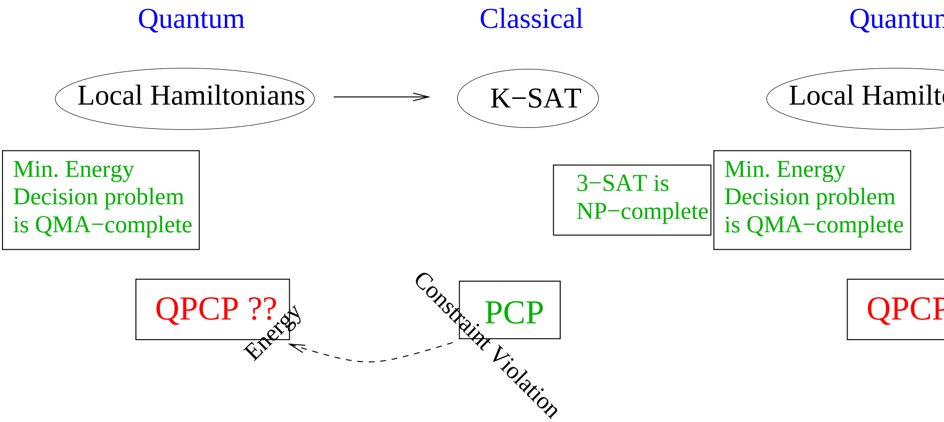
$$||\phi_\perp||^2 > C\epsilon.$$

Thus the square of the norm is shrunk by at least $C'\epsilon$ which is the desired conclusion.

A picture in the case of more than 2 layers



Implications of the Detectability Lemma



- Transfers intuition about constraint violation to Local Hamiltonian Complexity.
- A generalization of the lemma leads to quantum gap amplification. . .

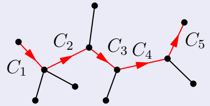
Quantum Gap Amplification

A recent new proof of PCP (Dinur) requires *Gap Amplification* as an important step:

Gap Amplification

Classical
From a given constraint problem, form a new constraint problem (larger constraints) with linear scaling in average number of violations:

$$\text{avg. \# of violations of } C' = \text{c (avg \# of violations of } C\text{)}.$$



$$C_{\mathbf{e}} = C_1 \wedge C_2 \wedge C_3 \wedge C_4 \wedge C_5$$

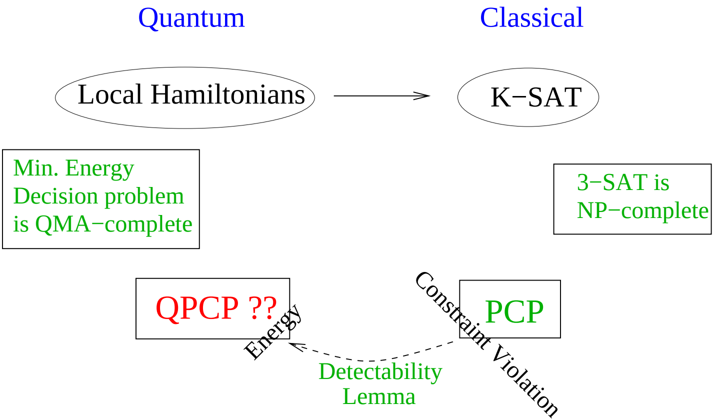
Quantum [Aharonov, Arad, Landau, Vazirani]
From a given local Hamiltonian H, form a new local Hamiltonian (larger local parameter) with with linear scaling in *normalized energy*:

$$\text{Energy of } \frac{1}{M'}H' = \text{c(Energy of } \frac{1}{M}H\text{)}.$$

Quantum Gap Amplification: Ingredients

- Detectability Lemma for violation of a constant number of constraints.
  - Requires refinement of XY decomposition that further breaks up the Y component.
  - Gets mildly more complicated.
- Classical Gap Amplification Proof
  - Detectability Lemma finds a layer upon which there is positive probability of detecting at least a constant number of constraints violations.
  - Now proceed as in the classical case for this layer of commuting constraints.

Where things stand



- QPCP? : still would require degree and alphabet reduction steps . . .
- The XY decomposition, exponential decay lemma, and detectability lemma as analysis tools within Hamiltonian complexity.