Tight Noise Thresholds for Quantum Computation with Perfect Stabilizer Operations

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Work with Wim van Dam [WvD,MH arXiv:0907.3189, To Appear PRL 10/2009]

Outline

Noisy Non-Stabilizer Gates:

(UQC $=$ Universal Quantum Computation)

- Upper Bounds via the "Clifford Polytope"
(UQC provably impossible when noise too strong)

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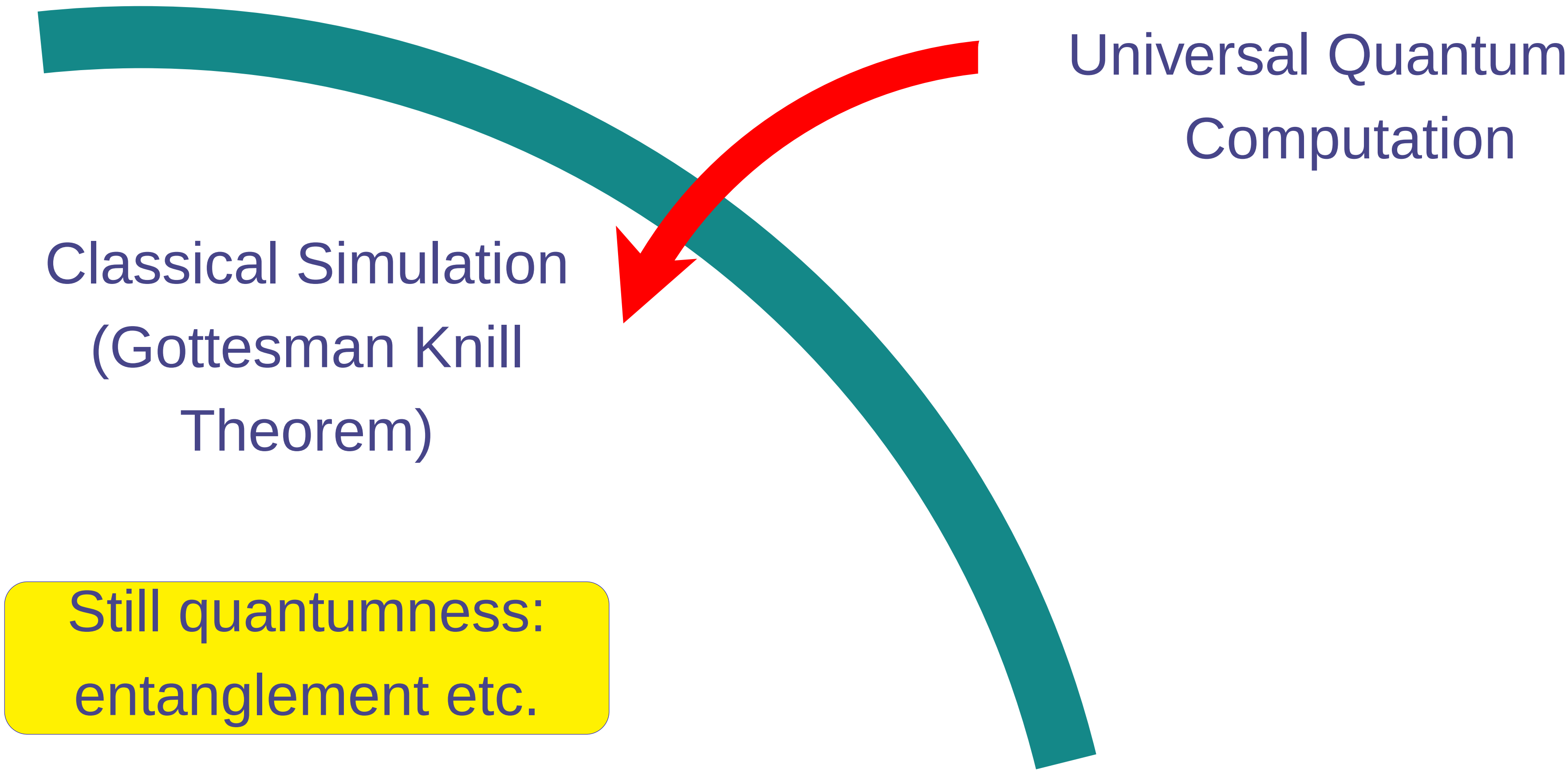
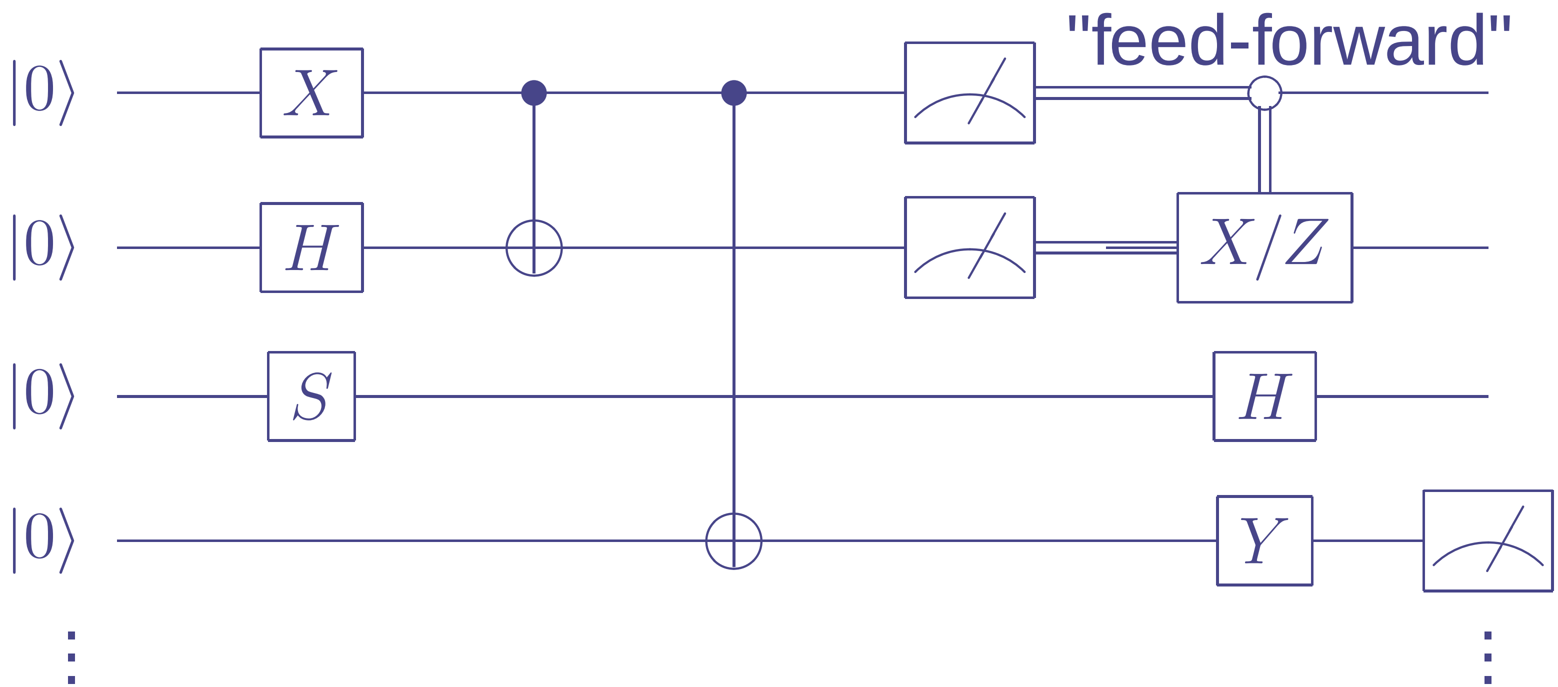
- Upper Bounds via the "Clifford Polytope"
  (UQC provably impossible when noise too strong)
- Lower Bounds* via "Magic State Distillation"
  (UQC provably possible provided noise not too strong)
- Our Result:
  Tight Threshold for Gates with Depolarizing Noise

Upper Bounds: General Idea

- Noise $\rightarrow$ Uselessness
- Useless $\Leftrightarrow$ Classically efficiently simulable



Gottesman Knill Theorem

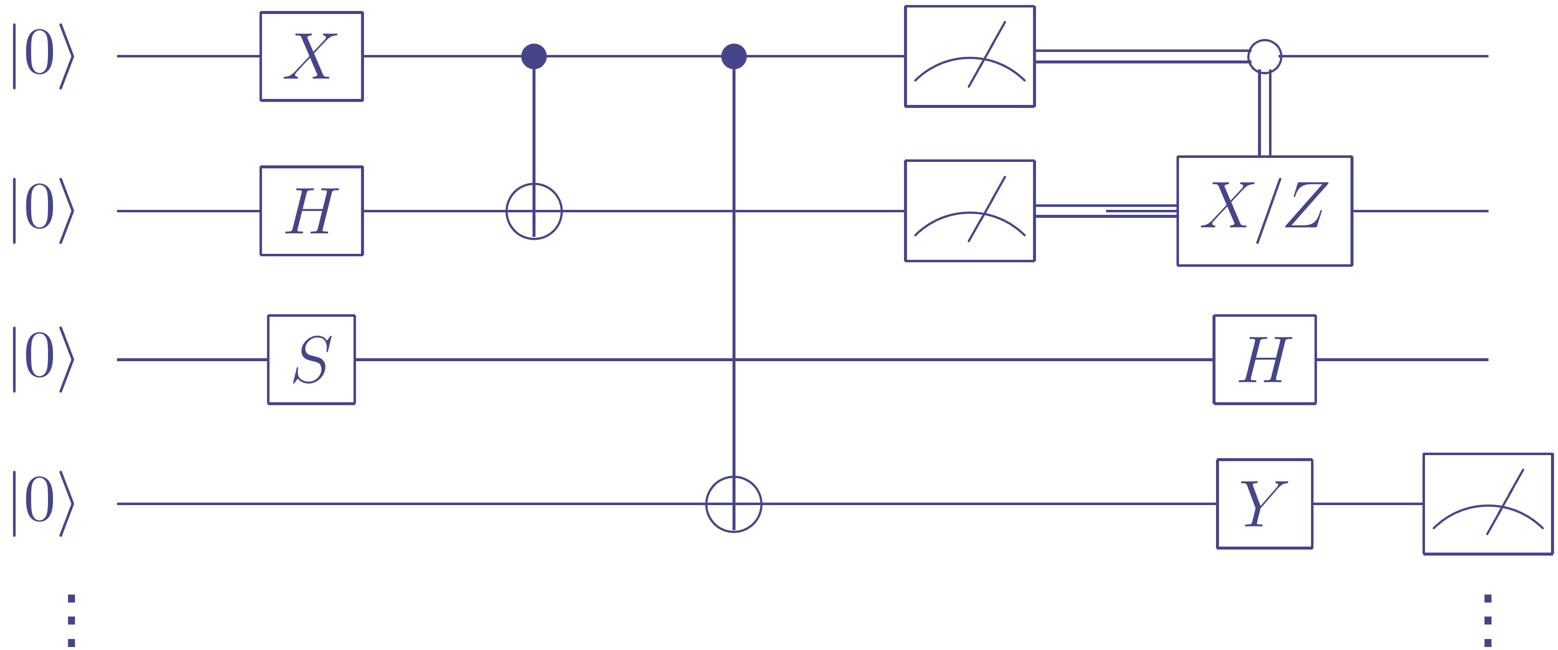


- n Input Qubits
- Simulate in $\mathcal{O}(n^2)$ time

S. Aaronson and D. Gottesman, "Improved simulation of stabilizer circuits," Phys. Rev. A, vol.70, 052328, 2004

$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \qquad H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Gottesman Knill Theorem



- n Input Qubits
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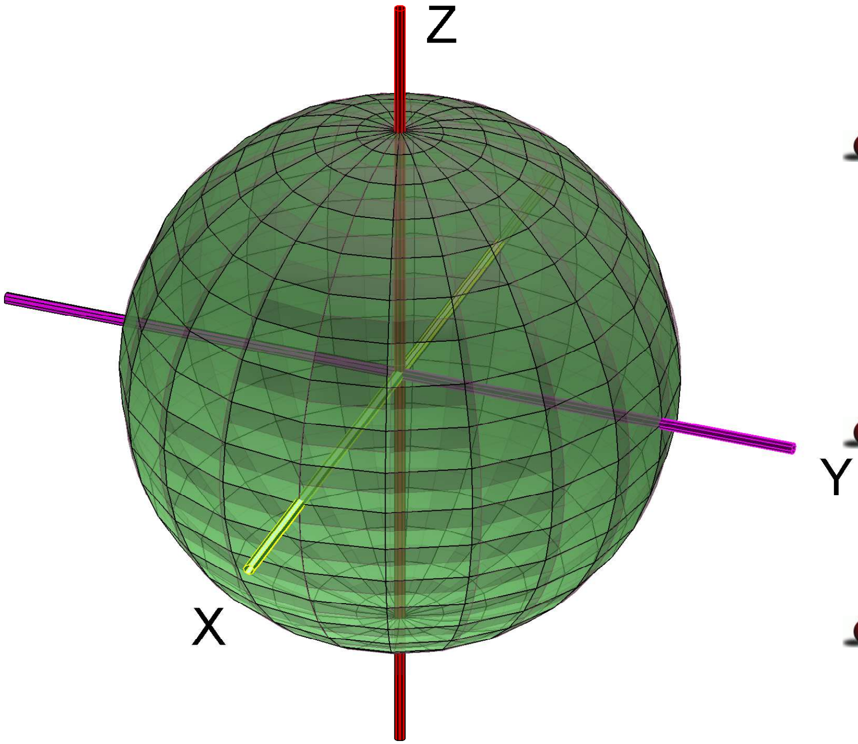
24 allowed single-qubit (**Clifford**) gates $= \{S, H, S^2, SH, SHS, S^3H \ldots\}$

Gottesman Knill Theorem

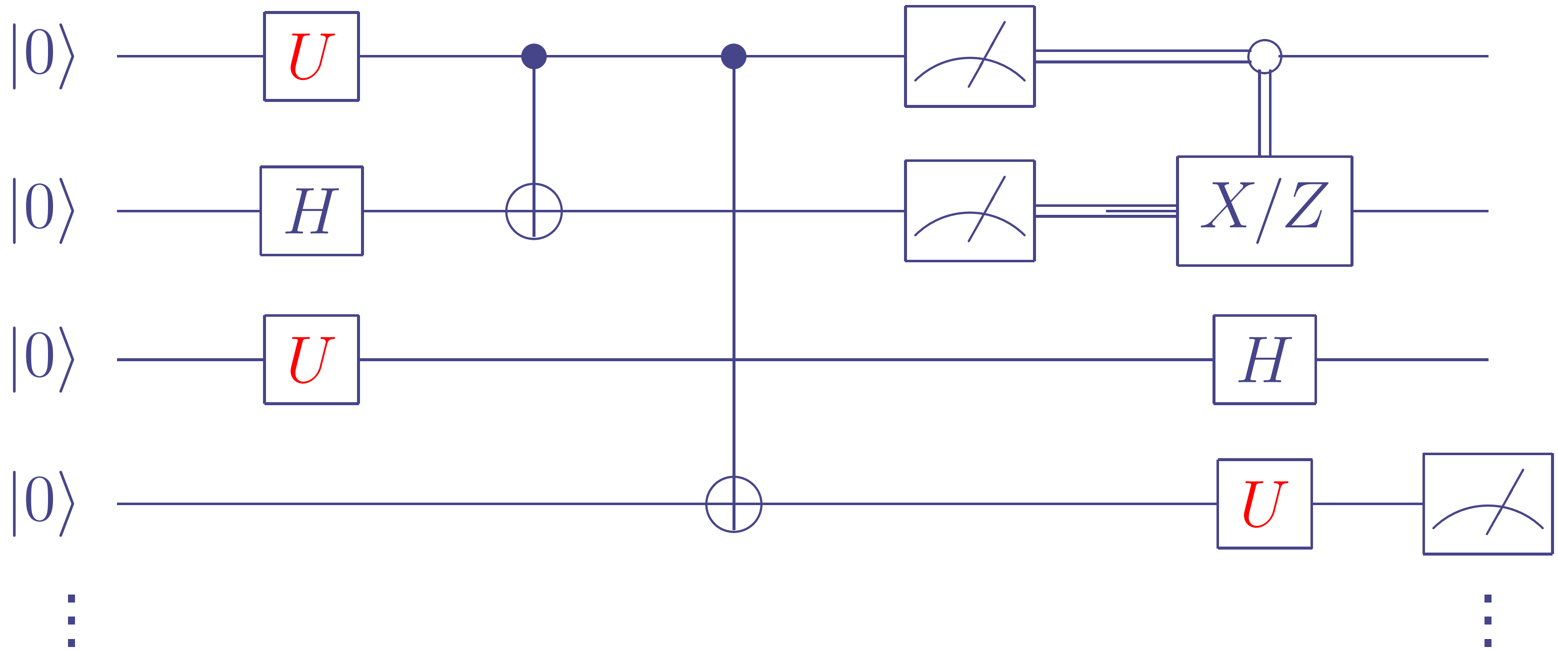


24 Clifford gates?

- Group with $SU(2)$ elements: $\{S, H, S^2, SH, SHS, S^3H \ldots\} = \langle S, H \rangle$
- Group with $SO(3)$ elements $= \langle R_x(\frac{\pi}{2}), R_y(\frac{\pi}{2}), R_z(\frac{\pi}{2}) \rangle$
- $24 = 6 \cdot 4 \cdot 1$

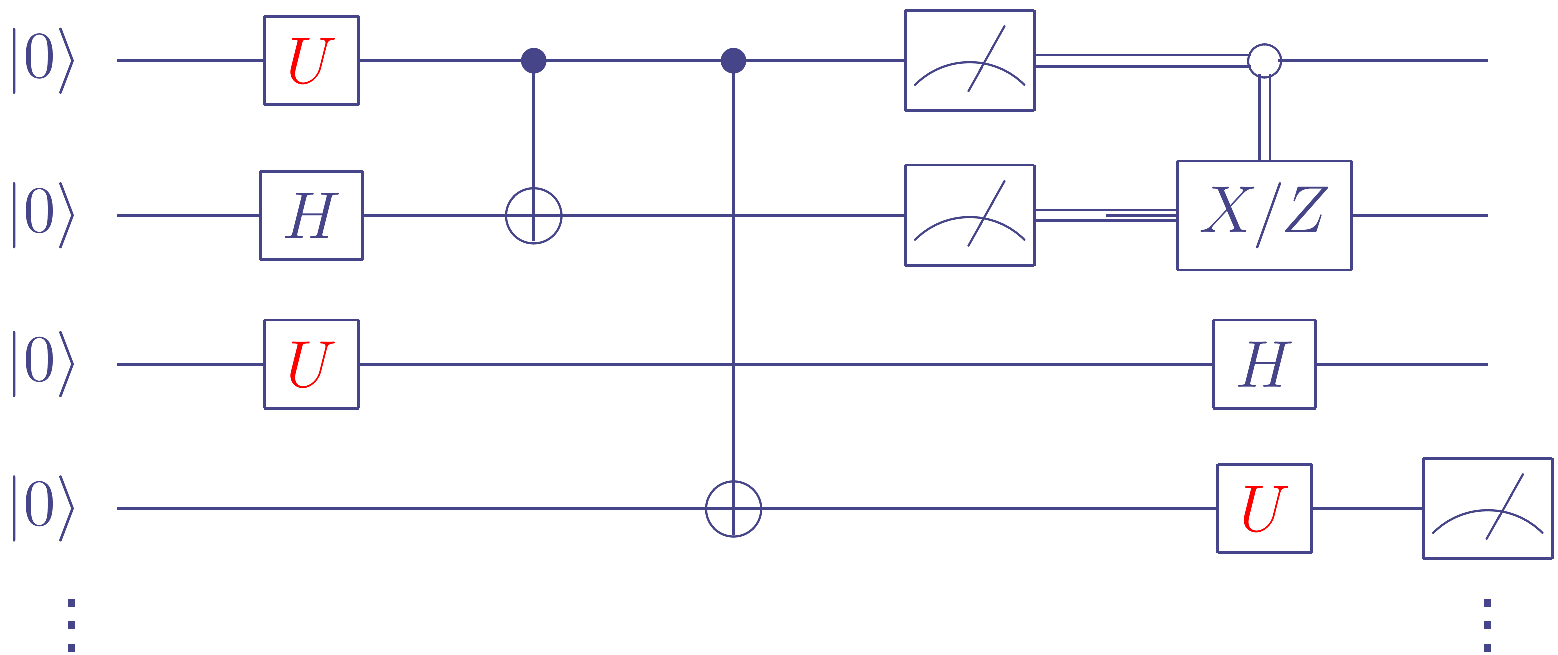
$CNOT$ is also in the Clifford group. . . creates entanglement.

Gottesman Knill Theorem



- n input qubits $|0\rangle$
- r non-Clifford gates U

Gottesman Knill Theorem



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Important Fact #1

- n input qubits $|0\rangle$
- r non-Clifford gates $\boxed{U}$

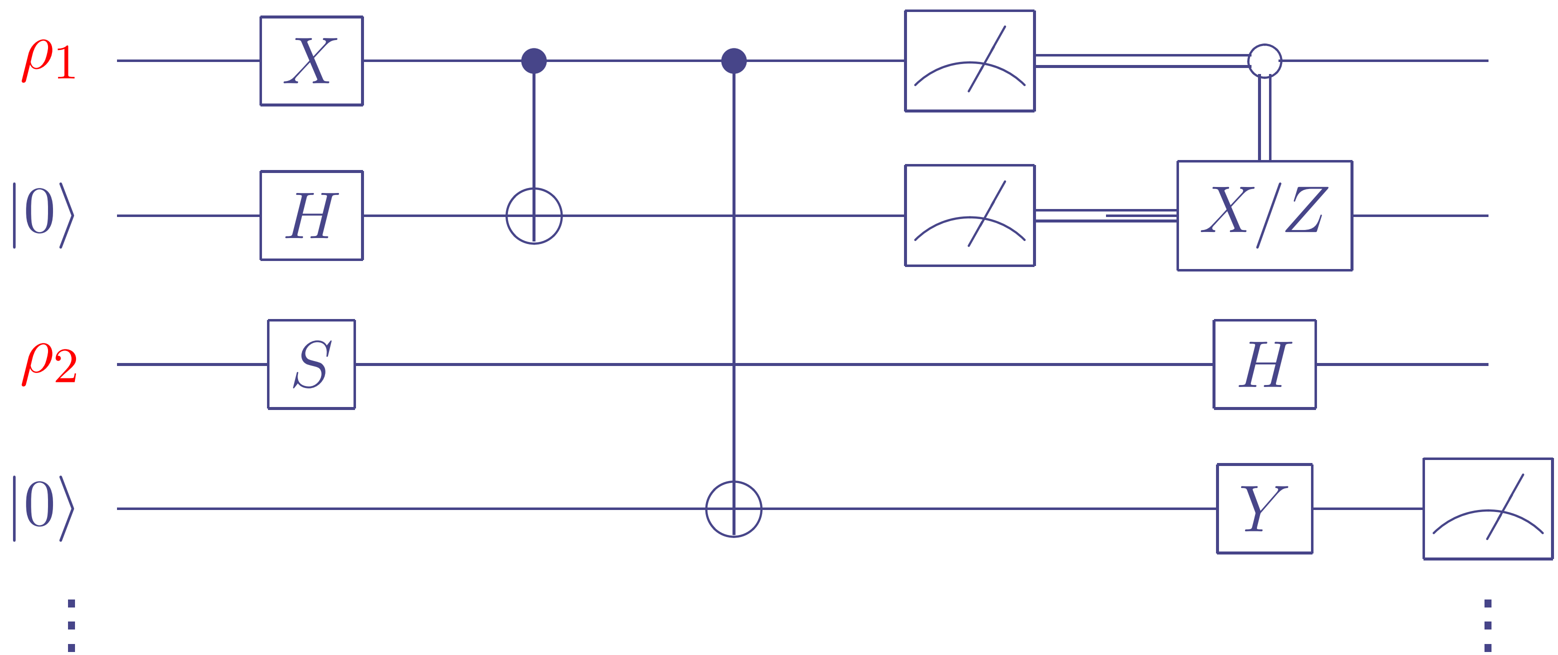
$\Rightarrow$ Simulation time and space $\mathcal{O}(2^{4b}n + n^2)$

Makes sense:

Clifford gates plus any single qubit $\boxed{\mathbf{U}}$ from outside the group enable UQC

Y. Shi, "Both Toffoli and controlled-Not need little help to do universal quantum computation", *Quantum Information and Computation*, 3(1):84–92, 2002

Gottesman Knill Theorem



- n input qubits $|0\rangle$ or ρ_i
- ρ_i non-stabilizer
- m measurements.

$\Rightarrow$ Simulation time $\mathcal{O}(2^{2m}n + n^2)$

S. Aaronson and D. Gottesman, "Improved simulation of stabilizer circuits," Phys. Rev. A, vol.70, 052328, 2004

Important Fact #2

- n input qubits $|0\rangle$ or ρ_i
- ρ_i are non-stabilizer qubits
- m measurements.

$\Rightarrow$ Simulation time $\mathcal{O}(2^{2m}n + n^2)$

Makes sense:

Clifford gates, 0/1 measurements (with feed-forward), plus appropriate non-stabilizer states ρ enable UQC. ("Magic states..")

Bravyi, S. and A. Kitaev, "Universal quantum computation with ideal Clifford gates and noisy ancillas" Phys. Rev. A **71**, 022316 2005

Upper Bounds via "Clifford Polytope"

General Idea

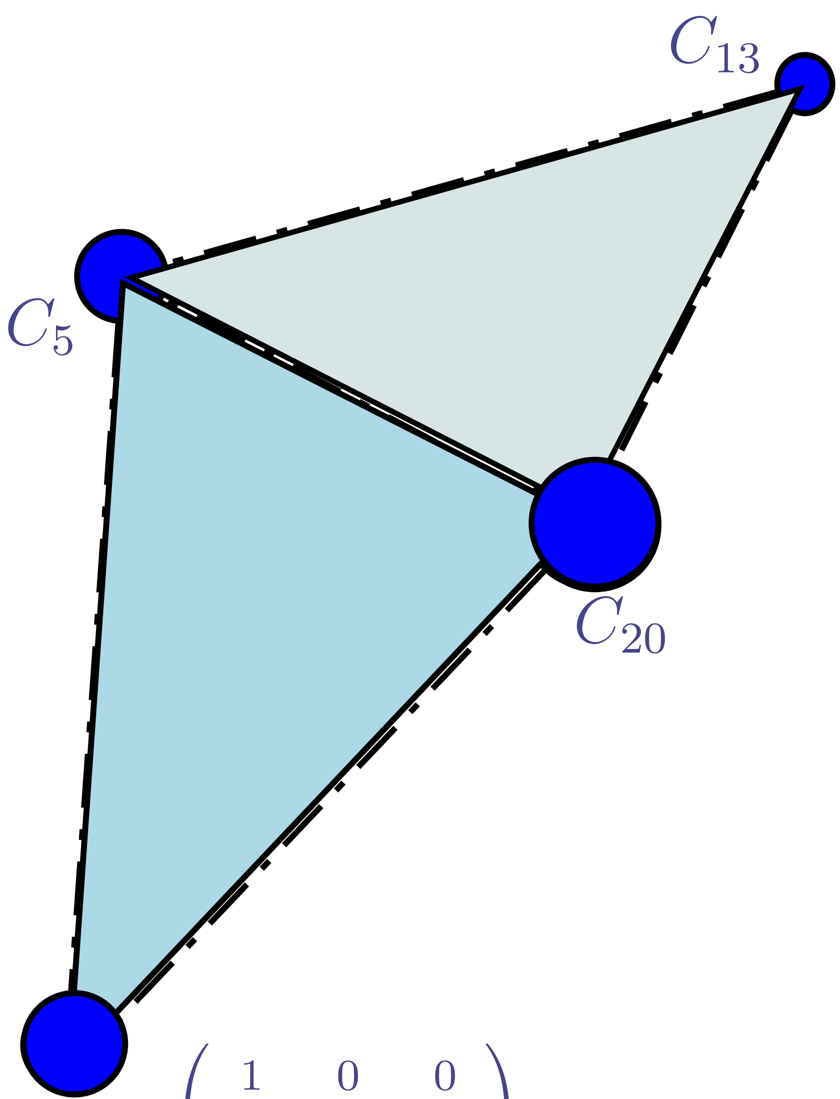
- Recall U outside Clifford group enables UQC.
- Noise during implementation of U means

$$\begin{aligned} SU(2) \text{ picture}: \quad & \mathcal{E}_{\text{TOTAL}}(\rho) &= (1-p)U\rho U^\dagger + p\mathcal{E}_{\text{NOISE}}(\rho) \\ SO(3) \text{ picture}: \quad & M &= (1-p)R + pN \end{aligned}$$

- For what noise rate, p, is $\mathcal{E}_{\text{TOTAL}}(\rho)$ implementable using Clifford operations only?
- Depends on U. Depends on $\mathcal{E}_{\text{NOISE}}(\rho)$.

H. Buhrman, R. Cleve, M. Laurent, N. Linden, A. Schrijver and F. Unger, "New limits on fault-tolerant quantum computation", FOCS 411(2006).

Clifford Polytope



$$C_7 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

M is a unital
$\mathbb{R}^{3\times 3}$ transformation
(acts on the Bloch vector).

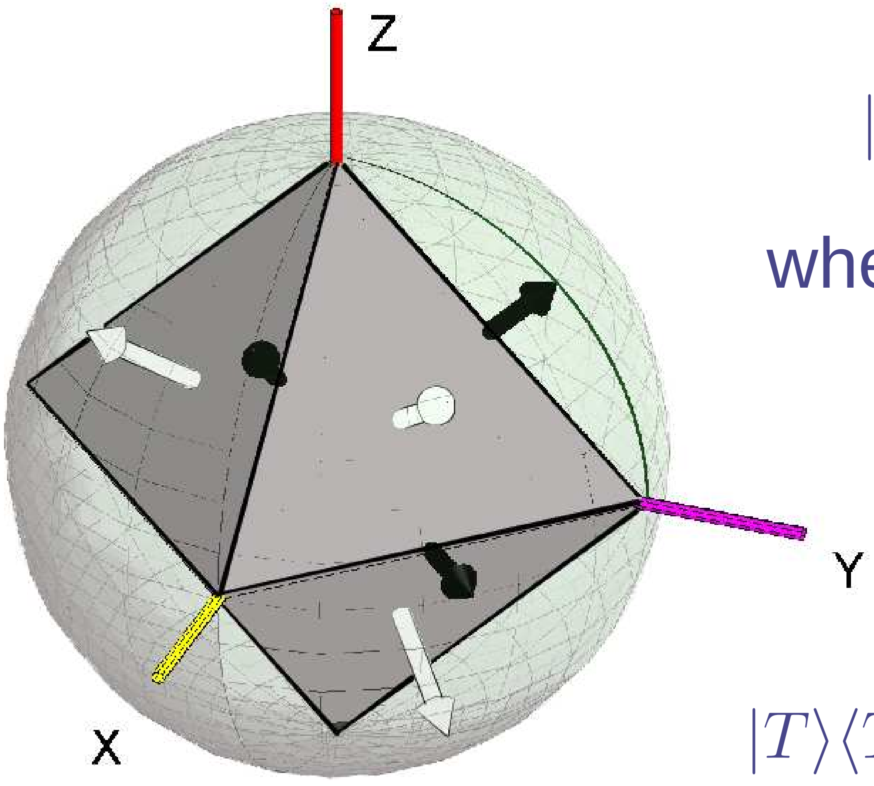
C_i are $SO(3)$ Clifford rotations.

p_i are probabilities:
$0 \le p_i \le 1$ and $\sum_{i=1}^{24} p_i = 1$

Is $M = \sum_{i=1}^{24} p_i C_i$?

$M \in$ Polytope
if and only if
$M \cdot F \le 1$
for all facets F.

Proving UQC with noisy U: Magic States



$$|H\rangle\langle H| = \frac{1}{2}\left(I + \frac{1}{\sqrt{2}}(\pm\sigma_i \pm \sigma_j)\right)$$

where $\sigma_i, \sigma_j \in \{\sigma_x, \sigma_y, \sigma_z\}, \sigma_i \neq \sigma_j$

$$|T\rangle\langle T| = \frac{1}{2}\left(I + \frac{1}{\sqrt{3}}(\pm\sigma_x \pm \sigma_y \pm \sigma_z)\right)$$

Or, for example, in ket form:

$|H\rangle = |0\rangle + e^{\frac{i\pi}{4}}|1\rangle$

$|T\rangle = \cos(\vartheta)|0\rangle + e^{\frac{i\pi}{4}}\sin(\vartheta)|1\rangle$ with $\cos(2\vartheta) = \frac{1}{\sqrt{3}}$

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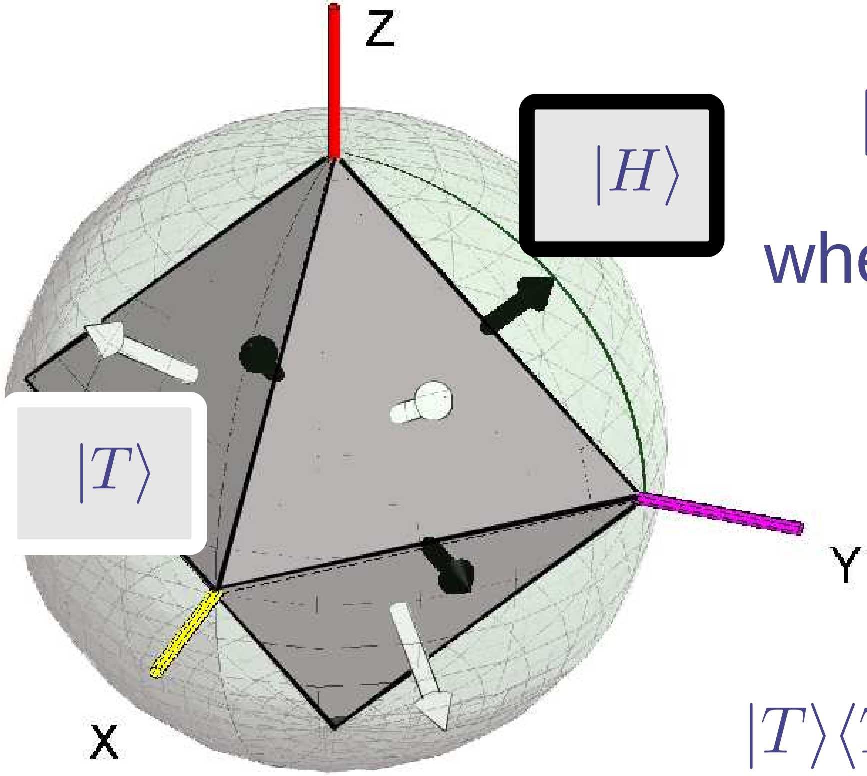

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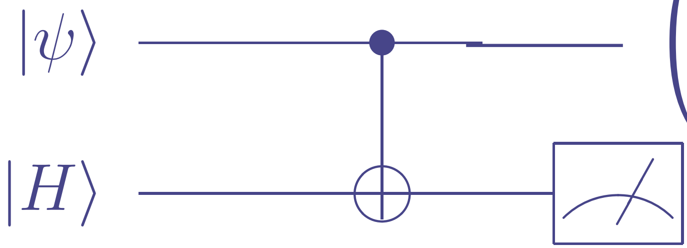
Proving UQC with noisy U: Magic States

Why are they "magic"?

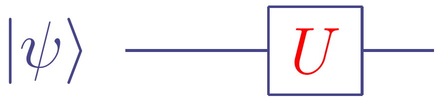
- Pure magic states + perfect stabilizer operations enable UQC.
- Impure magic states can be distilled towards pure magic states using stabilizer operations only.

Bravyi, S. and A. Kitaev, "Universal quantum computation with ideal Clifford gates and noisy ancillas" Phys. Rev. A **71**, 022316 2005

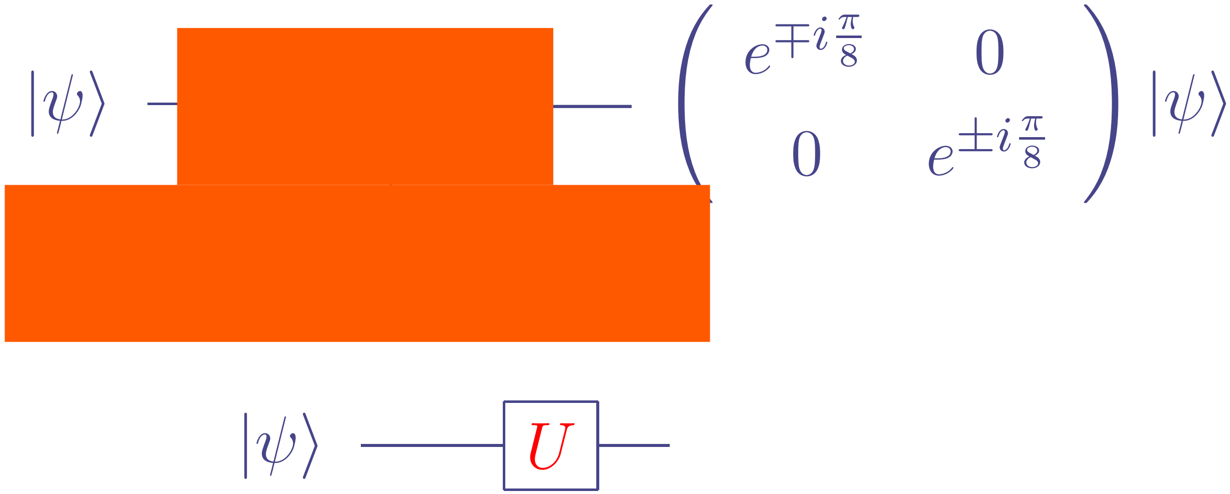
Magic States enable UQC



$$\begin{pmatrix} e^{\mp i\frac{\pi}{8}} & 0 \\ 0 & e^{\pm i\frac{\pi}{8}} \end{pmatrix} |\psi\rangle$$



Magic States enable UQC



- A similar method works when using $|T\rangle$-type states.

Magic States Distillation (Purification)

- Input n identical copies of ρ to error correction type procedure (uses stabilizer operations only).
- Provided fidelity of ρ w.r.t. $|T\rangle$, or fidelity w.r.t $|H\rangle$ high enough then the output of the distillation ρ_{out} can be made arbitrarily close to a magic state.

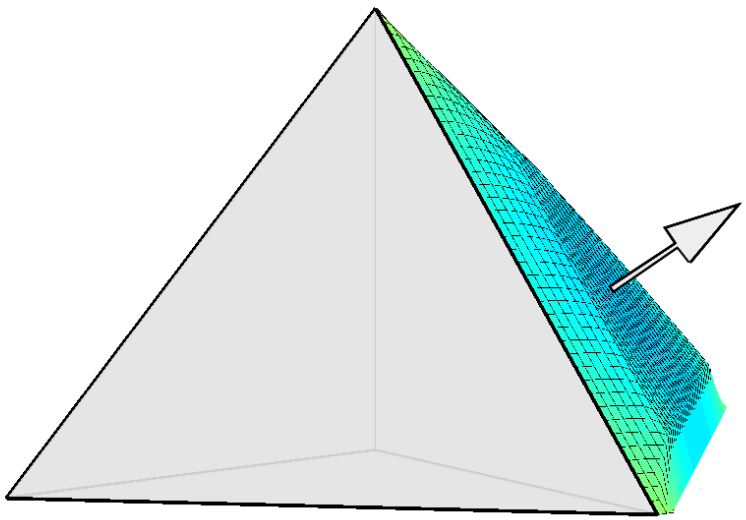
Bravyi, S. and A. Kitaev, "Universal quantum computation with ideal Clifford gates and noisy ancillas" Phys. Rev. A **71**, 022316 2005

Earl T. Campbell, and Dan E. Browne, "On the structure of protocols for magic state distillation" arXiv:0908.0838 (2009)

Magic States Distillation

- States with Bloch vectors satisfying $\max\{|x|+|z|, |x|+|y|, |y|+|z|\} > 1$ are distillable (tight up to the 12 edges of octahedron).

[Rei1] Ben W. Reichardt, "Improved magic states distillation for quantum universality", Quantum Information Processing 4 pp.251-264 (2005).



- There is currently an undistillable region just outside the octahedron in the $|T\rangle$ direction (Bloch vector: $1 < |x|+|y|+|z| < \frac{3}{\sqrt{7}}$).

[CB'09] Earl T. Campbell and Dan E. Browne, “Bound states for magic state distillation", arXiv:0908.0836 (2009)

General Idea

- Apply noisy U to an input stabilizer state.

$$\mathcal{E}_{\text{TOTAL}}(\rho) \;\; = (1-p)U\rho U^{\dagger} + p\mathcal{E}_{\text{NOISE}}(\rho)$$

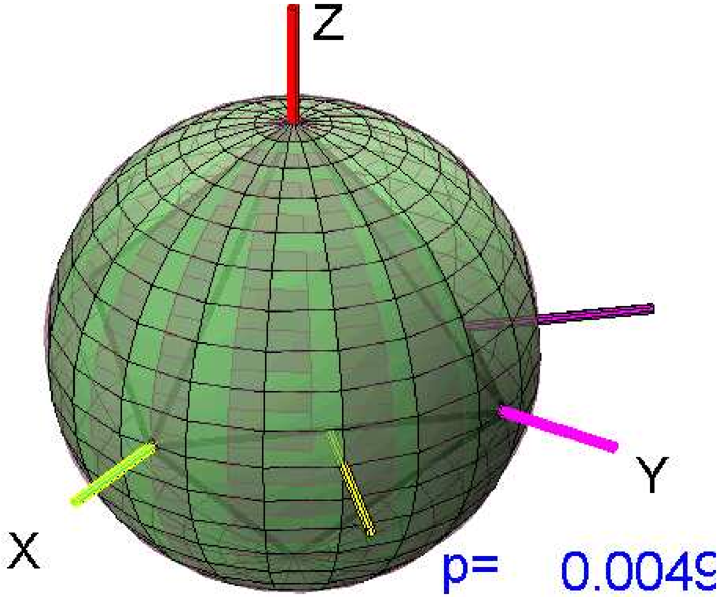
- If the resulting state is such that it can be distilled to $|T\rangle$ or $|H\rangle$, then the noisy U enables UQC.

⋆ Assuming stabilizer ops performed perfectly.

Analyze bounds for important gate: "$\pi/8$" gate. . .

[Rei2] Ben W. Reichardt, "Quantum universality by distilling certain one- and two-qubit states with stabilizer operations", quant-ph/0608085 (2006).

Lower Bounds⋆ via Magic States Distillation



$$U = \begin{pmatrix} e^{-i\frac{\pi}{8}} & 0 \\ 0 & e^{i\frac{\pi}{8}} \end{pmatrix}$$

Depolarizing Noise

- Apply noisy "$\pi/8$" gate to σ_x eigenstate. . . Outside octahedron for up to 29% depolarizing noise
- Upper Bound from Clifford polytope [BCLLSU]: $p \geq 0.453 \Rightarrow$ UQC

Can we get closer to 45%?

Lower Bounds* via Magic States Distillation

- Apply noisy "$\pi/8$" gate to half of $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

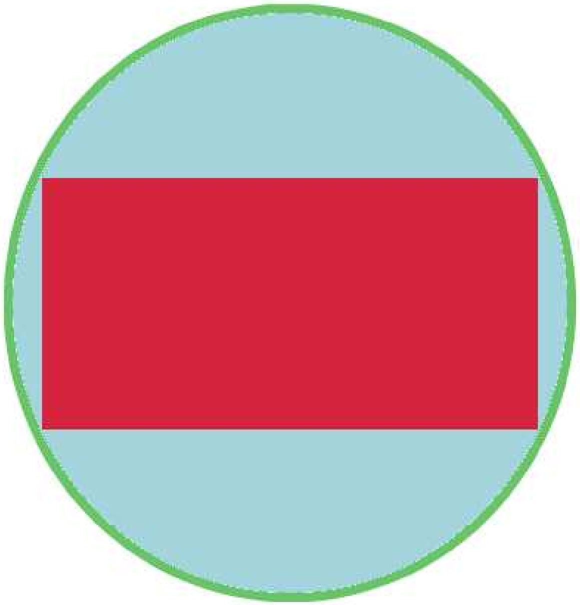
1. Perform parity measurement $\Pi = \frac{1}{2}(II + \sigma_z\sigma_z)$ on output .
2. Resulting state is $\rho = \frac{\widetilde{\rho}}{\text{Tr}[\widetilde{\rho}]}$ with

$$\widetilde{\rho} = \frac{(1-p)}{2}\begin{pmatrix} 1 + \frac{p}{2(1-p)} & e^{i\frac{\pi}{4}} \\ e^{-i\frac{\pi}{4}} & 1 + \frac{p}{2(1-p)} \end{pmatrix}$$

3. Check if the output state ρ has a Bloch vector that enables distillation.

$|\mathbf{x}| + |\mathbf{y}| > \mathbf{1}$ for $\mathbf{p} < \mathbf{0.453}$ . . . a tight bound [Rei2]

Lower Bounds⋆ via Magic States Distillation

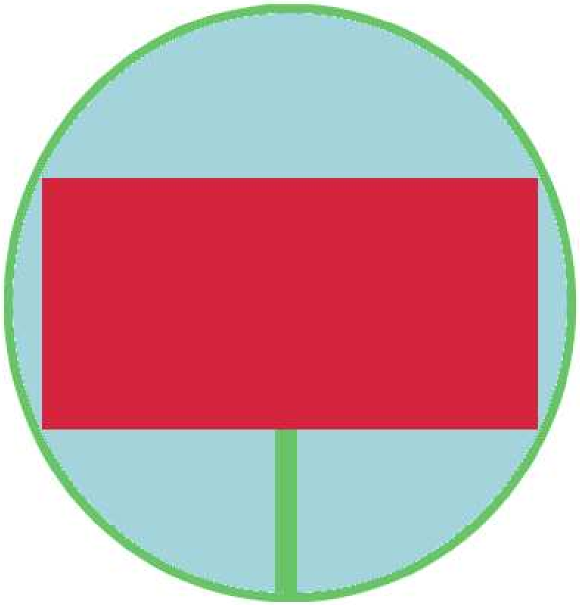


Depolarizing Noise:

$$\mathcal{E}(\rho) = (1-p)U\rho U^{\dagger} + p\frac{I}{2}$$

[BCLLSU]: $p \geq 0.453 \Rightarrow$ UQC (All U)

Lower Bounds* via Magic States Distillation



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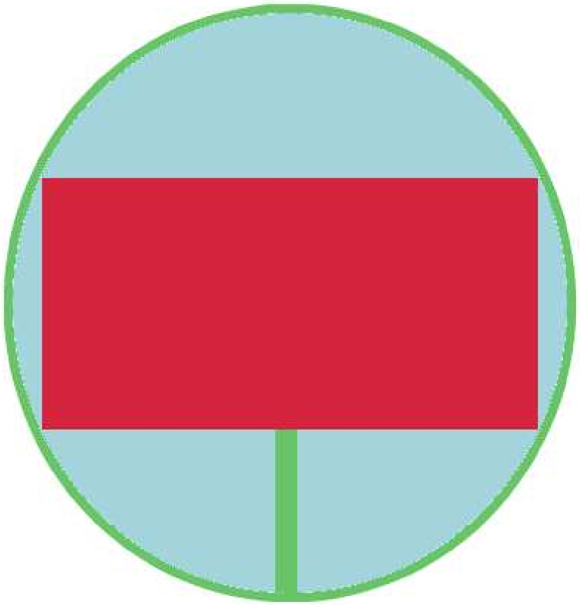
[Rei2]: $p < 0.453 \Rightarrow$ UQC ("$\pi/8$" gate)

"$\pi/8$" gate tight

Lower Bounds* via Magic States Distillation



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What about other gates?

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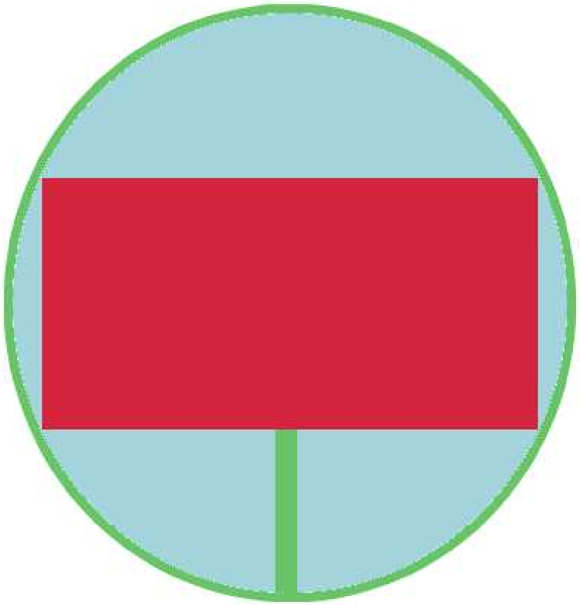

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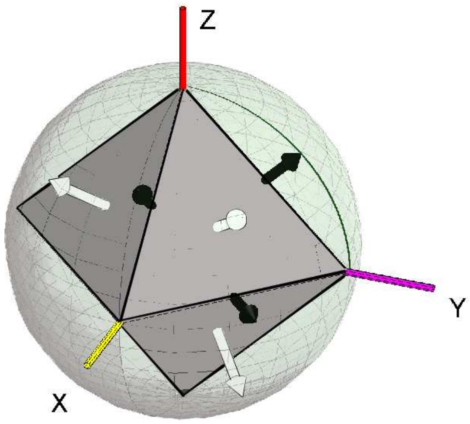
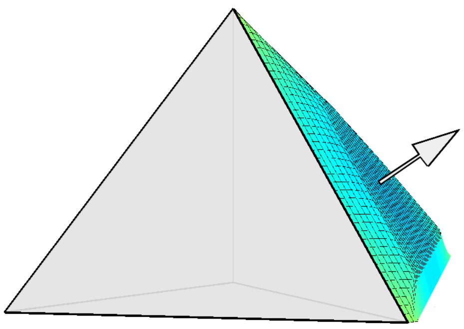
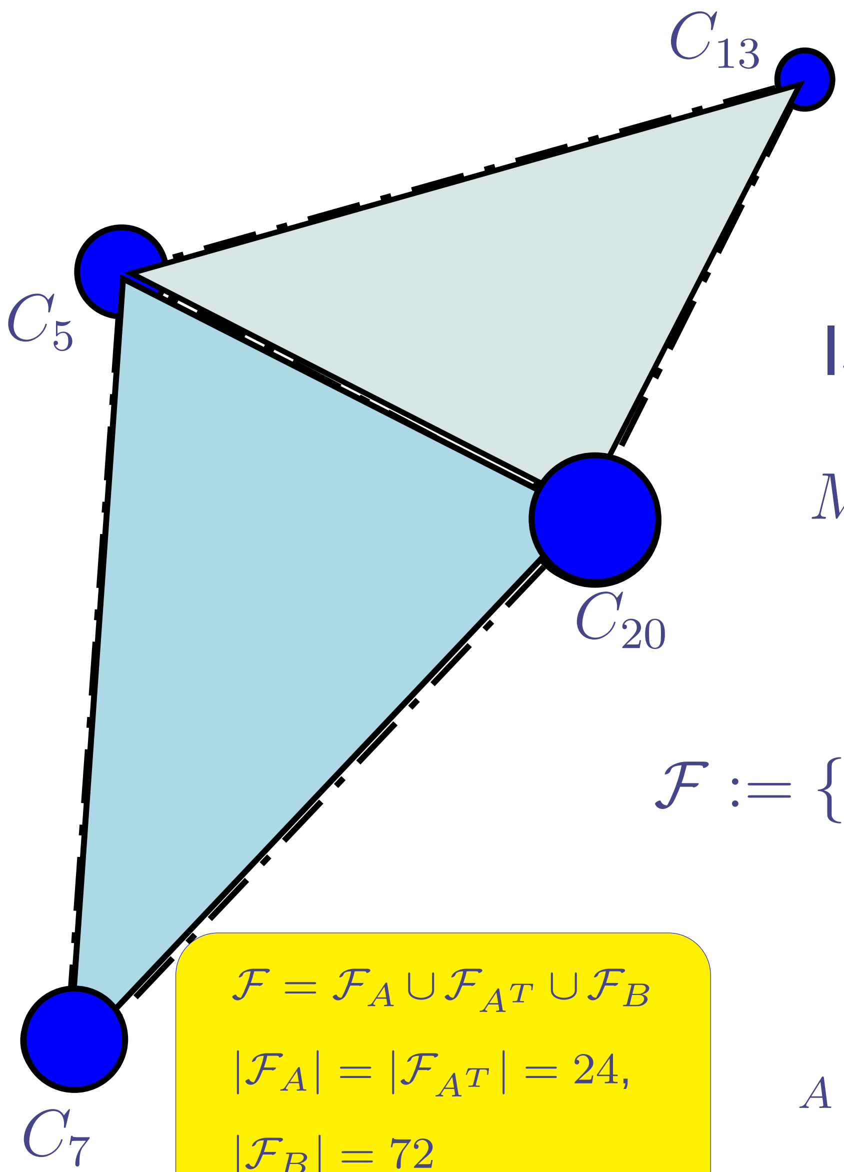
What about other gates?

Similar to ancilla question:





Clifford Polytope: Facets



$\mathcal{F} = \mathcal{F}_A \cup \mathcal{F}_{A^T} \cup \mathcal{F}_B$

$|\mathcal{F}_A| = |\mathcal{F}_{A^T}| = 24,$

$|\mathcal{F}_B| = 72$

M is an $\mathbb{R}^{3\times 3}$ transformation.

Is $M = \sum_{i=1}^{24} p_i C_i \;\; (\sum_{i=1}^{24} p_i = 1)$?

$M \in$ Polytope $: M \cdot F \leq 1 \quad \forall F \in \mathcal{F}$

where

$$\mathcal{F} := \{C_1 F C_2 | C_1, C_2 \in \mathcal{C}, F \in \{A, A^T, B\}\}$$

with

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 1 \end{pmatrix}.$$

Our Result

- Interpret Facets of Clifford Polytope
  - Interpret "A-type" Facets
  - Interpret "B-type" Facets

I. Tight Qubit Ancilla Threshold $\Longrightarrow$ Tight Noise Threshold (All Unital Operations)

- Show Depolarized $R \in SO(3)$ possess particular property

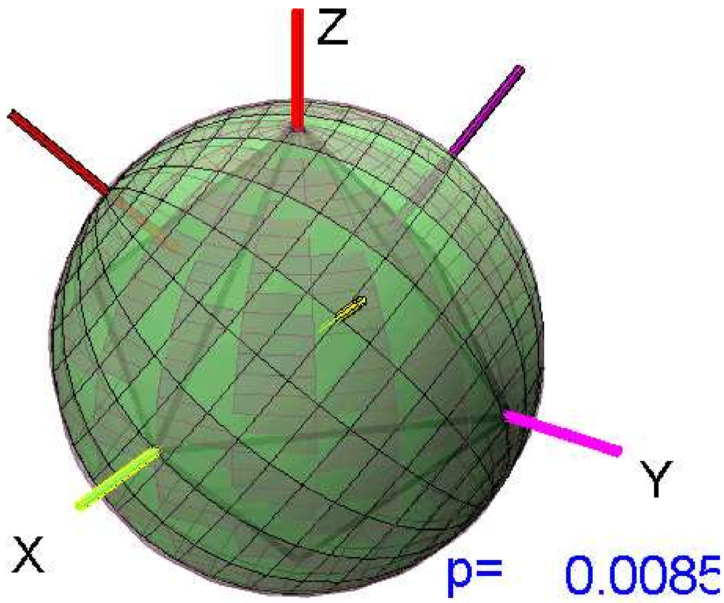
II. Tight Depolarizing Noise Threshold

Interpreting Polytope Facets ("A-type")

$$R_{|T\rangle} = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{2}-\frac{1}{2\sqrt{3}} & \frac{1}{2}-\frac{1}{2\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{2}-\frac{1}{2\sqrt{3}} & -\frac{1}{2}-\frac{1}{2\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

- $M = (1-p)R_{|T\rangle}$
- $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

$$\Rightarrow M \cdot A = \sqrt{3}(1-p)$$



Operation outside an "A-type" facet of polytope.	$\Leftrightarrow$	Qubit stabilizer state mapped outside octahedron.

Interpreting Polytope Facets ("B-type")

- Recall:

$$SU(2) \text{ picture}: \quad \mathcal{E}(\rho) = (1-p)U\rho U^\dagger + p\mathcal{E}_{\text{NOISE}}(\rho)$$
$$SO(3) \text{ picture}: \quad M = (1-p)R + pN$$

- Define: $\varrho = \mathcal{I} \otimes \mathcal{E}(|\Phi\rangle\langle\Phi|)$

$\left(\text{Jamiolkowski}: |\Phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)\right)$

- Pauli Decomposition:

$\varrho = \frac{1}{4}\sum_{i,j} \mathsf{c}_{ij}(\sigma_i \otimes \sigma_j) \qquad i,j \in \{I,X,Y,Z\}$

$$\Rightarrow \mathsf{c}_{IX} = \mathsf{c}_{IY} = \mathsf{c}_{IZ} = \mathsf{c}_{XI} = \mathsf{c}_{YI} = \mathsf{c}_{ZI} = 0, \quad \mathsf{c}_{II} \equiv 1$$

$$\Rightarrow M = \begin{pmatrix} \mathsf{c}_{XX} & -\mathsf{c}_{YX} & \mathsf{c}_{ZX} \\ \mathsf{c}_{XY} & -\mathsf{c}_{YY} & \mathsf{c}_{ZY} \\ \mathsf{c}_{XZ} & -\mathsf{c}_{YZ} & \mathsf{c}_{ZZ} \end{pmatrix}$$

Interpreting Polytope Facets ("B-type")

- Perform Stabilizer Measurement on ϱ

  $\Rightarrow$ Postselect to get single qubit state ρ' [Rei2]

e.g. Measurement $\Pi = \frac{1}{2}(II + YX)$ returns

$$\vec{r}(\rho') = \left(0, \frac{c_{XZ} - c_{ZY}}{c_{II} + c_{YX}}, -\frac{c_{XY} + c_{ZZ}}{c_{II} + c_{YX}}\right).$$

- Check if ρ' outside octahedron ($||\vec{r}(\rho')||_1 > 1$?)

$$|c_{\mathbf{XZ}} - c_{\mathbf{ZY}}| + |-(c_{\mathbf{XY}} + c_{\mathbf{ZZ}})| - c_{\mathbf{YX}} > c_{\mathbf{II}}? \qquad (c_{\mathbf{II}} \equiv \mathbf{1})$$

$$\begin{pmatrix} c_{XX} & -c_{YX} & c_{ZX} \\ c_{XY} & -c_{YY} & c_{ZY} \\ c_{XZ} & -c_{YZ} & c_{ZZ} \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 1 & 0 & -1 \end{pmatrix} > 1 \text{ ?}$$

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  $|c_{\mathbf{XZ}} - c_{\mathbf{ZY}}| + |-(c_{\mathbf{XY}} + c_{\mathbf{ZZ}})| - c_{\mathbf{YX}} > c_{\mathbf{II}}$? $\quad (c_{\mathbf{II}} \equiv \mathbf{1})$

$$\left(\quad M \quad \right) \cdot \left(\quad F \quad \right) > 1 \, ?$$

Part I Proven

I. Tight Qubit Ancilla Threshold $\Longrightarrow$ Tight Noise Threshold (All Unital Operations)

- Depending on which facet $M = (1-p)R + pN$ violates:

  $[\mathcal{F}_A]$: $\mathcal{E}$ applied to qubit stabilizer state $\rightarrow$ Outside Face

  $[\mathcal{F}_B]$: $\mathcal{E}$ applied to half $|\Phi\rangle$ then measure $\rightarrow$ Outside Edge

- Corollary: Any noise model that enters Clifford Polytope via "B-type" facet has tight threshold.

Tight Threshold for Depolarizing Noise

- The corollary applies to depolarizing noise.
- We can prove that whenever depolarized R is outside the Clifford polytope, it is outside a "B-type" facet.
- We know that "B-type" facets lead to tight thresholds.
- Since $M = (1-p)R$, suffices to prove

  $\forall A \in \mathcal{F}_A \cup \mathcal{F}_{A^T}, \quad \exists B \in \mathcal{F}_B$ such that $R \cdot (B - A) \geq 0.$

- Using Clifford symmetry, we just consider a subset of $R \in SO(3)$ without loss of generality.

Tight Threshold for Depolarizing Noise

- Using Clifford symmetry, only need to consider R s. t.

  $R \cdot A$ is maximized by $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$,

  and $\quad -R_{1,2} \geq |R_{i,j}| \qquad (i \in \{1,2,3\}, j \in \{2,3\})$

$$\Rightarrow R \in \left\{ \begin{pmatrix} + & - & + \\ + & + & - \\ + & + & + \end{pmatrix}, \begin{pmatrix} + & - & - \\ + & + & - \\ + & + & + \end{pmatrix}, \begin{pmatrix} + & - & - \\ + & + & - \\ + & - & + \end{pmatrix}, \begin{pmatrix} + & - & + \\ + & - & - \\ + & + & + \end{pmatrix} \right\}.$$

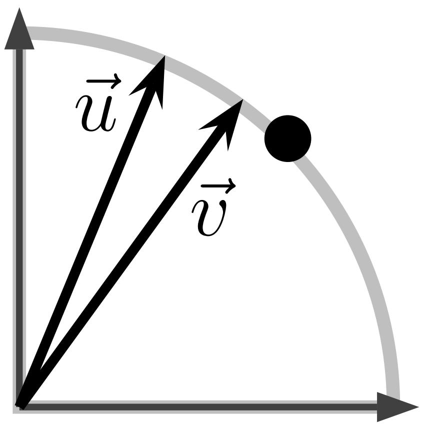
- For such R, we know which $B \in \mathcal{F}_B$ maximizes $R \cdot B$.

$$R \cdot (B - A) = \begin{pmatrix} + & - & \cdot \\ + & \cdot & - \\ + & \cdot & + \end{pmatrix} \cdot \left[\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right]$$

Tight Threshold for Depolarizing Noise

$$R \cdot (B - A) \geq 0 \quad \Leftrightarrow \begin{pmatrix} + & - & \cdot \\ + & \cdot & - \\ + & \cdot & + \end{pmatrix} \cdot \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix} \geq 0$$

- Define $\vec{u} = (R_{1,1}, R_{1,2})$ and $\vec{v} = (R_{2,3}, R_{3,3})$.
- Note that $\vec{u}$ and $\vec{v}$ have the same Euclidean norm.
- $R \cdot (B - A) \geq 0 \quad \Leftrightarrow \quad ||\vec{v}||_1 - ||\vec{u}||_1 \geq 0$



By assumption:
$|R_{1,2}| \geq |R_{2,3}|, |R_{3,3}|$
and hence
$||\vec{v}||_1 \geq ||\vec{u}||_1$

Open Questions

- Close the gap for ancilla distillation?
- Non-unital Noise (e.g. Amplitude Damping)